RANK OF A MATRIX

Def :- A matrix is said to be of rank r when (i) it has at least one non-zero order of r, and (ii) every minor of order higher than r vanishes.

Briefly, the rank of a matrix is the largest order of any non vanishing minor of the matrix

Note :- (i) If a matrix has a non-zero minor of order r, its rank is $\geq r$

(ii) If all minors of a matrix of order $r+1$ are zero, its rank is $\leq r$.

The rank of matrix A should be denoted by $\rho(A)$

Elementary transformation of a matrix :-

The following operation, three of which refer to rows and three to column are known as elementary operation.

(I) The interchange of any two rows or columns

(II) The multiplication of any row (column) by a non zero number.

(III) The addition of a constant multiple of the elements of any row (column) to the corresponding elements of any other row (column)

Notation-1 – The elementary row transformation will be denoted by the following symbols :

(i) R_{ij} for the interchange of the ith and jth row.

(ii) kR_i, for multiplication of the ith row by k.

(iii) $R_i + pR_j$, for addition to ith row, p times the jth row.

Elementary transformation donot change the either the order or rank of a matrix.

Equivalent matrix :- Two matrices A and B are said to be equivalent if one can be obtained from the other by a sequence of elementary transformations. Two equivalent matrices have the same order and the same rank. The symbol $\sim$ is used for equivalence.

Note Rank = No. of non zero row in upper triangular matrix.

Determine the rank of the following matrix.

1. $$\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$$

Soln → $R_2 + 2R_1$, $R_3 + R_1$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix}$$

PAGE No.
DATE / /

$R_3 - \frac{1}{2}R_2$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

Here the no. of non-zero row is 2.

$\therefore$ rank of a matrix is 2.

2. $$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

Soln $\rightarrow$ $R_2 - 2R_1$, $R_3 - 3R_1$, $R_4 - 6R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

operate $R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

operate $R_4 - R_3$

PAGE No.
DATE / /

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

operate $R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here the no. of non - zero in upper triangular matrix is 3.

$\therefore$ rank of a matrix is 3.

Gauss Jordan method of finding the inverse :-

Those elementary row transformation which reduce a given square matrix A to the unit matrix when applied to unit matrix I give the inverse of A.

Q. (1) Using the Gauss Jordan method, find the inverse of the matrix

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

Sol^n Writing the same matrix side by side with the unit matrix of order 3, we have

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

operate $R_2 - R_1$, $R_3 + 2R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$$

operate $\frac{1}{2}R_2$ and $\frac{1}{2}R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 1 & 1 & 0 & \frac{1}{2} \end{array}\right]$$

operate $R_1 - R_2$, $R_3 + R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 6 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -2 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

PAGE No.
DATE / /

operate $R_1 + 3R_3$, $R_2 - \frac{3}{2}R_3$ and $\left(-\frac{1}{2}\right)R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 3/2 \\ 0 & 1 & 0 & -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$$

Hence the inverse of the given matrix is

$$\begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -3/4 \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Q. 2

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$$

Soln → Writing the same matrix side by side with the unit matrix of order 3, we have

$$\left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 5 & 2 & -3 & 0 & 0 & 1 \end{array}\right]$$

PAGE No.
DATE / /

$$\sim \begin{bmatrix} 1 & 0 & -3/4 & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 10 & -1 & -4 \end{bmatrix}$$

$$R_2 - \frac{1}{2}R_3 \quad , \quad R_1 + \frac{3}{4}R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 8 & -1 & -3 \\ 0 & 1 & 0 & -5 & 1 & 2 \\ 0 & 0 & 1 & 10 & -1 & -4 \end{bmatrix}$$

Hence the inverse of the given matrix is

$$\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$$

Normal form :- Every non-zero matrix A of rank r, can be reduced by a sequence of elementary transformation, to the form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$ called the normal form of A.

Note - (i) The rank of a matrix A is r if and only if it can be reduced to the normal form

(ii) Corresponding to every matrix A of rank r, there exists non-singular matrices P and Q such that PAQ is in the normal form.

