UNIT-II

UNIT – II •

Infinite Impulse Response Filter design • (IIR): Analog & Digital Frequency transformation. Designing by

impulse invariance & Bilinear method. • Butterworth and Chebyshev Design Method.

What is meant by a filter!

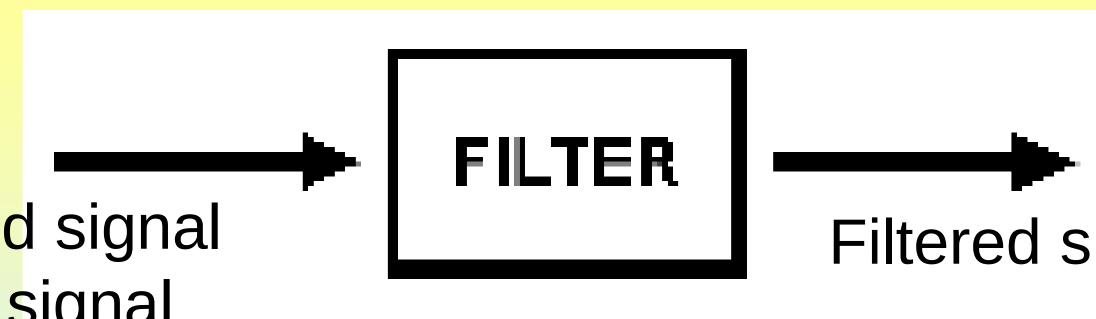
The DTFT is remembered again:

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw}) e^{jwn} dw$$

X[n] is expressed as a summation of sinusoids with scaled amplitude. Using a system with a frequency selective to these inputs, then it is possible to pass some frequencies and attenuate the others. **Such a system is called a Filter**.

The function of a *filter* is to remove unwanted parts of the signal, such as random noise, or to extract useful parts of the signal, such as the components lying within a certain frequency range.



Classification of filters as analog or digital

Digital filters

A **digital** filter uses a digital processor to perform numerical calculations on sampled values of the signal. The processor may be a general-purpose computer such as a PC, or a specialized DSP (Digital Signal Processor) chip.

Analog filters

An **analog** filter uses analog electronic circuits made up from components such as resistors, capacitors and op amps to produce the required filtering effect. Such filter circuits are widely used in such applications as noise reduction, video signal enhancement, graphic equalizers in hi-fi systems, and many other areas.

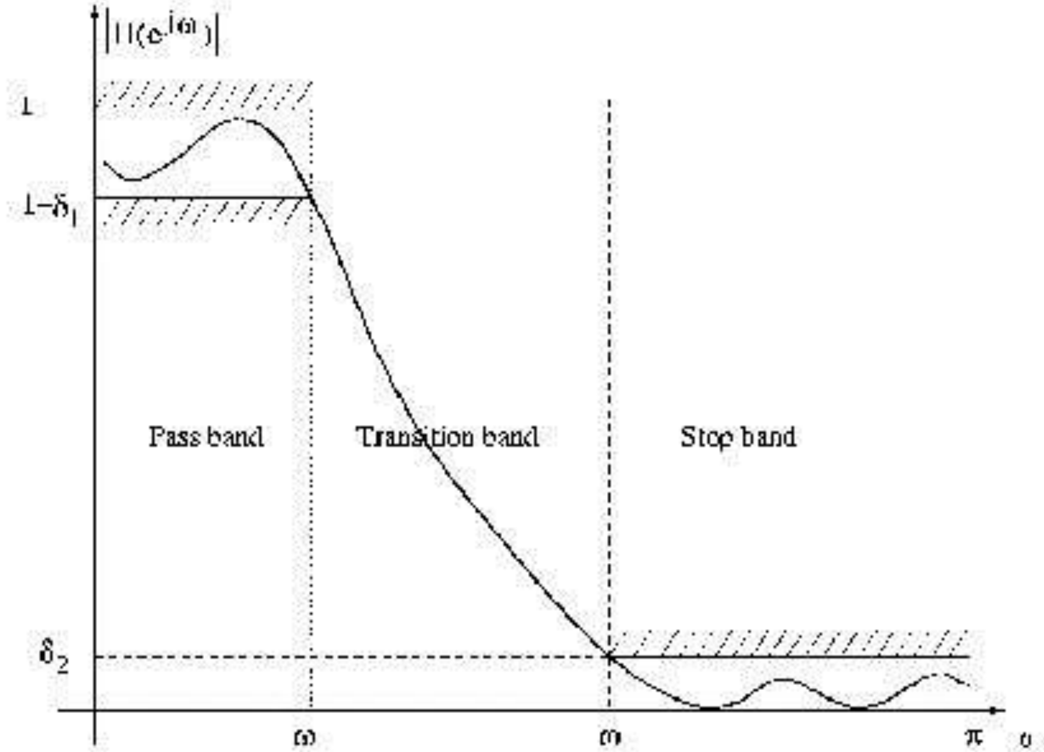
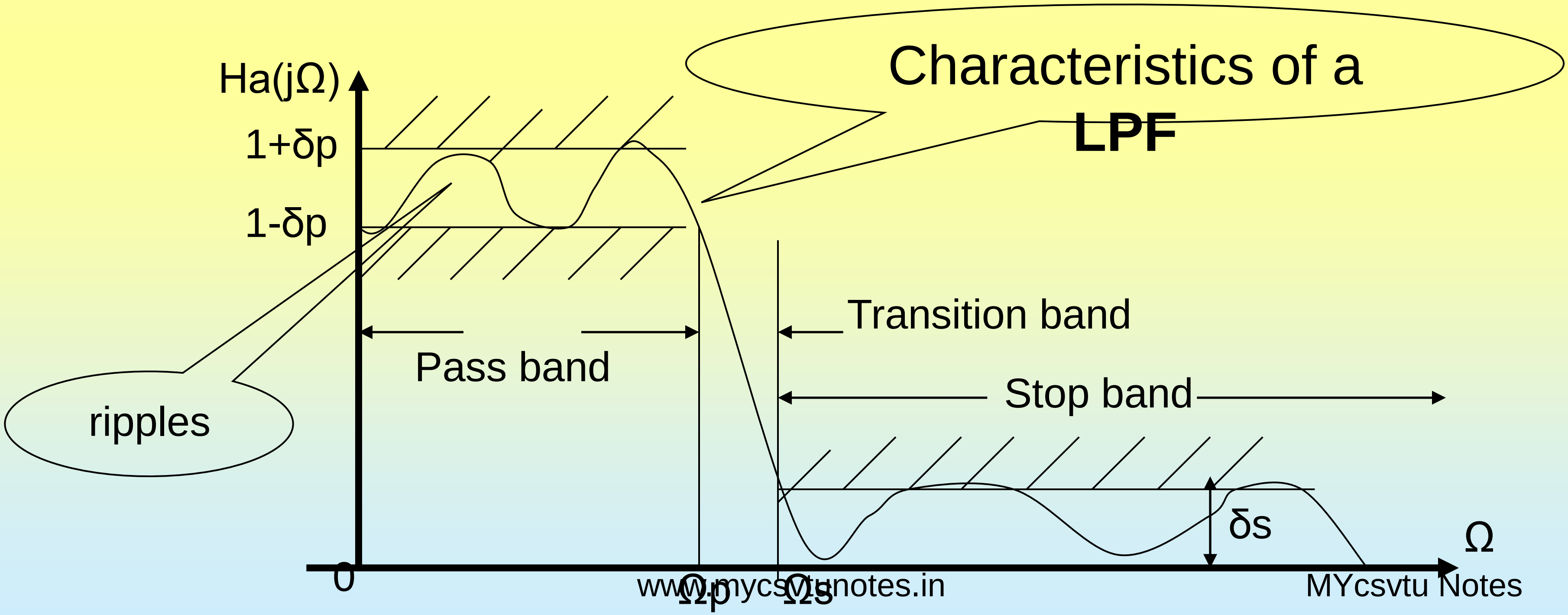


Figure 4.1: Tolerance limits for approximation of ideal low-pass filter

Design of analog filters

We need to discuss some of the famous techniques for analog filter design due to two main reasons:

- We need them as a prefilter or antialiasing filter before the A/D conversion,
- Some techniques for digital filter design are based on the transformation of some analog techniques.



$1-\delta p \le |Ha(j\Omega)| \le 1+\delta p \qquad 0 \le |\Omega| \le \Omega p$

Pass band

$|Ha(j\Omega)| \le \delta s \qquad \Omega s \le |\Omega| \le \infty$

Stop band

$$k = \frac{\Omega_p}{\Omega_s}$$

Transition ratio, or selectivity parameter, it is larger than unity

$a_p = -20\log_{10}(1-\delta p)$ dB

Peak pass band ripple

$a_s = -20\log_{10}(\delta s)$ dB

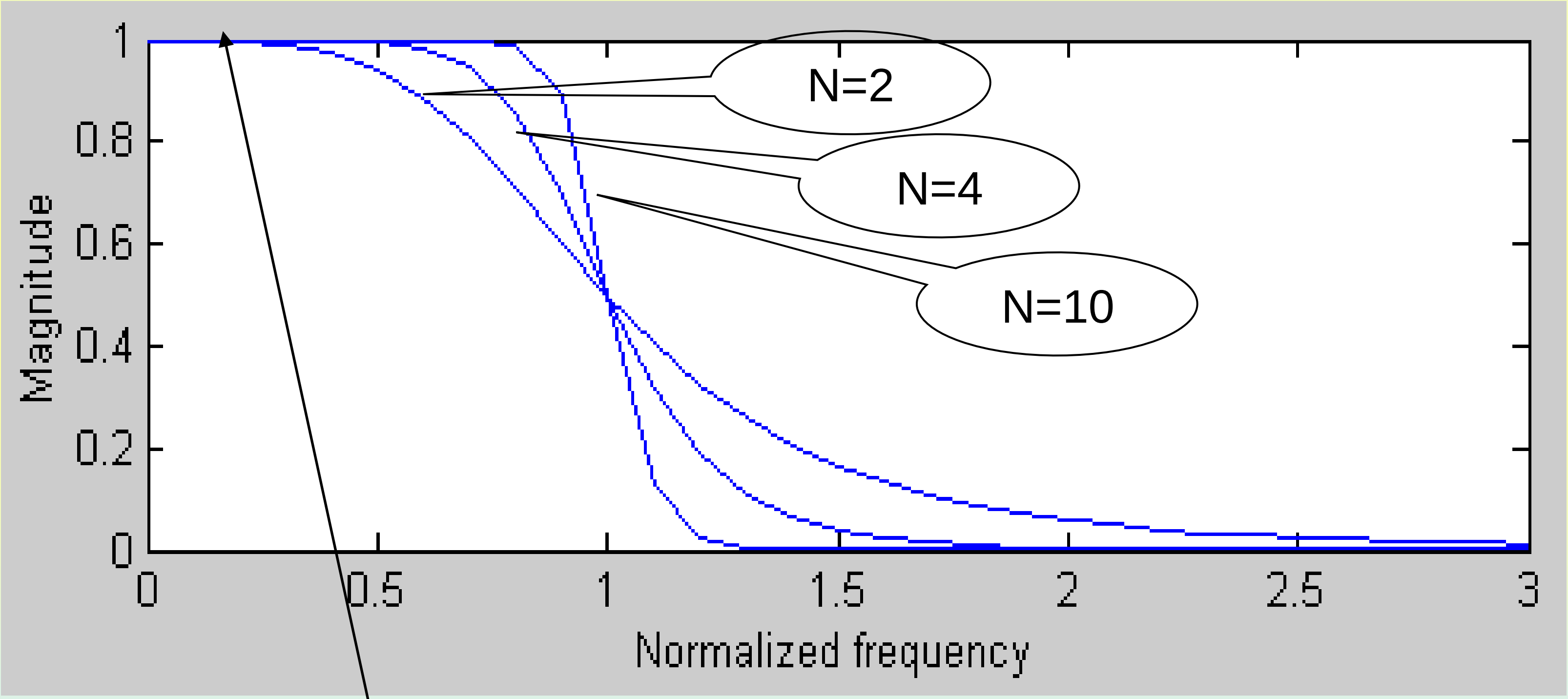
Minimum stop band attenuation

Such filter is completely characterized by Ωc, the 3dB point, Ωs, Ωp, δp, and δs.