Q. (1) Reduce the following matrix into its normal form and hence find its rank.

$$\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Sol^n operate $R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix}$$

$R_2 - 3R_1$, $R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix}$$

operate $C_2 - C_1$, $C_3 - C_1$, $C_4 - 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix}$$

operate $C_4 - 2C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -2 & 0 \\ 0 & -1 & -5 & 0 \end{bmatrix}$$

operate $R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & -6 & -2 & 0 \end{bmatrix}$$

operate $R_3 - 6R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & 0 & 28 & 0 \end{bmatrix}$$

operate $C_3 - 5C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 28 & 0 \end{bmatrix}$$

operate $\frac{1}{28} R_3$, $-1 R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \text{normal form.}$$

Here the no. of non zero row is 3

∴ The rank of ~~matri~~ given matrix is 3.

Q-(2) Find the non-singular matrices P and Q such that PAQ is in the normal form for the matrices!

(i) $$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Sol^n → We write $A = I_3 A I_3$

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

operate $R_2 - R_1$, $R_3 - 3R_1$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

operate $C_2 + C_1$, $C_3 + C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

operate $\frac{1}{2}R_2$, $\frac{1}{4}R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{3}{4} & 0 & \frac{1}{4} \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

operate $C_3 - C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{3}{4} & 0 & \frac{1}{4} \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

operate $R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Here non singular matrices P and

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1/2 & 0 \\ -1/4 & -1/2 & 1/4 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

such that PAQ is in the normal form.

PAGE No.
DATE / /

Consistency of Linear System of equations

(I) **Consistent** :- A system of equation is said to be consistent, if then have one or more solution. i.e.

$$x + 2y = 4 \qquad\qquad x + 2y = 4$$
$$3x + 2y = 2 \qquad\qquad 3x + 6y = 12$$

unique soln   infinite solutions

II **Inconsistent** :- If a system of equation has no solution, it is said to be inconsistent i.e.

$$x + 2y = 4$$
$$3x + 6y = 5$$

Now consider the linear system of equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$- - - - - - - - -$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

or

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

or $\quad AX = B$

and $K = [A, B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ - & - & - & - & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$

is called the augmented matrix.

PAGE No.
DATE / /

(a) Consistent equations :- If Rank A = Rank C

(i) Unique solution : Rank A = Rank C = n (number of unknowns)

(ii) Infinite solution

Rank A = Rank C = $r < n$

~~$\Rightarrow (n-r)$ independent variables.~~

(b) Inconsistent equations :-

If Rank A $\neq$ Rank C

PAGE No.	
DATE	/ /

~~Problems 2.9~~

Q.1 Investigate for consistency of the following equations and if possible find the solutions:

$$4x - 2y + 6z = 8$$

$$x + y - 3z = -1$$

$$15x - 3y + 9z = 21$$

Sol^n → The given linear system of equation can be written in the matrix form as

$$\begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix}$$

Its augmented matrix is

$$K = \begin{bmatrix} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{bmatrix}$$

operate $R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{bmatrix}$$

operate $R_2 - 4R_1$, $R_3 - 15R_1$

$$\sim \begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & -18 & 54 & 36 \end{bmatrix}$$

$R_3 - 3R_2$

$$\sim \begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

operate $\frac{1}{6} R_2$

$$\sim \begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

~~Rank~~

Now Here the rank of the coefficient and augmented matrix are same and is 2

$\therefore$ The given linear system of eq^n are consistent. Since ~~rank is~~ $r = 2 < 3$ (no. of unknowns) $\therefore$ Therefore the given system of eq^n possess infinite no. of sol^n.

Now the given matrix form can be reduced into linear system of equation such that

$$x + y - 3z = -1$$

$$-y + 3z = 2$$

put $z = c$ $\therefore$ $y = 3c - 2$ for all c

and $x + 3c - 2 - 3c = -1$

$$x = -1 + 2 \Rightarrow x = 1$$

Thus infinite no. of sol^n are

$x = 1$, $y = 3c - 2$, $z = c$ for all c.

Q. 3 Investigate for what value of λ and μ the simultaneous equations

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

have (i) no sol^n (ii) a unique sol^n (iii) an infinite number of sol^n.

$Sol^n \rightarrow$ The given linear system of eq^n can be written in the matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

Its augmented matrix is

PAGE No.
DATE / /

$$k = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$$

operate $R_2 - R_1,\ R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix}$$

operate $R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$$

Case I No Solution :- If $\lambda = 3$ and $\mu = 10$
Then given linear System of eq^n are inconsistent and possess ~~no.~~ no sol^n.

Case II Unique solution :- If $\lambda \neq 3$ then the rank of the coefficient matrix and augmented are same and is equal to the no. of unknowns, and μ may have any value.