Butterworth Approximation

Ωc is the cut off frequency or the -3db cut ff frequency

$$|H_a(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega c)^{2N}}$$



It has a maximally flat magnitude at zero frequency. This clear from (2N-1) differentiation of its function gives zeros.

Type 1 Chebyshev Approximation

$$\left|H_a(j\Omega)\right|^2 = \frac{1}{1+\eta^2 T_N^2(\Omega/\Omega p)}$$

Type 1 Chebyshev filter. It is equiripple in the pass band and monotonically decreasing in the stop band

$T_N(\Omega)=$ $\cos(N\cos^{-1}\Omega)$ $\quad |\Omega| \le 1$

$=$ $\cosh(N\cosh^{-1}\Omega)$ $\quad |\Omega| > 1$

η^2 represent the ripples in the pas band

CHEBY1 Chebyshev type I digital and analog filter design.

[num,den] = CHEBY1(N,R,Wn) designs an Nth order lowpass digital Chebyshev filter with R decibels of ripple in the pass band. CHEBY1 returns the filter coefficients in length N+1 vectors (numerator) and (denominator).

Type 2 Chebyshev Approximation

Type 2 Chebyshev filter. It is equiripple in the stop band and monotonically decreasing in the pass band

$$\left|H_a(j\Omega)\right|^2 = \frac{1}{1+\eta^2\left[\dfrac{T_N(\Omega_s/\Omega_p)}{(\Omega_s/\Omega)}\right]^2}$$

CHEBY2 Chebyshev type II digital and analog filter design.

[B,A] = CHEBY2(N,R,Wn) designs an Nth order lowpass digital Chebyshev filter with the stop band ripple R decibels down and stop band edge frequency Wn. CHEBY2 returns the filter coefficients in length N+1 vectors B (numerator) and A (denominator).

Elliptic Approximation

$$\left|H_a(j\Omega)\right|^2 = \frac{1}{1+\eta^2 R_N^2(\Omega/\Omega_P)}$$

Elliptic filter. It is equiripple in the stop band and in the pass band

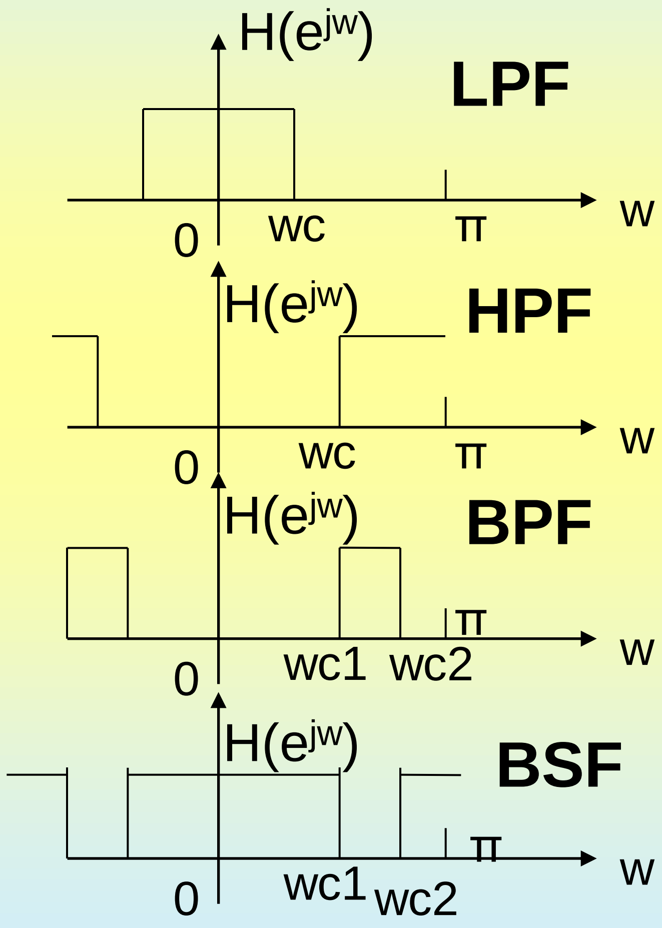
$R_N(\Omega)$ is a rational function satisfies the property

$R_N(1/\Omega)=1/R_N(\Omega)$

ELLIP Elliptic or Cauer digital and analog filter design.

[B,A] = ELLIP(N,Rp,Rs,Wn) designs an Nth order lowpass digital elliptic filter with Rp decibels of ripple in the passband and a stopband Rs decibels down. ELLIP returns the filter coefficients in length N+1 vectors B (numerator) and A (denominator).

Classification of filters According to frequency response



LPF
- Pass band $0 \le w \le wc$
- Stop band $wc \le w \le \pi$

HPF
- Stop band $0 \le w \le wc$
- Pass band $wc \le w \le \pi$

BPF
- Pass band $wc1 \le w \le wc2$
- Stop band $0 \le w \le wc1,\ wc2 \le w \le \pi$

BSF
- Stop band $wc1 \le w \le wc2$
- Pass band $0 \le w \le wc1,\ wc2 \le w \le \pi$

Wc, wc1, and wc2 are called the cut off frequencies.

Design Of Digital Filters

Advantages of using digital filters

1. A digital filter is *programmable*, i.e. its operation is determined by a program stored in the processor's memory. This means the digital filter can easily be changed without affecting the circuitry (hardware). An analog filter can only be changed by redesigning the filter circuit.

2. Digital filters are easily *designed*, *tested* and *implemented* on a general-purpose computer or workstation.

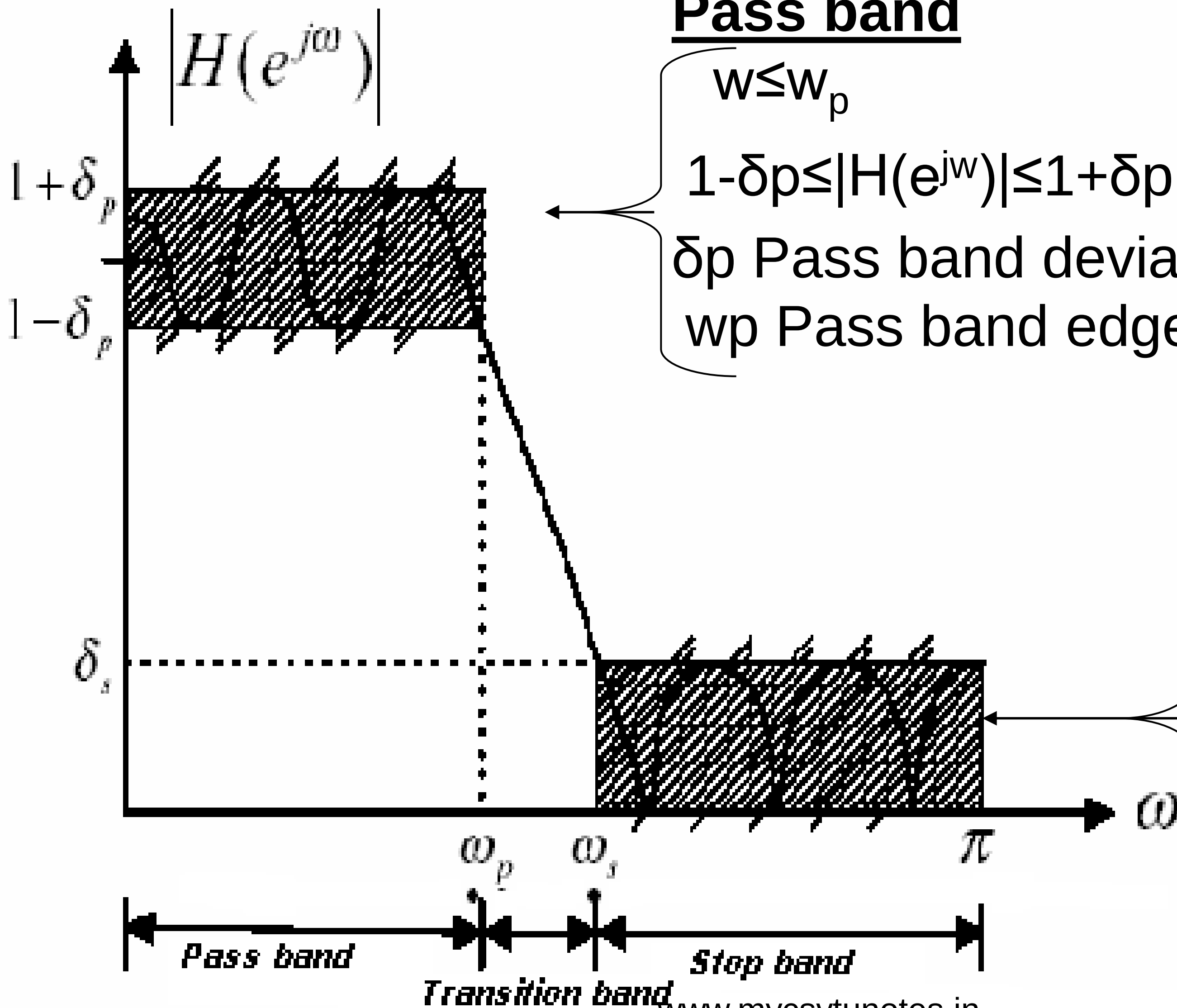
3. The characteristics of analog filter circuits (particularly those containing active components) are subject to drift and are dependent on temperature. **Digital filters do not suffer from these problems,** and so are extremely *stable* with respect to both time and temperature.

4. Unlike their analog counterparts, **digital filters can handle *low frequency* signals accurately**. As the speed of DSP technology continues to increase, digital filters are being applied to high frequency signals in the RF (radio frequency) domain, which in the past was the exclusive preserve of analog technology.

5. Digital filters are very much more *versatile* in their ability to process signals in a variety of ways; this includes the ability of some types of digital filter to adapt to changes in the characteristics of the signal.

6. Fast DSP processors can handle complex combinations of filters in parallel or cascade (series), making the hardware requirements relatively *simple* and *compact* in comparison with the equivalent analog circuitry.

1- Filter Characteristics Specification



Pass band

$w \le w_p$

$1-\delta p \le |H(e^{jw})| \le 1+\delta p$

δp Pass band deviation

wp Pass band edge frequency

Stop band

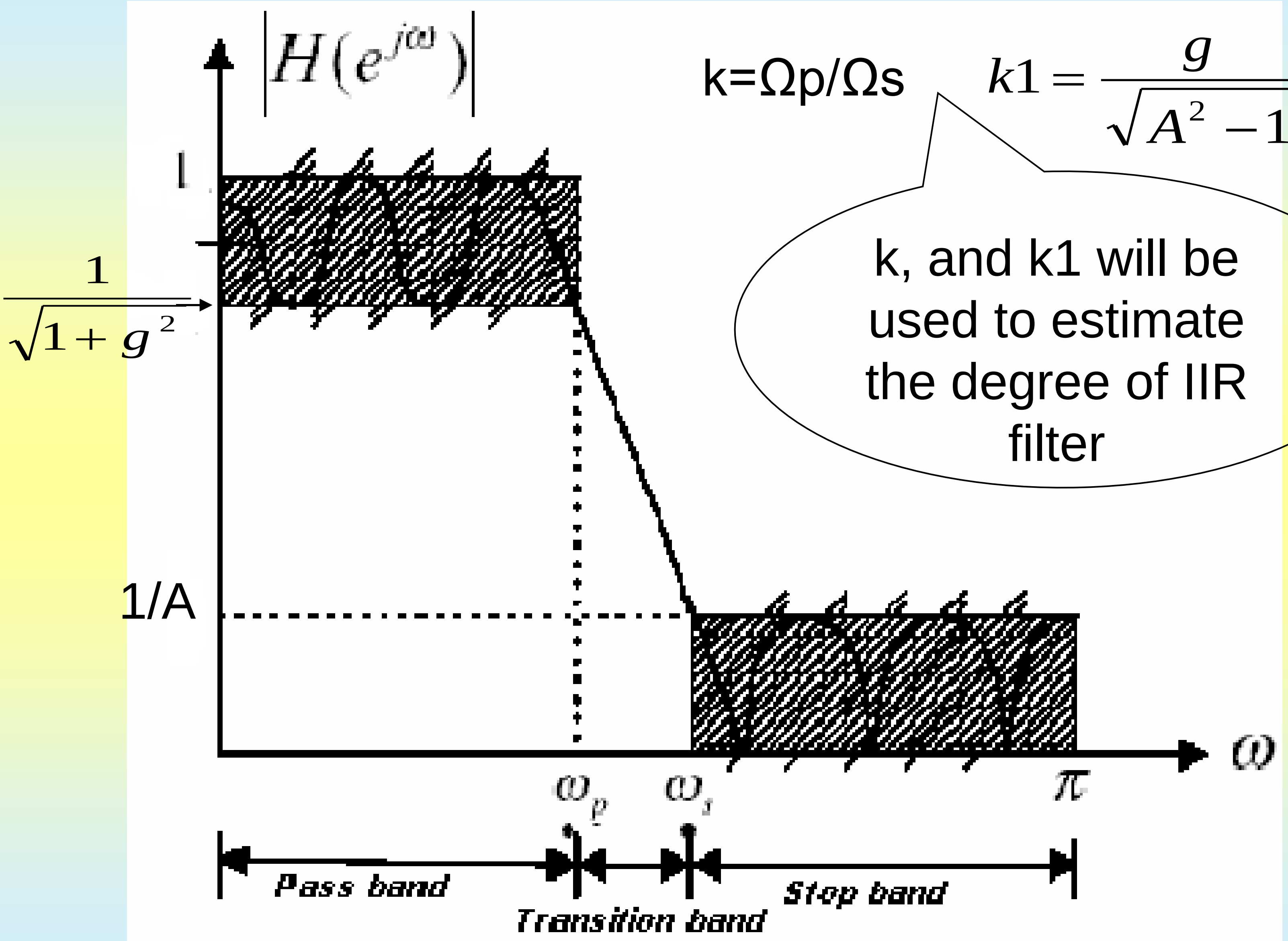
$w_s \le w \le \pi$

$|H(e^{jw})| \le \delta s$

δs Stop band deviation

ws Stop band edge frequency

Normalized LPF specs



Classification of filters according to impulse response length

Finite Impulse Response, **FIR** filters

$$h[n] = \sum_{n=0}^{M-1} a^n u[n]$$

Infinite Impulse Response, **IIR** filters

$$h[n] = \sum_{n=0}^{\infty} a^n u[n]$$

2- Selection of filter type

FIR or IIR

1. FIR can have an exactly linear phase response.

2. FIR realized nonrecursively is always stable.

3. Quantization effects are less severe in FIR than in IIR.

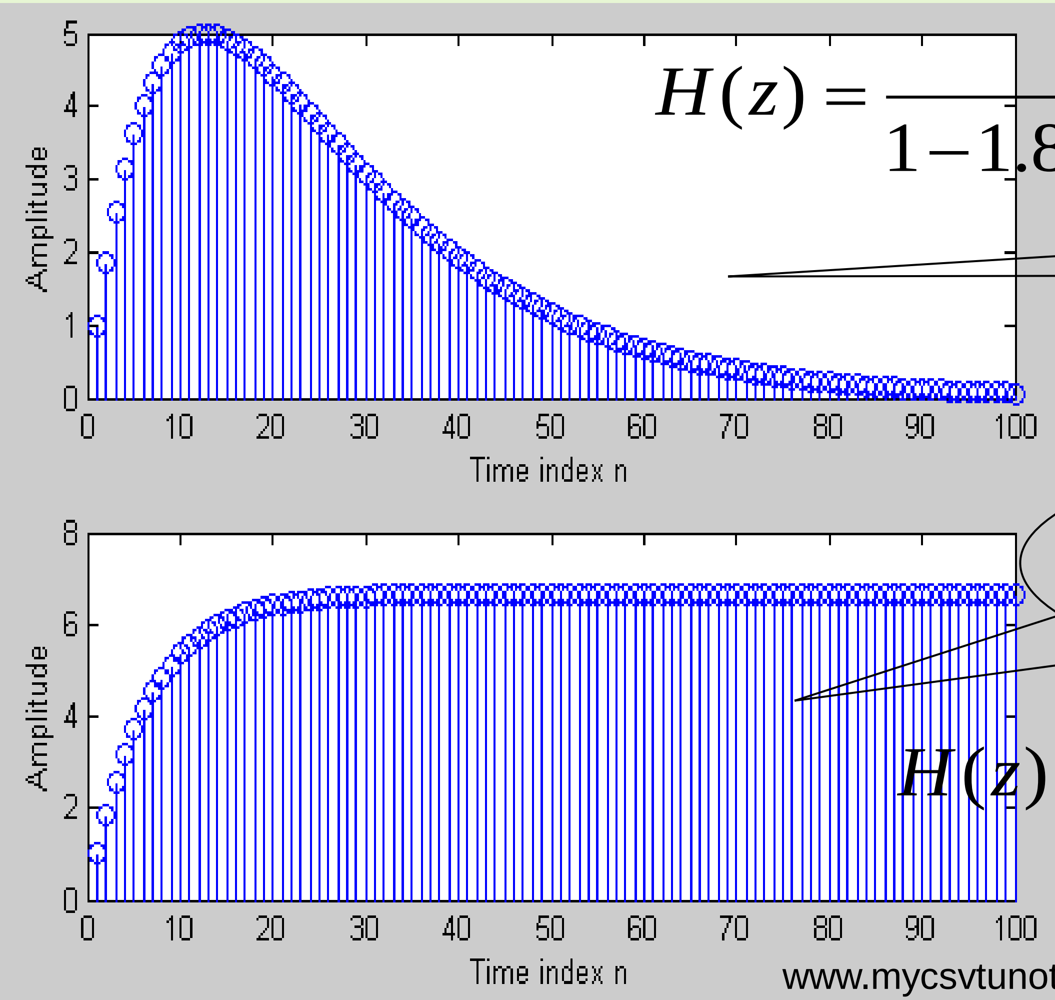
4. FIR requires more coefficients for sharp cutoff than IIR.

5. Analog filters can be transformed into IIR.

6. FIR is easier to synthesize if CAD support is available.

An FIR system is always stable, But an IIR system may be stable or not, and it must be designed properly.

An originally stable IIR filter with precession coefficients may become unstable after implementation due to unavoidable quantization error in its coefficients. !!!



$$H(z) = \frac{1}{1 - 1.845z^{-1} + 0.850586z^{-1}}$$

Stable IIR filter

After quantization unstable IIR filter

$$H(z) = \frac{1}{1 - 1.85z^{-1} + 0.85z^{-1}}$$

Filter degree

1- IIR filter

Butterworth filter

$$N = \frac{\log_{10}(1/k1)}{\log_{10}(1/k)}$$

Chebyshev filter

$$N = \frac{\cosh^{-1}(1/k1)}{\cosh^{-1}(1/k)}$$

Elliptic filter

$$N \cong \frac{2\log_{10}(4/k1)}{\log_{10}(1/\rho)}, \rho = \rho0 + 2(\rho0)^5 + 15(\rho0)^9 + 150(\rho0)^{13}$$

$$\rho0 = \frac{1-\sqrt{k'}}{2\left(1+\sqrt{k'}\right)} \qquad k' = \sqrt{1-k^2}$$

2- FIR filter

$$N \cong \frac{-20\log_{10}\left(\sqrt{\delta_p \delta_s}\right) - 13}{14.6(w_s - w_p)/2\pi}$$

There are another approximation for very narrow pass band and very wide pass band

Example

LPF with 1dB at wp=1kHz, and 40dB at ws=5kHz

$$10\log_{10}\left(\frac{1}{1+g^2}\right)=-1 \Rightarrow$$ g^2=0.25895

$$10\log_{10}\left(\frac{1}{A^2}\right)=-40 \Rightarrow$$ A^2=10000

1- Butterworth filter $$N=\frac{\log_{10}(1/k1)}{\log_{10}(1/k)} \Rightarrow$$ N=3.281=4

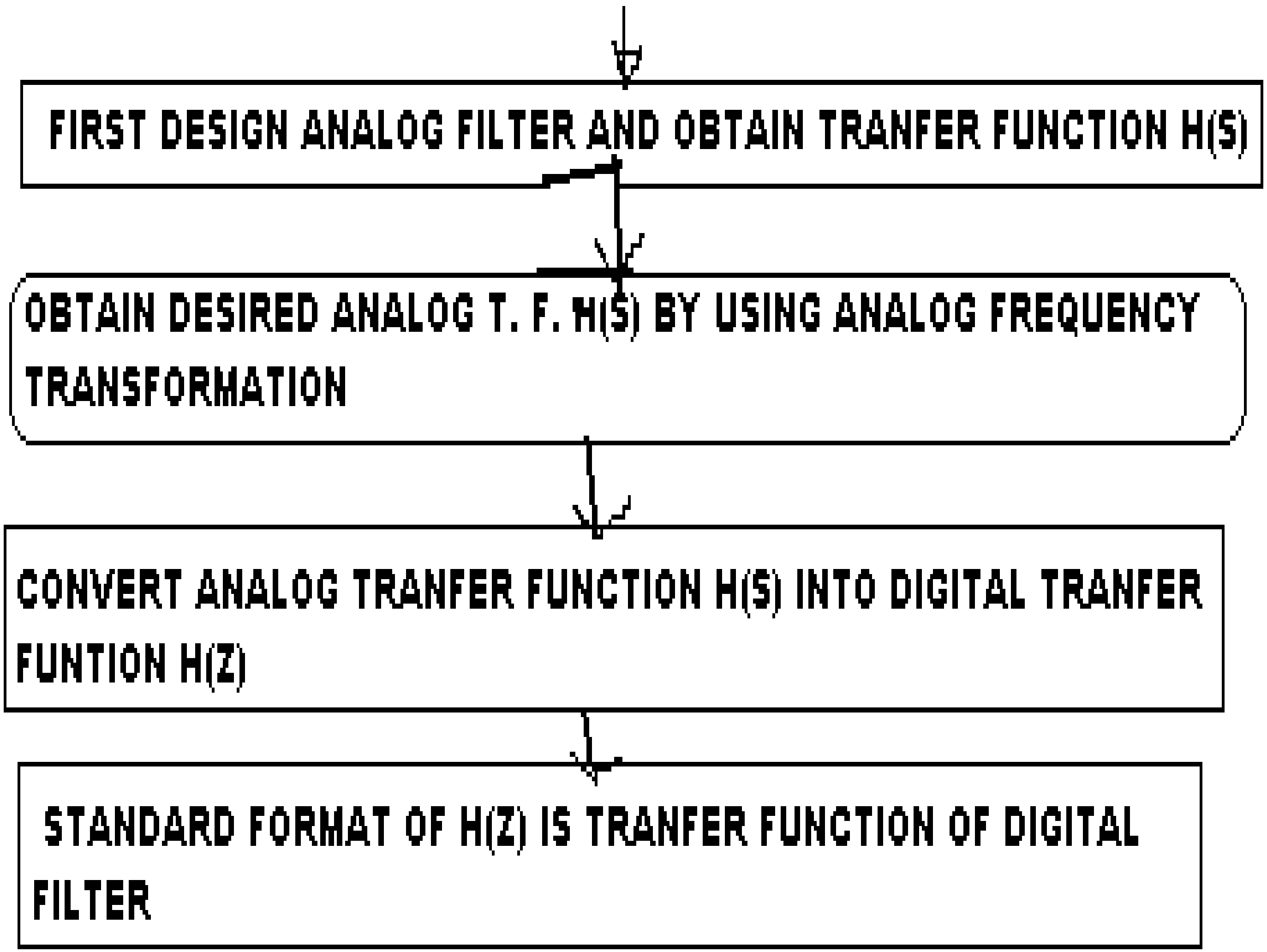
2- Chebyshev filter $$N=\frac{\cosh^{-1}(1/k1)}{\cosh^{-1}(1/k)} \Rightarrow$$ N=2.6059=3