Case III An infinite number of Sol^n :- If $\lambda = 3$ and $\mu = 10$ then the rank of the coefficient matrix and augmented matrix are same and is less than the number of unknowns Then System of eq^n possess infinite ~~no.~~ number of Solutions.

Test for consistency and solve

$$x + 2y + z = 3$$
$$2x + 3y + 2z = 5$$
$$3x - 5y + 5z = 2$$
$$3x + 9y \cancel{+} -z = 4$$

$\underline{\text{Sol}^n}$ Given equation can be written in the matrix form as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & -5 & 5 \\ 3 & 9 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 2 \\ 4 \end{bmatrix}$$

Its augmented matrix is

$$K = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & 2 & 5 \\ 3 & -5 & 5 & 2 \\ 3 & 9 & -1 & 4 \end{bmatrix}$$

operate $R_2 - 2R_1$, $R_3 - 3R_1$, $R_4 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & -11 & 2 & -7 \\ 0 & 3 & -4 & -5 \end{bmatrix}$$

operate $R_3 - 11R_2$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 2 & 5 \\ 0 & 3 & -4 & -5 \end{bmatrix}$$

operate $R_3 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 2 & 5 \end{bmatrix}$$

operate $R_3 + 3R_2$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -4 & -8 \\ 0 & 0 & 2 & \end{bmatrix}$$

operate $R_3 - 11R_2$ and $R_4 + 3R_2$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & -8 \end{bmatrix}$$

operate $R_4 + 2R_3$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here the rank of the coefficient matrix and augmented are same and equal to the no. of Unknown therefore the linear system of eqn possess a unique soln.

PAGE No.
DATE / /

Now the given matrix form can be reduced into linear system of eq^n such that

$$x + 2y + z = 3$$
$$-y = -1 \Rightarrow y = 1$$
$$2z = 4 \Rightarrow z = 2$$

$\therefore \; x = -1$

$\therefore$ Unique sol^n is $x = -1, y = 1, z = 2$.

Find the values of a and b for which the equation

$$x + ay + z = 3$$
$$x + 2y + 2z = b$$
$$x + 5y + 3z = 9$$

are consistent. When will these equation have a unique sol^n.

$sol^n \rightarrow$ Given linear system of equation can be written in the matrix form as

$$\begin{bmatrix} 1 & a & 1 \\ 1 & 2 & 2 \\ 1 & 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ b \\ 9 \end{bmatrix}$$

Its augmented matrix is

$$K = \begin{bmatrix} 1 & a & 1 & 3 \\ 1 & 2 & 2 & b \\ 1 & 5 & 3 & 9 \end{bmatrix}$$

operate $R_2 - R_1$, $R_3 - R_1$

$$\sim \begin{bmatrix} 1 & a & 1 & 3 \\ 0 & 2-a & 1 & b-3 \\ 0 & 5-a & 2 & 6 \end{bmatrix}$$

operate $(5-a)R_2$,
$(2-a)R_3$

$$\sim \begin{bmatrix} 1 & a & 1 & 3 \\ 0 & (5-a)(2-a) & (5-a) & (b-3)(5-a) \\ 0 & (2-a)(5-a) & 2(2-a) & 6(2-a) \end{bmatrix}$$

or

$$\sim \begin{bmatrix} 1 & a & 1 & 3 \\ 0 & (5-a)(2-a) & (5-a) & 5b-ab-15+3a \\ 0 & (2-a)(5-a) & 4-2a & 12-6a \end{bmatrix}$$

operate
$R_3 - R_2$

$$\sim \begin{bmatrix} 1 & a & 1 & 3 \\ 0 & (5-a)(2-a) & (5-a) & 5b-ab-15+3a \\ 0 & 0 & -1-a & -5b+ab+27-9a \end{bmatrix}$$

Now the system of equations are possess consistent if $a \neq -1$ and if $a=-1$, $b=6$

Now If $a \neq -1$, b has any value, equation will be consistent and have a Unique soln.

Q. 7 Show that the equation

$3x + 4y + 5z = a$
$4x + 5y + 6z = b$
$5x + 6y + 7z = c$

do not have a soln unless $a+c=2b$.

Soln → The given system of equation can be written in the matrix form as

$$\begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

~~operate $4R_1$, $3R_2$~~

Its augmented matrix is

$$K = \begin{bmatrix} 3 & 4 & 5 & a \\ 4 & 5 & 6 & b \\ 5 & 6 & 7 & c \end{bmatrix}$$

operate $4R_1$, $3R_2$

$$\sim \begin{bmatrix} 12 & 16 & 20 & 4a \\ 12 & 15 & 18 & 3b \\ 5 & 6 & 7 & c \end{bmatrix}$$

operate $R_2 - R_1$

$$\sim \begin{bmatrix} 12 & 16 & 20 & 4a \\ 0 & -1 & -2 & 3b-4a \\ 5 & 6 & 7 & c \end{bmatrix}$$

operate $\frac{1}{4} R_1$

$$\sim \begin{bmatrix} 3 & 4 & 5 & a \\ 0 & -1 & -2 & 3b-4a \\ 5 & 6 & 7 & c \end{bmatrix}$$

operate $5R_1$, $3R_3$

$$\sim \begin{bmatrix} 15 & 20 & 25 & 5a \\ 0 & -1 & -2 & 3b-4a \\ 15 & 18 & 21 & 3c \end{bmatrix}$$

operate $R_3 - R_1$

$$\sim \begin{bmatrix} 15 & 20 & 25 & 5a \\ 0 & -1 & -2 & 3b-4a \\ 0 & -2 & -4 & 3c-5a \end{bmatrix}$$

operate $\frac{1}{5} R_1$

$$\sim \begin{bmatrix} 3 & 4 & 5 & a \\ 0 & -1 & -2 & 3b-4a \\ 0 & -2 & -4 & 3c-5a \end{bmatrix}$$

operate $R_3 - 2R_2$

$$\sim \begin{bmatrix} 3 & 4 & 5 & a \\ 0 & -1 & -2 & 3b-4a \\ 0 & 0 & 0 & -6b+3a+3c \end{bmatrix}$$

Now the system of eqn possess a soln

Unless $-6b+3a+3c=0$

$\Rightarrow 3a+3c = 6b$

$\Rightarrow a+c = 2b$

which is required.

System of linear homogenous equations:-

Consider the homogenous linear equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{aligned}\right\} \text{---(i)}$$

Find the rank r of the coefficient matrix A by reducing it to the triangular form by elementary operations

1. If $r = n$ the equation (i) have only a trivial zero soln.

$$x_1 = x_2 = x_3 = \cdots = x_n = 0$$

2. If $r < n$ the eqn (i) have $(n-r)$ linearly independent solutions.

3. When $m = n$ (i.e. the number of equations = the number of variables), the necessary and sufficient Condition for all solution other than $x_1 = x_2 = \cdots = x_n = 0$, is that the determinant of the coefficient matrix is zero. In this case the equations are said to be consistant and such a solutions is called non-trivial soln. The determinant is called the eliminant of the equations.

= show that the system of equations

$2x_1 - 2x_2 + x_3 = \lambda x_1$,

$2x_1 - 3x_2 + 2x_3 = \lambda x_2$,

$-x_1 + 2x_2 \quad = \lambda x_3$ can possess a non-trivial soln only if $\lambda = 1$, $\lambda = -3$. obtain the general soln in each case.

soln → The given equation will be consistent or posses a non trivial solution if

$$\begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -(3+\lambda) & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\{\lambda(3+\lambda)-4\} + 2\{-2\lambda+2\} + 1\{4-(3+\lambda)\} = 0$$

$$\Rightarrow (2-\lambda)\{3\lambda + \lambda^2 - 4\} - 4\lambda + 4 + 4 - 3 - \lambda = 0$$

$$\Rightarrow 6\lambda + 2\lambda^2 - 8 - 3\lambda^2 - \lambda^3 + 4\lambda - 4\lambda + 5 - \lambda = 0$$

$$\Rightarrow -\lambda^3 - \lambda^2 + 5\lambda - 3 = 0$$

$$\Rightarrow (\lambda-1)(-\lambda^2 - 2\lambda + 3) = 0$$

PAGE No.
DATE / /

Case (I) ~~If $\lambda = 1$~~ If $\lambda = 1$ then given system of equation becomes

$$x_1 - 2x_2 + x_3 = 0$$

$$2x_1 - 4x_2 + 2x_3 = 0$$

$$-x_1 + 2x_2 - x_3 = 0$$

Its coefficient matrix is

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix}$$

operate $R_2 - 2R_1$, $R_3 + R_1$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank of the coefficient matrix is 1.