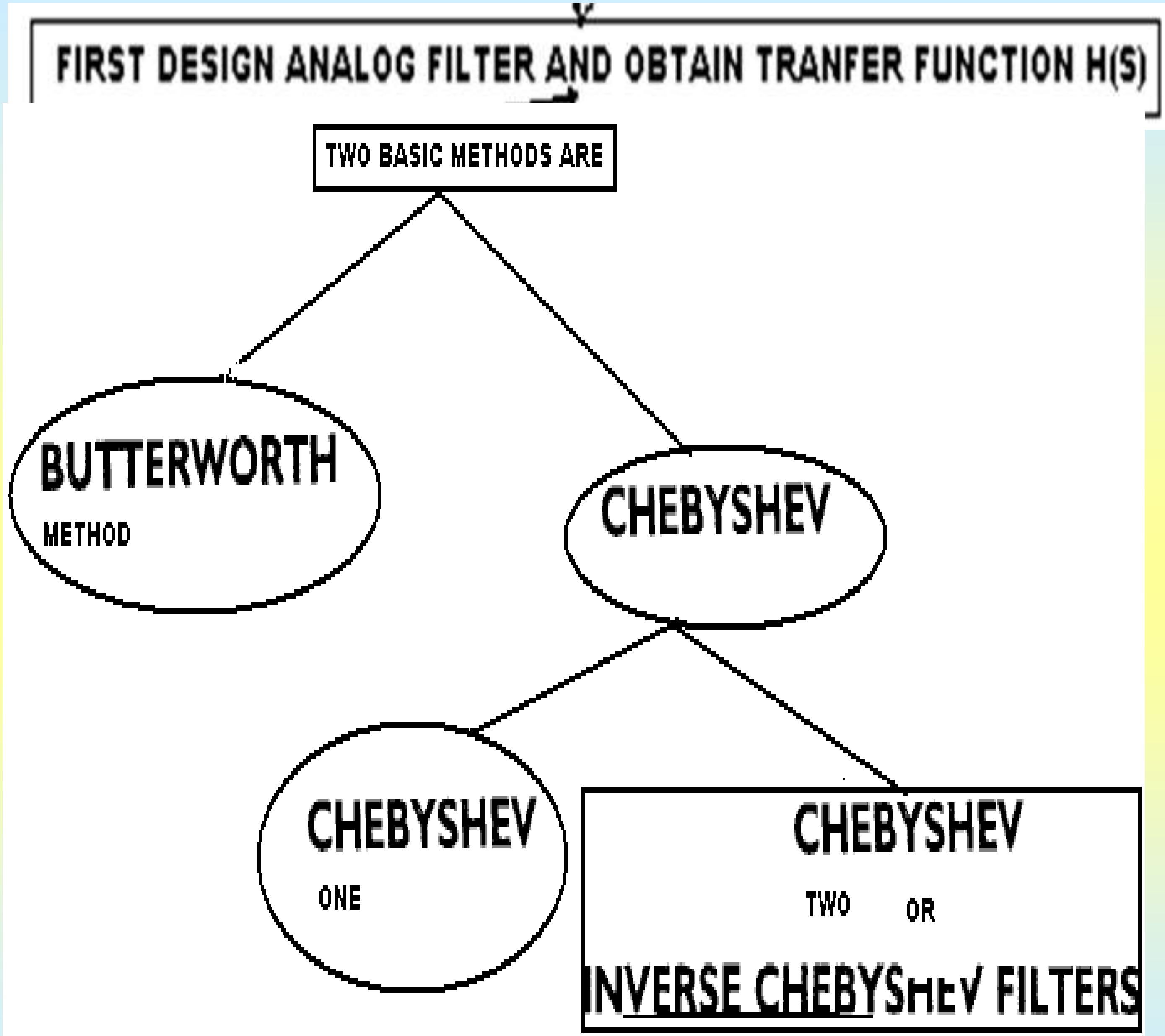
3- FIR filter $$N\cong\frac{-20\log_{10}\left(\sqrt{\delta_p\delta_s}\right)-13}{14.6(w_s-w_p)/2\pi} \Rightarrow$$ N=7

agk

DESIGN OF

Infinite Impulse Response (IIR) Filters


FIRST DESIGN ANALOG FILTER AND OBTAIN TRANFER FUNCTION H(S)
OBTAIN DESIRED ANALOG T. F. H(S) BY USING ANALOG FREQUENCY TRANSFORMATION
CONVERT ANALOG TRANFER FUNCTION H(S) INTO DIGITAL TRANFER FUNTION H(Z)
STANDARD FORMAT OF H(Z) IS TRANFER FUNCTION OF DIGITAL FILTER


FIRST DESIGN ANALOG FILTER AND OBTAIN TRANFER FUNCTION H(S)
TWO BASIC METHODS ARE
BUTTERWORTH
METHOD
CHEBYSHEV
CHEBYSHEV
ONE
CHEBYSHEV
TWO OR
INVERSE CHEBYSHEV FILTERS

OBTAIN DESIRED ANALOG T. F. H(S) BY USING ANALOG FREQUENCY TRANSFORMATION

Analog frequency transformation

Type	*Transformation*
Low-pass	$s \to \frac{\Omega_c}{\Omega_c^*} s$
High-pass	$s \to \frac{\Omega_c \Omega_c^*}{s}$
Bandpass	$s \to \Omega_c \frac{s^2 + \Omega_1 \Omega_2}{s(\Omega_2 - \Omega_1)}$
Bandstop	$s \to \Omega_c \frac{s(\Omega_2 - \Omega_1)}{s^2 + \Omega_1 \Omega_2}$

CONVERT ANALOG TRANFER FUNCTION H(S) INTO DIGITAL TRANFER FUNTION H(Z)

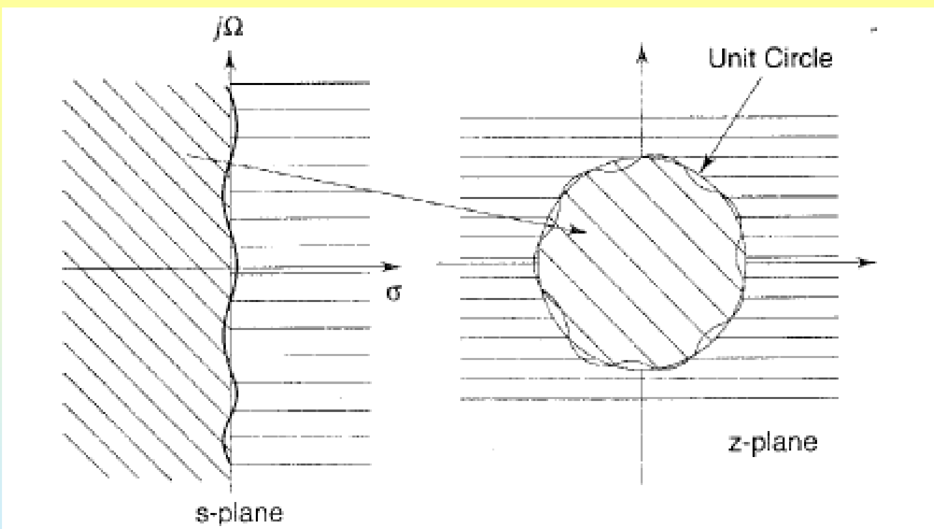
METHODS

- **IIR FILTER DESIGN BY APPROXIMATION OF DERIVATIVES**
- **IIR FILTER DESIGN BY IMPULSE INVARIANT METHOD**

  $$\omega = \Omega T$$

- **IIR FILTER DESIGN BY THE BILINEAR TRANSFORMATION**

  $$\omega = 2\tan^{-1}\frac{\Omega T}{2}$$



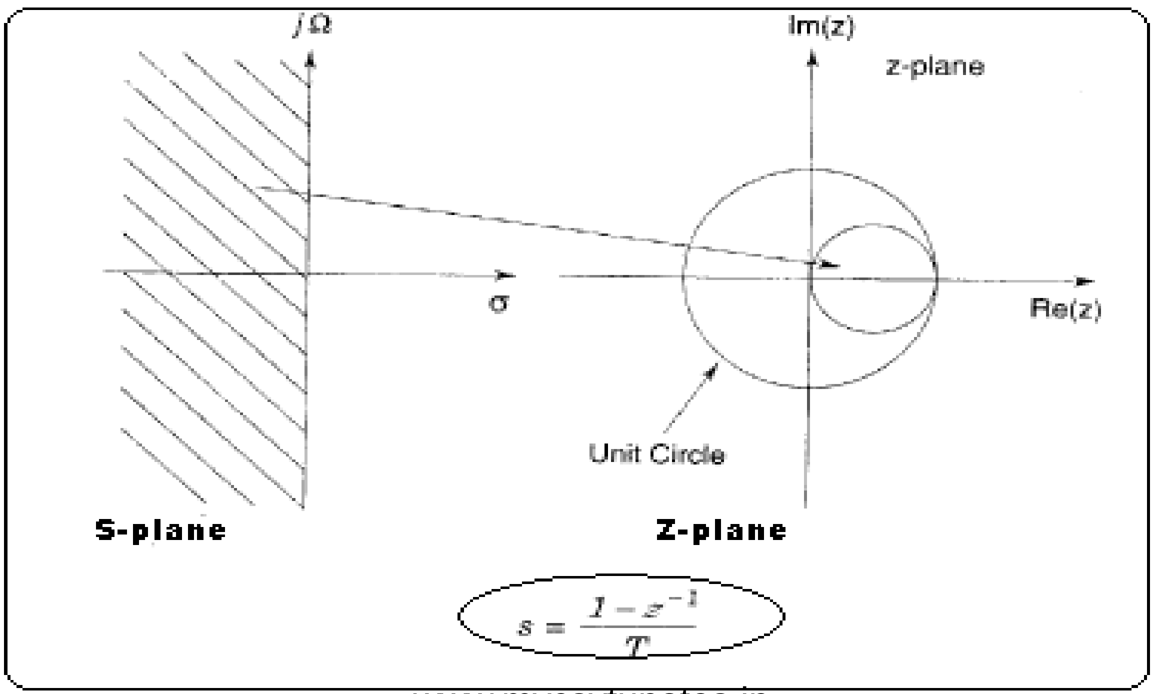
Mapping from s-plane to z-plane

CONVERT ANALOG TRANFER FUNCTION H(S) INTO DIGITAL TRANFER FUNTION H(Z)