or $x_1 - 2x_2 + x_3 = 0$

put $x_3 = s$, $x_2 = t$ $\therefore x_1 = 2t - s$

Case II $\lambda = -3$, then given system of equation becomes

$$5x_1 - 2x_2 + x_3 = 0$$

$$2x_1 + 0x_2 + 2x_3 = 0$$

$$-x_1 + 2x_2 + 3x_3 = 0$$

PAGE No.
DATE / /

Its coefficient matrix is

$$A = \begin{bmatrix} 5 & -2 & 1 \\ 2 & 0 & 2 \\ -1 & 2 & 3 \end{bmatrix}$$

operate $\frac{1}{2}R_2$

$$\sim \begin{bmatrix} 5 & -2 & 1 \\ 1 & 0 & 1 \\ -1 & 2 & 3 \end{bmatrix}$$

operate $R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 5 & -2 & 1 \\ -1 & 2 & 3 \end{bmatrix}$$

operate $R_2 - 5R_1$, $R_3 + R_1$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & -4 \\ 0 & 2 & 4 \end{bmatrix}$$

operate $R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

operate $\frac{1}{2}R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank of the matrix A is 2

$\therefore$ $x_1 + 0x_2 + x_3 = 0$

$-x_2 - 2x_3 = 0$

put $x_3 = t$

$\therefore x_2 = -2t$

and $x_1 = -t$

CHARACTERISTIC EQUATION

If A is any square matrix of order n, we can form the matrix $A - \lambda I$, where I is the nth order unit matrix. The determinant of this matrix equated to zero i.e.

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & & a_{nn}-\lambda \end{vmatrix} = 0.$$

is called the characteristic equation of A. On expanding the determinant, the characteristic equation takes the form

$$(-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \cdots + k_n = 0$$

where k's are expressible in term of the elements a_{ij}. The roots of this equation are called the characteristic roots or latent roots or eigen-values of the matrix A.

Eigen Vectors → If A be a square matrix and λ be its one eigen value, then corresponding to this λ if there exists a vector x such that, $(A-\lambda I)x = 0$ then x is called characteristic vector of A corresponding to characteristic root λ.

This is called eigen vector or invarient of A also.

PAGE No.
DATE / /

Q.1 Find the eigen values and corresponding eigen vectors of the matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Soln → The characteristic equation of matrix A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (8-\lambda)[(7-\lambda)(3-\lambda) - 16] + 6[-6(3-\lambda)+8]$$
$$+2[24-14+2\lambda] = 0$$

$$\Rightarrow \lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\Rightarrow \lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\Rightarrow \lambda(\lambda-3)(\lambda-15) = 0$$

$\Rightarrow \lambda = 0, 3, 15$ so eigen values of matrix A are 0, 3, 15.

Let $X = [x_1, x_2, x_3]'$ be the eigen vector, corresponding to eigen value $\lambda = 0$ of the matrix A, then

$$(A - 0I)X = 0$$

$$\Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

PAGE No.
DATE / /

$$8x_1 + 6x_2 + 2x_3 = 0 \quad --(i)$$
$$-6x_1 + 7x_2 - 4x_3 = 0 \quad --(ii)$$
$$2x_1 - 4x_2 + 3x_3 = 0 \quad --(iii)$$

From eq^n (i) and (ii), we get

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36}$$

$$\Rightarrow \frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2} = K_1$$

$$\Rightarrow x_1 = K, \quad x_2 = 2K_1, \quad x_3 = 2K_1$$

Above values satisfies eq^n (iii) for all values of K_1, so eigen vector corresponding to eigen value $\lambda = 0$ is

$$X = [K_1, 2K_1, 3K_1]'$$
$$= K_1 [1 \ 2 \ 3]'$$

The eigen vector corresponding to eigen value $\lambda = 3$ of the matrix A, is the non-zero sol^n of the equation

$$(A - 3I)X = 0$$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

PAGE NO.:
DATE: / /

$$5x_1 - 6x_2 + 2x_3 = 0 \quad ---(iv)$$
$$-6x_1 + 4x_2 - 4x_3 = 0 \quad ---(v)$$
$$2x_1 - 4x_2 + 0x_3 = 0 \quad ---(vi)$$