IIR FILTER DESIGN BY APPROXIMATION OF DERIVATIVES

$$H(z) = H_a(s)\big|_{s=(1-z^{-1})/T}$$

$$s^i = \left(\frac{1-z^{-1}}{T}\right)^i$$



Example 8.1 Use the backward difference for the derivative to convert the analog low-pass filter with system function

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$$H(s) = \frac{1}{s+2}$$

Example 8.2 Use the backward difference for the derivative and convert the analog filter with system function

$$H(s) = \frac{1}{s^2+16}$$

Example 8.3 An analog filter has the following system function. Convert this filter into a digital filter using backward difference for the derivative.

$$H(s) = \frac{1}{(s+0.1)^2+9}$$

Example 8.1

ANSWER OF 8.1,8.2 &8.3

If $T = 1s$,

$$H(z) = \frac{1}{3 - z^{-1}}$$

Example 8.2

$$H(z) = \frac{1}{z^{-2} - 2z^{-1} + 17}$$

Example 8.3

If $T = 1s$,

$$H(z) = \frac{0.0979}{1 - 0.2155\, z^{-1} + 0.09792\, z^{-2}}$$

IIR FILTER DESIGN BY IMPULSE INVARIANT METHOD

T is the sampling interval $h(n) = h_a(nT)$

Some of the properties of the impulse invariant transformation

$$\frac{1}{(s+s_i)^m} \rightarrow \frac{(-1)^{m-1}}{(m-1)!} \frac{d^{m-1}}{ds^{m-1}} \left[\frac{1}{1-e^{-sT} z^{-1}} \right]; s \rightarrow s_i$$

$$\frac{s+a}{(s+a)^2+b^2} \rightarrow \frac{1-e^{-aT}(\cos bT) z^{-1}}{1-2e^{-aT}(\cos bT) z^{-1} + e^{-2aT} z^{-2}}$$

$$\frac{b}{(s+a)^2+b^2} \rightarrow \frac{e^{-aT}(\sin bT) z^{-1}}{1-2e^{-aT}(\cos bT) z^{-1} + e^{-2aT} z^{-2}}$$

REMEMBER ABOVE FORMULA

$$H_a(s) = \sum_{i=1}^{M} \frac{A_i}{s - p_i}$$ filter's system function

taking the inverse Laplace transform

$$h_a(t) = \sum_{i=1}^{M} A_i e^{p_i t} u_a(t)$$

$u_a(t)$ is the unit step function in continuous time

$$h(n) = h_a(nT) = \sum_{i=1}^{M} A_i e^{p_i n t} u_a(nT)$$

taking the z-transform

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

$$H(z) = \sum_{n=0}^{\infty} \left[\sum_{i=1}^{M} A_i e^{p_i nT} u_a(nT) \right] z^{-n}$$

OR

$$H(z) = \sum_{i=1}^{M} \left[\sum_{n=0}^{\infty} A_i e^{p_i nT} u_a(nT) \right] z^{-n}$$

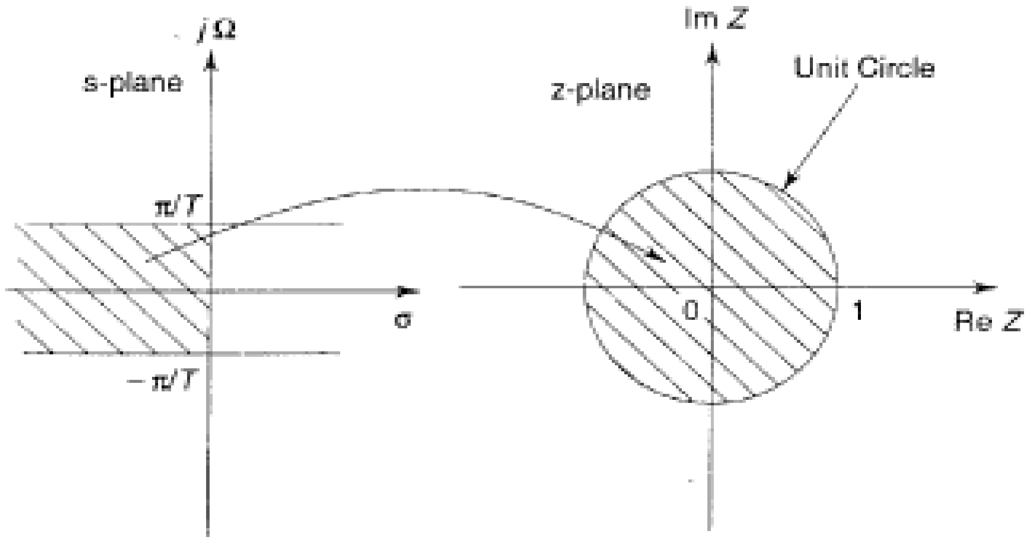
$$H(z) = \sum_{i=1}^{M} \frac{A_i}{1 - e^{p_i T} z^{-1}}$$

$$\frac{1}{s - p_i} \longrightarrow \frac{1}{1 - e^{p_i T} z^{-1}}$$

$$z = e^{sT} \qquad r e^{j\omega} = e^{\sigma T} e^{j\Omega T} \qquad r = e^{\sigma T}$$

$$\omega = \Omega T$$

$$z = e^{p_i T}$$



The Mapping of $z = e^{sT}$

Example 8.4 Convert the analog filter into a digital filter whose system function is

$$H(s) = \frac{s + 0.2}{(s + 0.2)^2 + 9}$$

Use the impulse invariant technique. Assume $T = 1$s.

Example 8.5 For the analog transfer function

$$H(s) = \frac{1}{(s + 1)(s + 2)}$$

determine $H(z)$ using impulse invariant technique. Assume $T = 1$s.

Example 8.6 Determine $H(z)$ using the impulse invariant techniq for the analog system function

$$H(s) = \frac{1}{(s + 0.5)(s^2 + 0.5s + 2)}$$

SOLUTION OF PROBLEM NUMBER **Example 8.4**

$$H(s) = \frac{s+a}{(s+a)^2 + b^2}$$

$$H(z) = \frac{1 - e^{-aT}(\cos bT)\, z^{-1}}{1 - 2e^{-aT}(\cos bT)\, z^{-1} + e^{-2aT}\, z^{-2}}$$

$$= \frac{1 - e^{-0.2T}(\cos 3T)\, z^{-1}}{1 - 2e^{-0.2T}(\cos 3T)\, z^{-1} + e^{-0.4T}\, z^{-2}}$$

$T = 1\text{s},$

$$H(z) = \frac{1 - (0.8187)(-0.99)\, z^{-1}}{1 - 2(0.8187)(-0.99)\, z^{-1} + 0.6703\, z^{-2}}$$

$$H(z) = \frac{1 + (0.8105)\, z^{-1}}{1 + 1.6210\, z^{-1} + 0.6703\, z^{-2}}$$

Example 8.5

$$H(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{A(s+2) + B(s+1)}{(s+1)(s+2)}$$

$$H(z) = \frac{1}{1 - e^{-T} z^{-1}} - \frac{1}{1 - e^{-2T} z^{-1}}$$

$$= \frac{z^{-1}[e^{-T} - e^{-2T}]}{1 - (e^{-T} + e^{-2T}) z^{-1} + e^{-3T} z^{-2}}$$

Since $T = 1\text{s}$,

$$H(z) = \frac{0.2326 z^{-1}}{1 - 0.5032 z^{-1} + 0.0498 z^{-2}}$$

Example 8.6

$$H(s) = \frac{1}{(s+0.5)(s^2+0.5s+2)} = \frac{A}{s+0.5} + \frac{Bs+C}{s^2+0.5s+2}$$

$$A(s^2+0.5s+2) + (Bs+C)(s+0.5) = 1$$

$A + B = 0$

$0.5A + 0.5B + C = 0$

$2A + 0.5C = 1$

$A = 0.5,\ B = -$[redacted] and $C = 0$.

$$H(s) = \frac{0.5}{s+0.5} - \frac{0.5s}{s^2+0.5s+2}$$

$$= \frac{0.5}{s+0.5} - 0.5\left(\frac{s}{(s+0.25)^2+(1.3919)^2}\right)$$

$$= \frac{0.5}{s+0.5} - 0.5\left(\frac{s+0.25}{(s+0.25)^2+(1.3919)^2} - \frac{0.25}{(s+0.25)^2+(1.3919)^2}\right)$$

$$= \frac{0.5}{s+0.5} - 0.5\left(\frac{s+0.25}{(s+0.25)^2+(1.3919)^2}\right)$$

$$+ 0.0898\left(\frac{1.3919}{(s+0.25)^2+(1.3919)^2}\right)$$

$$H(z) = \frac{0.5}{1-e^{-0.5T}z^{-1}} - 0.5\left[\frac{1-e^{-0.25T}(\cos 1.3919\,T)\,z^{-1}}{1-2e^{-0.25T}(\cos 1.3919\,T)\,z^{-1}+e^{-0.5T}z^{-2}}\right]$$

$$+ 0.0898\left[\frac{e^{-0.25T}(\sin 1.3919\,T)\,z^{-1}}{1-2e^{-0.25T}(\cos 1.3919\,T)\,z^{-1}+e^{-0.5T}z^{-2}}\right]$$

Letting $T = 1$s,

$$H(z) = \frac{0.5}{1-0.6065\,z^{-1}} - 0.5\left(\frac{1-0.1385\,z^{-1}}{(1+0.277\,z^{-1}+0.606\,z^{-2}}\right)$$

$$+ 0.0898\left(\frac{0.7663\,z^{-1}}{(1-0.277\,z^{-1}+0.606\,z^{-2}}\right)$$

$$H(z) = \frac{0.30325\,z^2+0.2194\,z}{z^3-0.8835\,z^2+0.774\,z-0.3675}$$

ANSWER

IIR FILTER DESIGN BY THE BILINEAR TRANSFORMATION

above both methods are useful for **LOW & BANDPASS FILTERS** WHOSE RESONANT FREQUENCIES ARE LOW, THESE TECHNIQUES ARE NOT SUITABLE FOR HIGH & BAND REJECT FILTERS. OVERCOME ABOVE LIMITATION USING **BILINEAR TRANSFORMATION**.

THIS IS ONE TO ONE MAPPING FROM S-PLANE TO Z-PLANE.

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) = \frac{2}{T}\left(\frac{z - 1}{z + 1}\right)$$

$$s = \frac{2}{T}\left(\frac{z - 1}{z + 1}\right) = \frac{2}{T}\left(\frac{re^{j\omega} - 1}{re^{j\omega} + 1}\right)$$

$$\omega = 2\tan^{-1}\frac{\Omega T}{2}$$

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

frequency warping.

Example 8.10 Using bilinear transformation obtain $H(z)$ if

$$H(s) = \frac{1}{(s+1)^2}$$

and $T = 0.1\text{s}$.

Solution For the bilinear transformation

$$H(z) = H(s)\big|_{s = \frac{2}{T}\frac{(z-1)}{(z+1)}} = \frac{1}{\left(\frac{2}{T}\frac{(z-1)}{(z+1)} + 1\right)^2}$$

Substituting $T = 0.1\text{s}$,

$$H(z) = \frac{1}{\left(20\frac{(z-1)}{(z+1)} + 1\right)^2} = \frac{(z+1)^2}{(21z-19)^2}$$

$$H(z) = \frac{0.0476\,(1+z^{-1})^2}{(1-0.9048\,z^{-1})^2}$$

Example 8.7 Convert the analog filter with system function

$$H(s) = \frac{s + 0.1}{(s + 0.1)^2 + 9}$$

into a digital IIR filter using bilinear transformation. The digital filter should have a resonant frequency of $\omega_r = \frac{\pi}{4}$.

Example 8.8 Apply bilinear transformation to

$$H(s) = \frac{2}{(s + 1)(s + 3)}$$

with $T = 0.1$s.

Example 8.9 A digital filter with a 3 dB bandwidth of 0.25π is to be designed from the analog filter whose system response is

$$H(s) = \frac{\Omega_c}{s + \Omega_c}$$

Use bilinear transformation and obtain $H(z)$.

Example 8.7

AGK

$\Omega_c = 3.$

$$\Omega_c = \frac{2}{T} \tan \frac{\omega_r}{2}$$

$$T = \frac{2}{\Omega_c} \tan \frac{\omega_r}{2} = \frac{2}{3} \tan \frac{\pi}{8} = 0.276 \text{ s}$$

$$H(z) = \frac{\frac{2}{T} \frac{(z-1)}{(z+1)} + 0.1}{\left[\frac{2}{T} \frac{(z-1)}{(z+1)} + 0.1\right]^2 + 9}$$

$$= \frac{(2/T)(z-1)(z+1) + 0 \cdot 1(z+1)^2}{[(2/T)(z-1) + 0.1(z+1)]^2 + 9(z+1)^2}$$

Substituting $T = 0.276$ s,

$$H(z) = \frac{1 + 0.027 z^{-1} - 0.973 z^{-2}}{8.572 - 11.84 z^{-1} + 8.177 z^{-2}}$$

Example 8.8

$$H(z) = H(s)\big|_{s = \frac{2}{T}\frac{(z-1)}{(z+1)}}$$

$$= \frac{2}{\left(\frac{2}{T}\frac{(z-1)}{(z+1)} + 1\right)\left(\frac{2}{T}\frac{(z-1)}{(z+1)} + 3\right)}$$

Using $T = 0.1\text{s}$,

$$H(z) = \frac{2}{\left(20\frac{(z-1)}{(z+1)} + 1\right)\left(20\frac{(z-1)}{(z+1)} + 3\right)}$$

$$= \frac{2(z+1)^2}{(21z-19)(23z-17)}$$

$$H(z) = \frac{0.0041\left(1+z^{-1}\right)^2}{1 - 1.644z^{-1} + 0.668z^{-2}}$$

AGK_RCET_6ET

Example 8.9

$$\Omega_c = \frac{2}{T} \tan \frac{\omega_r}{2} = \frac{2}{T} \tan 0.125\,\pi = 0.828/T$$

$$H(z) = H(s)\big|_{s = \frac{2}{T}\frac{(z-1)}{(z+1)}}$$

$$= \frac{\Omega_c}{\frac{2}{T}\frac{(z-1)}{(z+1)} + \Omega_c} = \frac{\frac{0.828}{T}}{\frac{2}{T}\frac{(z-1)}{(z+1)} + \frac{0.828}{T}}$$

$$= \frac{0.828\,(z+1)}{2\,(z-1) + 0.828\,(z+1)}$$

$$H(z) = \frac{1 + z^{-1}}{3.414 - 1.414\,z^{-1}}$$

DESIGN ANALOG FILTER AND OBTAIN TRANFER FUNCTION H(S)

1) BUTTERWORTH FILTER

BUTTERWORTH FILTERS

The Butterworth low-pass filter has a magnitude response given by:

$$|H(j\Omega)| = \frac{A}{\left[1+(\Omega/\Omega_c)^{2N}\right]^{0.5}}$$

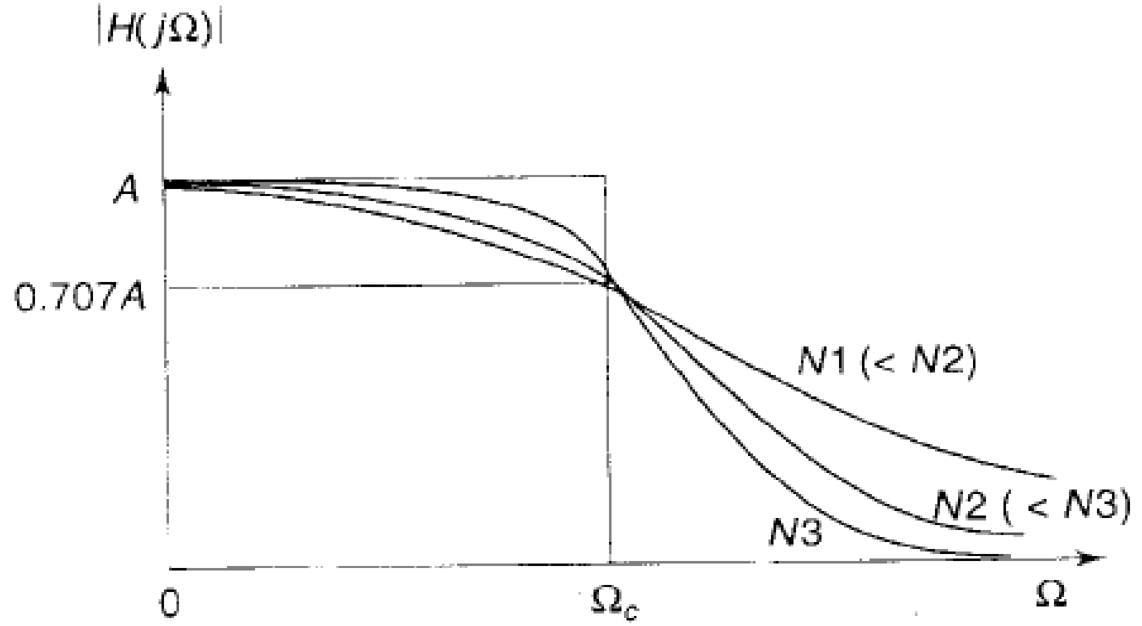


Fig Magnitude Response of a Butterworth Low-pass Filter

DESIGN OF ANALOG BUTTERWORTH FILTER

DESIGN METHOD:

AGK_RCET_BHILAI

STEP-I:- STANDARD SPECIFICATION IS

$$A = 1,$$

MAGNITUDE

$$\delta_1 \le \left| H(e^{j\omega}) \right| \le 1, \ 0 \le \omega \le \omega_1$$

$$\left| H(e^{j\omega}) \right| \le \delta_2, \ \omega_2 \le \omega \le \pi$$

CONVERT DIGITAL FRAQUENCIES INTO ANALOG FREQUENCIES ,WHICH IS DEPEND ON METHODS.FOR BILINEAR TRANSFORMATION USE FOLLOWING RELATION:

$$\Omega = \frac{2}{T} \tan \frac{\omega}{2}.$$

OTHERWISE USE

$$\omega = \Omega T$$

WHERE:-

$$\delta_1^2 \le \frac{1}{1 + (\Omega_1/\Omega_c)^{2N}} \le 1$$

$$\frac{1}{1 + (\Omega_2/\Omega_c)^{2N}} \le \delta_2^2$$

3 dB cut-off frequency Ω_c.

STEP II:- Determine order of filter N

$$N = \frac{1}{2}\frac{\log\left\{\left[(1/\delta_2^2 - 1)/(1/\delta_1^2 - 1)\right]\right\}}{\log(\Omega_2/\Omega_1)}$$

Where

N is chosen to be next nearest integer to the value of N

$$\Omega_c = \frac{\Omega_1}{\left[(1/\delta_1^2) - 1\right]^{1/2N}}$$

STEP III:- Determine analog tranfer funtion H(s).

There are two possibility with N, it is even or odd function

$N = 2, 4, 6, \ldots$ $\qquad b_k = 2\sin\left[(2k-1)\pi/2N\right]$ and $c_k = 1$

$$H(s) = \prod_{k=1}^{N/2} \frac{B_k\,\Omega_c^2}{s^2 + b_k\,\Omega_c\,s + c_k\,\Omega_c^2}$$

$N = 3, 5, 7, \ldots$ $\qquad b_k = 2\sin\left[(2k-1)\pi/2N\right]$ and $c_k = 1$

$$H(s) = \frac{B_0\,\Omega_c}{s + c_0\,\Omega_c} \prod_{k=1}^{(N-1)/2} \frac{B_k\,\Omega_c^2}{s^2 + b_k\,\Omega_c\,s + c_k\,\Omega_c^2}$$

The parameter B_k can be obtained from

$$A = \prod_{k=1}^{N/2} B_k \text{, for even } N$$

and

$$A = \prod_{k=1}^{(N-1)/2} B_k \text{, for odd } N$$

Solve above and Obtain analog TF H(s)

STEP IV: Use frequency tranformation and obtain desired TF if any(It is depend on question)

Type	*Transformation*
Low-pass	$s \rightarrow \frac{\Omega_c}{\Omega_c^*} s$
High-pass	$s \rightarrow \frac{\Omega_c \Omega_c^*}{s}$
Bandpass	$s \rightarrow \Omega_c \frac{s^2 + \Omega_1 \Omega_2}{s(\Omega_2 - \Omega_1)}$
Bandstop	$s \rightarrow \Omega_c \frac{s(\Omega_2 - \Omega_1)}{s^2 + \Omega_1 \Omega_2}$

Note: When cutoff frequency is real value function then no need of step IV because solution of step III is analog tranfer function of lowpass filter.

STEP V: After determining analog TF convert then into Digital TF,which is depend on question

system function of the equivalent digital filter is obtained from

$H(s)$ using the specified transformation technique.

Always represent H(Z) in standard format

for example

$$H(z) = \frac{-1.4509 - 0.2321z^{-1}}{1 - 0.1310z^{-1} + 0.3006z^{-2}}$$

Poles of a normalised Butterworth filter

This is another method of design

The poles in the left-half of the s-plane are given by,

$$s_n = \sigma_n + j\Omega_n = e^{j(2n+N-1)\pi/2N} = je^{j(2n-1)\pi/2N}$$

$$s_n = -\sin\left(\frac{2n-1}{2N}\right)\pi + j\cos\left(\frac{2n-1}{2N}\right)\pi$$

$$s'_n = s_n(\Omega_c)^{-1/N}$$

Example 8.11 Determine $H(z)$ for a Butterworth filter satisfying the following constraints

$$\sqrt{0.5} \le |H(e^{j\omega})| \le 1 \qquad 0 \le \omega \le \pi/2$$

$$|H(e^{j\omega})| \le 0.2 \qquad 3\pi/4 \le \omega \le \pi$$

with $T = 1$s. Apply impulse invariant transformation.

Example 8.12 Design a digital Butterworth filter that satisfies the constraint using bilinear transformation. Assume $T = 1$s.

$$0.9 \le |H(e^{j\omega})| \le 1 \qquad 0 \le \omega \le \pi/2$$

$$|H(e^{j\omega})| \le 0.2 \qquad 3\pi/4 \le \omega \le \pi$$

Example 8.11

Solution Given $\delta_1 = \sqrt{0.5} = 0.707$, $\delta_2 = 0.2$, $\omega_1 = \pi/2$ and $\omega_2 = 3\pi/4$.

STEP I:-

$$\Omega_1 = \frac{\omega_1}{T} = \frac{\pi}{2} \quad \text{and} \quad \Omega_2 = \frac{\omega_2}{T} = \frac{3\pi}{4}$$

STEP II:-

$$= \frac{1}{2}\frac{\log\{24/1\}}{\log(1.5)} = 3.91$$

N = 4

Determination of – 3 dB cut-off frequency Ω_c

$$\frac{\pi/2}{\left[(1/0.707^2) - 1\right]^{1/8}} = \frac{\pi}{2}$$

STEP III:-

$$H(s) = \prod_{k=1}^{N/2} \frac{B_k\, \Omega_c^2}{s^2 + b_k\, \Omega_c\, s + c_k\, \Omega_c^2}$$

$$= \left(\frac{B_1\, \Omega_c^2}{s^2 + b_1\, \Omega_c\, s + c_1\, \Omega_c^2}\right)\left(\frac{B_2\, \Omega_c^2}{s^2 + b_2\, \Omega_c\, s + c_2\, \Omega_c^2}\right)$$

$$H(s) = \left(\frac{2.467}{s^2 + 1.2022\,s + 2.467}\right)\left(\frac{2.467}{s^2 + 2.9025 + 2.467}\right)$$

NO NEED OF STEP IV .WE DESIGN LOW PASS FILTER.

STEP V:- APPLY IMPULSE INVARIENT METHOD TO CONVERT ANALOG TF H(S) TO DIGITAL TH OR DETERMINE H(Z):

$$H(s) = \left(\frac{As + B}{s^2 + 1.2022\,s + 2.467}\right) + \left(\frac{Cs + D}{s^2 + 2.9025 + 2.467}\right)$$

$$H(s) = -\left(\frac{1.4509\,s + 1.7443}{s^2 + 1.2022\,s + 2.467}\right) + \left(\frac{1.4509\,s + 4.2113}{s^2 + 2.9025\,s + 2.467}\right)$$

Let $H(s) = H_1(s) + H_2(s)$,

$$H_1(s) = -\left(\frac{1.4509\,s + 1.7443}{s^2 + 1.2022\,s + 2.467}\right) = -1.4509\left(\frac{s + 1.2022}{(s + 0.601)^2 + 1.451^2}\right)$$

$$= (-1.4509)\left(\frac{s + 0.601}{(s + 0.601)^2 + 1.451^2}\right) - (0.601)\left(\frac{1.451}{(s + 0.601)^2 + 1.451^2}\right)$$

$$H2(S) = (1.4509)\left(\frac{s+1.45}{(s+1.45)^2+0.604^2}\right) + (3.4903)\left(\frac{0.604}{(s+1.45)^2+0.604^2}\right)$$

$$= (1.4509)\frac{1-e^{-1.45T}(\cos 0.604T)z^{-1}}{1-2e^{-1.45T}(\cos 0.604T)z^{-1}+e^{-2.9T}z^{-2}}$$

$$+(3.4903)\frac{e^{-1.45T}(\sin 0.604T)z^{-1}}{1-2e^{-1.45T}(\cos 0.604T)z^{-1}+e^{-2.9T}z^{-2}}$$

$$H(z) = H_1(z) + H_2(z).$$

$$H(z) = \frac{-1.4509-0.2321z^{-1}}{1-0.1310z^{-1}+0.3006z^{-2}} + \frac{1.4509+0.1848z^{-1}}{1-0.3862z^{-1}+0.055z^{-2}}$$

xample 8.12

$$= \frac{2}{T}\tan\frac{\omega_1}{2} = 2\tan\frac{\pi}{4} = 2 \text{ and } \Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} = 2\tan\frac{3\pi}{8} = 4.828$$

efore, $\Omega_2/\Omega_1 = 2.414$

$$N \geq \frac{1}{2}\frac{\log\left\{[(1/\delta_2^2 - 1)/(1/\delta_1^2 - 1)]\right\}}{\log(\Omega_2/\Omega_1)}$$

N = 3

$$\Omega_c = \frac{\Omega_1}{\left[(1/\delta_1^2) - 1\right]^{1/2N}} = \frac{2}{\left[(1/0.9^2) - 1\right]^{1/6}} = 2.5467$$

$$H(s) = \left(\frac{2.5467}{s + 2.5467}\right)\left(\frac{6.4857}{s^2 + 2.5467s + 6.4857}\right)$$

$$H(z) = \frac{16.5171(z+1)^3}{70.83z^3 + 31.1205z^2 + 27.2351z + 2.948}$$

$$H(z) = \frac{0.2332(1 + z^{-1})^3}{1 + 0.4394z^{-1} + 0.3845z^{-2} + 0.0416z^{-3}}$$

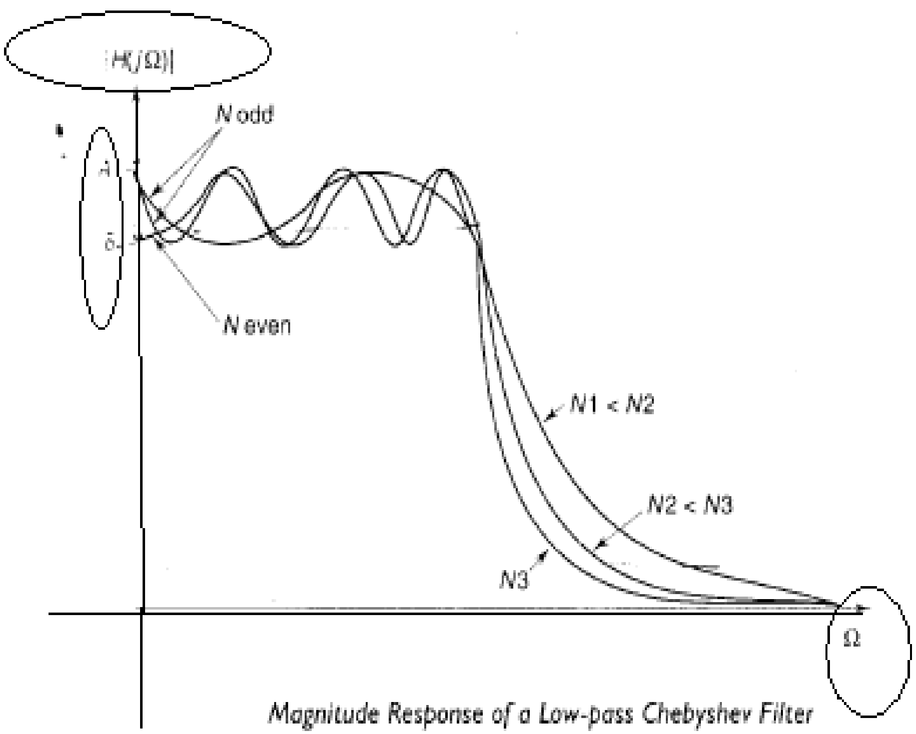
2)Design Chebyshev filters

CHEBYSHEV FILTERS

$$|H(j\Omega)| = \frac{A}{\left[1+\varepsilon^2 C_N^2(\Omega/\Omega_c)\right]^{0.5}}$$

Where___

$$C_N(x) = \begin{cases} \cos(N\cos^{-1}x), & \text{for } |x| \le 1 \\ \cosh(N\cosh^{-1}x), & \text{for } |x| \ge 1 \end{cases}$$



Magnitude Response of a Low-pass Chebyshev Filter

DESIGN STEP OF CHEBYSHEV FILTER TYPE IS AS FOLLOWS:

STEP 1:- DESIGN SPECIFICATIONS ARE

$$\delta_1 \le |H(e^{j\omega})| \le 1 \qquad 0 \le \omega \le \omega_1$$
$$|H(e^{j\omega})| \le \delta_2 \qquad \omega_2 \le \omega \le \pi$$

WHERE

$$\delta_1^2 \le \frac{1}{1+\varepsilon^2 C_N^2(\Omega_1/\Omega_c)} \le 1$$

$$= \frac{1}{1+\varepsilon^2 C_N^2(\Omega_2/\Omega_c)} \le \delta_2^2$$

Assuming $\Omega_c = \Omega_1$

$$\delta_1^2 \le \frac{1}{1+\varepsilon^2}$$

STEP II :-

The order of the analog filter, N

$$N \ge \frac{\cosh^{-1}\left\{\frac{1}{\varepsilon}\left[\frac{1}{\delta_2^2} - 1\right]^{0.5}\right\}}{\cosh^{-1}(\Omega_2/\Omega_1)}$$

Choose N to be next nearest integer

STEP III:- DETERMINE H(S)

The transfer function of Chebyshev filters are

FOR N= EVEN

$$H(s) = \prod_{k=1}^{N/2} \frac{B_k \, \Omega_c^2}{s^2 + b_k \, \Omega_c \, s + c_k \, \Omega_c^2} \qquad N = 2, 4, 6, \ldots$$

or

FOR N=ODD

$$H(s) = \frac{B_0 \, \Omega_c}{s + c_0 \, \Omega_c} \prod_{k=1}^{(N-1)/2} \frac{B_k \, \Omega_c^2}{s^2 + b_k \, \Omega_c \, s + c_k \, \Omega_c^2} \qquad N = 3, 5, 7, \ldots$$

WHERE:

$$b_k = 2y_N \sin\,[(2k-1)\pi/2N]$$

$$c_k = y_N^2 + \cos^2 \frac{(2k-1)\pi}{2N}$$

$$c_0 = y_N$$

The parameter y_N is given by

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\varepsilon^2}+1\right)^{0.5} + \frac{1}{\varepsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\varepsilon^2}+1\right)^{0.5} + \frac{1}{\varepsilon}\right]^{-\frac{1}{N}}\right\}$$

The parameter B_k can be obtained from

$$\frac{A}{(1+\varepsilon^2)^{0.5}} = \prod_{k=1}^{N/2} \frac{B_k}{c_k}, \text{ for } N \text{ even}$$

and

$$A = \prod_{k=0}^{\frac{N-1}{2}} \frac{B_k}{c_k} \text{ for } N \text{ odd.}$$

STEP IV :

Table gives the analog frequency transformations formulae.

Table *Analog frequency transformation*

Type	*Transformation*
Low-pass	$s \to \frac{\Omega_c}{\Omega_c^*} s$
High-pass	$s \to \frac{\Omega_c \Omega_c^*}{s}$
Bandpass	$s \to \Omega_c \frac{s^2 + \Omega_1 \Omega_2}{s(\Omega_2 - \Omega_1)}$
Bandstop	$s \to \Omega_c \frac{s(\Omega_2 - \Omega_1)}{s^2 + \Omega_1 \Omega_2}$

STEP V: After determining analog TF convert then into Digital TF,which is depend on question

system function of the equivalent digital filter is obtained from $H(s)$ using the specified transformation technique.

Always represent H(Z) in standard format

for example

$$H(z) = \frac{-1.4509 - 0.2321z^{-1}}{1 - 0.1310z^{-1} + 0.3006z^{-2}}$$

Normalised Chebys... Filter

$$|H(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 C_N^2(\Omega/\Omega_c)}$$

$$1 + \varepsilon^2 C_N^2(-js) = 0$$

$$C_N(-js) = \pm\frac{j}{\varepsilon} = \cos[N\cos^{-1}(-js)]$$

$$s_n = -\sin x \sinh y + j\cos x \cosh y = \sigma_n + j\Omega_n$$

$$n = \begin{cases} 1, 2, \ldots, (N+1)/2 & \text{for } N \text{ odd} \\ 1, 2, \ldots, N/2 & \text{for } N \text{ even} \end{cases}$$

$$\frac{\sigma_n^2}{\sinh^2 y} + \frac{\Omega_n^2}{\cosh^2 y} = 1$$

$$x = (2n-1)\frac{\pi}{2N} \quad n = 1, 2, \ldots, N$$

$$y = \pm\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)$$

$$s'_n = s_n\,\Omega_c$$

Example 8.13 Design a digital Chebyshev filter to satisfy the constraints

$$0.707 \le |H(e^{j\omega})| \le 1, \qquad 0 \le \omega \le 0.2\pi$$
$$|H(e^{j\omega})| \le 0.1, \qquad 0.5\pi \le \omega \le \pi$$

using bilinear transformation and assuming $T = 1s$.

Table *Digital frequency transformation*

	Transformation	*Design Parameter*
LOW PASS	$z^{-1} \rightarrow \frac{z^{-1} - a}{1 - a z^{-1}}$	$a = \frac{\sin[(\omega_c - \omega_c^*)/2]}{\sin[(\omega_c + \omega_c^*)/2]}$
HIGHPASS	$z^{-1} \rightarrow -\frac{z^{-1} + a}{1 + a z^{-1}}$	$a = -\frac{\cos[(\omega_c - \omega_c^*)/2]}{\cos[(\omega_c + \omega_c^*)/2]}$
BANDPASS	$z^{-1} \rightarrow -\frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$	$a_1 = -2\alpha K/(K+1)$ $a_2 = (K-1)/(K+1)$ $\alpha = \frac{\cos[(\omega_2 + \omega_1)/2]}{\cos[(\omega_2 - \omega_1)/2]}$ $K = \cot\left(\frac{\omega_2 - \omega_1}{2}\right)\tan\left(\frac{\omega_c}{2}\right)$
BANDREJECT	$z^{-1} \rightarrow \frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$	$a_1 = -2\alpha/(K+1)$ $a_2 = (1-K)/(1+K)$ $\alpha = \frac{\cos[(\omega_2 + \omega_1)/2]}{\cos[(\omega_2 - \omega_1)/2]}$ $K = \tan\left(\frac{\omega_2 - \omega_1}{2}\right)\tan\left(\frac{\omega_c}{2}\right)$

Example 8.13

$$\Omega_c = \Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2} = 2\tan 0.1\pi = 0.6498$$

$$\varepsilon = \left[\frac{1}{\delta_1^2} - 1\right]^{0.5} = \left[\frac{1}{0.707^2} - 1\right]^{0.5} = 1$$

$$\Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} = 2\tan 0.25\,\pi = 2$$

Let $N = 2$.

$$H(s) = \prod_{k=1}^{N/2} \frac{B_k\,\Omega_c^2}{s^2 + b_k\,\Omega_c\,s + c_k\,\Omega_c^2} = \frac{B_1\,\Omega_c^2}{s^2 + b_1\,\Omega_c\,s + c_1\,\Omega_c^2}$$

$$= \frac{1}{2}\left\{[2.414]^{\frac{1}{2}} - [2.414]^{-\frac{1}{2}}\right\} = 0.455$$

$$b_1 = 2y_2 \sin[(2k-1)\pi/2N] = 0.6435$$

$$\frac{B_1}{c_1} = 0.707, \text{ and hence } B_1 = 0.5$$

$$c_1 = y_2^2 + \cos^2\frac{(2k-1)\pi}{2N} = 0.707$$

$$H(s) = \frac{0.5\,(0.6498)^2}{s^2 + (0.6435)\,(0.6498)\,s + (0.707)\,(0.6498)^2}$$

$$H(z) = \frac{0.2111}{\left(2\,\frac{(z-1)}{(z+1)}\right)^2 + 0.4181\left(2\,\frac{(z-1)}{(z+1)}\right) + 0.2985}$$

$$= \frac{0.2111\,(z+1)^2}{5.1347\,z^2 - 7.403\,z - 3.4623}$$

Rearranging,

$$H(z) = \frac{0.041\,(1 + z^{-1})^2}{1 - 1.4418\,z^{-1} + 0.6743\,z^{-2}}$$

INVERSE CHEBYSHEV FILTERS

$$|H(j\Omega)| = \frac{\varepsilon C_N(\Omega_2/\Omega)}{\left[1+\varepsilon^2 C_N^2(\Omega_2/\Omega)\right]^{0.5}}$$

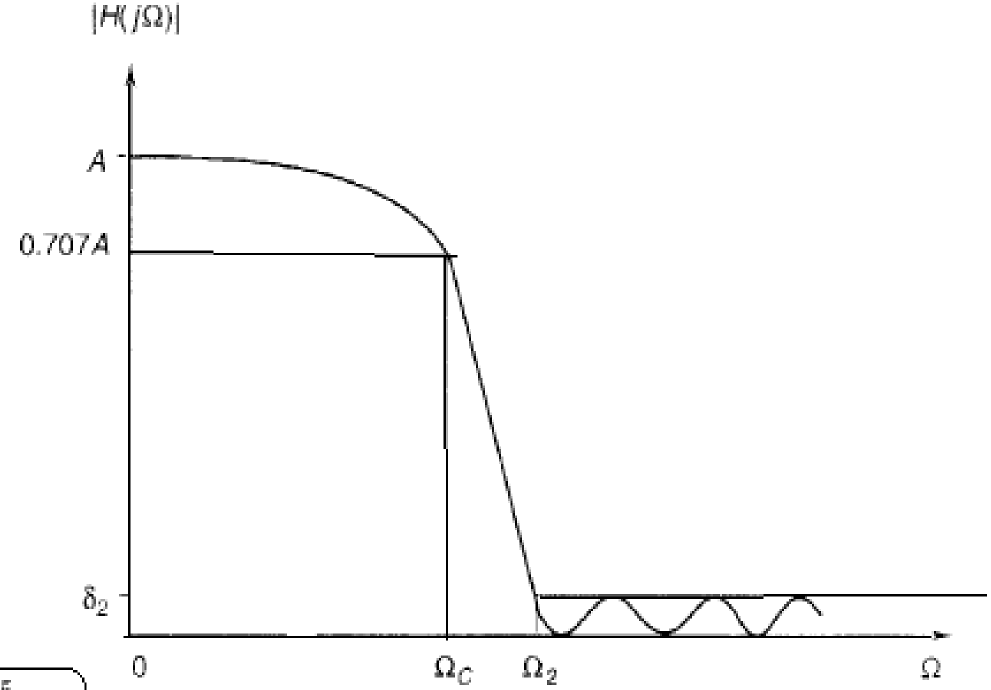
$$0.707 \le |H(j\Omega)| \le 1, \qquad 0 \le \Omega \le \Omega_c$$
$$|H(j\Omega)| \le \delta_2, \qquad \Omega \ge \Omega_2$$

$$\delta_2^2 = \frac{\varepsilon^2}{1+\varepsilon^2}$$

When $\Omega = \Omega_2$,

$$N = \frac{\cosh^{-1}(1/\varepsilon)}{\cosh^{-1}(\Omega_2/\Omega_c)} = \frac{\cosh^{-1}\left[\frac{1}{\delta_2^2} - 1\right]^{0.5}}{\cosh^{-1}(\Omega_2/\Omega_c)}$$

value of N is chosen to be the nearest integer greater

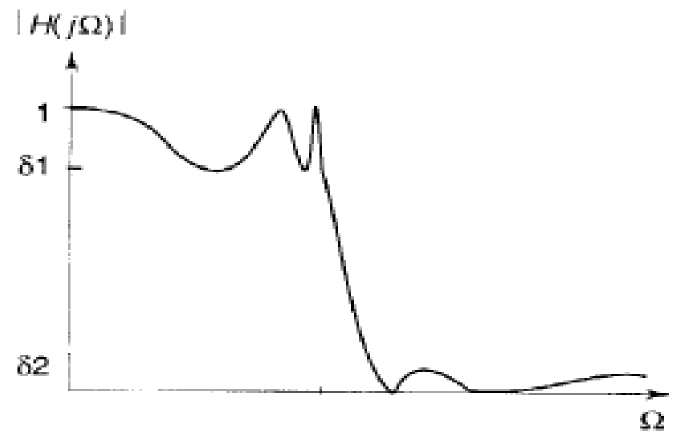


Magnitude Response of the Low-pass Inverse Chebyshev Filter

ALL PROCEDURE OF TYPE-II IS SIMILAR TO TYPE –I ACCEPT ABOVE

ELLIPTIC FILTERS

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 U_N\,(\Omega/\Omega_c)}$$



Magnitude Response of a Low-pass Elliptic Filter

Example 8.14 A prototype low-pass filter has the system response $H(s) = \frac{1}{s^2+2s+1}$. Obtain a bandpass filter with $\Omega_0 = 2$ rad/s and $Q = 10$. $\Omega_0^2 = \Omega_1 \cdot \Omega_2$ and $Q = \frac{\Omega_0}{\Omega_2 - \Omega_1}$,

$$s \to \Omega_c \frac{s^2 + \Omega_1 \Omega_2}{s(\Omega_2 - \Omega_1)}, \text{ i.e.}$$

Solution

$$s = \Omega_c \frac{s^2 + \Omega_0^2}{s(\Omega_0/Q)} = \Omega_c \frac{s^2 + 2^2}{s(2/10)} = 5\,\Omega_c\left(\frac{s^2+4}{s}\right)$$

$$H(s) = \frac{0.04\,s^2}{\Omega_c^2\, s^4 + 0.4\,\Omega_c\, s^3 + (8\,\Omega_c^2 + 0.01)\,s^2 + 1.6\,\Omega_c\, s + 16\,\Omega}$$

Example 8.15 Transform the prototype low-pas
function

$$H(s) = \frac{\Omega_c}{s + \Omega_c}$$

into a high-pass filter with cutoff frequency Ω_c^*.

8.26 Describe elliptic filters ?

8.27 Compare the passband and stopband characteristics major types of analog filters.

8.28 Why is frequency transformation needed ?

8.29 What are the different types of frequency transformation.

8.12 An analog filter has the following system function. Convert this filter into a digital filter using the impulse invariant technique.

$$H(s) = \frac{1}{(s+0.1)^2 + 9}$$

8.13 Convert the analog filter to a digital filter whose system function is

$$H(s) = \frac{1}{(s+2)^3}$$

8.14 Convert the analog filter to a digital filter whose system function is

$$H(s) = \frac{36}{(s+0.1)^2 + 36}$$

The digital filter should have a resonant frequency of $\omega_r = 0.2\pi$. Use impulse invariant mapping.

8.15 What is bilinear transformation ?

8.16 Compare bilinear transformation with other transformations based on their stability.

8.17 Obtain the transformation formula for the bilinear transformation.

8.18 An analog filter has the following system function. Convert this filter into a digital filter using bilinear transformation.

$$H(s) = \frac{1}{(s+0.2)^2 + 16}$$

8.19 Convert the analog filter to a digital filter whose system function is

$$H(s) = \frac{1}{(s+2)^2 (s+1)}$$

using bilinear transformation.

8.30 Design a digital Chebyshev filter to meet the constraint

$$0.8 \le |H(e^{j\omega})| \le 1, \qquad 0 \le \omega \le 0.2\pi$$
$$|H(e^{j\omega})| \le 0.2, \qquad 0.6\pi \le \omega \le \pi$$

using (i) bilinear transformation and (ii) impulse invariant transformation.

8.31 Design a digital Butterworth filter to meet the constraint

$$0.8 \le |H(e^{j\omega})| \le 1, \qquad 0 \le \omega \le 0.2\pi$$
$$|H(e^{j\omega})| \le 0.2, \qquad 0.26\pi \le \omega \le \pi$$

using (i) bilinear transformation and (ii) impulse invariant transformation.

8.32 Design a digital Butterworth filter to meet the constraint

$$0.9 \le |H(e^{j\omega})| \le 1, \qquad 0 \le \omega \le 0.25\pi$$
$$|H(e^{j\omega})| \le 0.2, \qquad 0.6\pi \le \omega \le \pi$$

using (i) bilinear transformation and (ii) impulse invariant transformation.

8.33 Design and realise a digital LPF using bilinear transformation to satisfy the following requirements
(a) monotonic stopband and passband
(b) -3 dB cut-off frequency at 0.6π radians, and
(c) magnitude down at 16 dB at 0.75π radians.

8.34 Determine the normalised low-pass Butterworth analog poles for $N = 10$.

8.35 Determine the normalised Chebyshev analog low-pass poles for $N = 6$.

END OF PPT
NAMASKAR