Solving (iv) and (v) we get

$$\frac{x_1}{24-8} = \frac{x_2}{-12+20} = \frac{x_3}{20-36}$$

$$\Rightarrow \frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2} = k_2$$

$$\Rightarrow x_1 = 2k_2, \quad x_2 = k_2, \quad x_3 = -2k_2$$

so eigen vector corresponding to eigen value $\lambda = 3$ is

$$x = k_2 [2 \quad 1 \quad 2]'$$

The eigen vector corresponding to eigen value $\lambda = 15$ of the matrix A, is the non-zero solution of the ~~eq~~ equation

Teacher's Signature

PAGE NO. :
DATE : / /

$$(A - 15I)X = 0$$

$$\begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-7x_1 - 6x_2 + 2x_3 = 0$ — (vii)

$-6x_1 - 8x_2 - 4x_3 = 0$ — (viii)

$2x_1 - 4x_2 - 12x_3 = 0$ — (ix)

Solving equation (viii) and (ix), we get

$$\frac{x_1}{96-16} = \frac{x_2}{-8-72} = \frac{x_3}{24+16}$$

$$\Rightarrow \frac{x_1}{80} = \frac{x_2}{-80} = \frac{x_3}{40}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1} = k_3$$

$x_1 = 2k_3, \; x_2 = -2k_3, \; x_3 = k_3$

Teacher's Signature:

So the eigen vector corresponding

PAGE NO.:
DATE: / /

to eigen value $\lambda = 15$ is

$$X = [2k_3, -2k_3, k_3]'$$

$$= k_3 [2, -2, 1]'$$

Q-2 Find all the eigen values and eigen vectors of the matrix

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Sol^n The characteristic equation of matrix A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$$\Rightarrow (\lambda - 8)(\lambda - 2)^2 = 0$$

$$\Rightarrow \lambda = 2, 2, 8.$$

Teacher's Signature

PAGE NO. :
DATE : / /

So the eigen values of the matrix A are $2, 2, 8$.

The eigen vector of $X = [x_1, x_2, x_3]'$ corresponding to eigen value $\lambda = 8$ is the non-zero sol^n of the equation

$$[A - 8I]X = 0$$

$$\Rightarrow \begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

operate $R_2 - R_1$, $R_3 + R_1$

$$\begin{bmatrix} -2 & 2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Teacher's Signature

operate $R_3 - R_2$

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the rank of ~~Coe~~ coefficient matrix is 2, so the following equation will have $3-2=1$, linearly independent solution

$$-2x_1 - 2x_2 + 2x_3 = 0 \quad \text{--(i)}$$
$$-3x_2 - 3x_3 = 0 \quad \text{--(ii)}$$

Equation (ii) $\Rightarrow x_2 = -x_3$

Let $x_3 = 1$, $x_2 = -1$

Equation (i) $\Rightarrow x_1 = 2$

so $X_1 = [2, -1, 1]'$ is the eigen vector corresponding to eigen value $\lambda = 8$

The eigen vectors ~~Coaro~~ corresponding to eigen value $\lambda = 2$ is the non-zero

PAGE NO. :
DATE : / /

Solution of the equation

$$[A-2I]X=0$$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} -2 & 1 & -1 \\ 4 & -2 & 2 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply $R_2+2R_1,\ R_3+R_1$

$$\begin{bmatrix} -2 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the rank of coefficient matrix is 1, so the following eqn $-2x_1+x_2-x_3=0$

Teacher's Signature

—CNT)

PAGE NO.:
DATE: / /

will have $3-1=2$, linearly independent soln

Equation (III) $\Rightarrow x_1=-1$, $x_2=0$ and so $x_3=2$

and $x_1=1$, $x_2=-2$ and so $x_3=0$

we get two linearly independence solution of above equation as

$$x_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Q. 2 Find the characteristic equation of the matrix A, show that A satisfies its characteristic equation, also find A^{-1}

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Teacher's Signature

PAGE NO. :
DATE : / /

Soln → The characteristic equation of the matrix A is

$$|A-\lambda I|=0$$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

which is the ~~requir~~ required characteristic equation, Now we shall show that A satisfies its characteristic equation i.e.

$$A^3 - 6A^2 + 9A - 4I = 0$$

We know that

Teacher's Signature

PAGE NO.:
DATE: / /

We know that

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A^2 = A \cdot A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

So $A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} -$$

$$6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Teacher's Signature

PAGE NO. :
DATE : / /

$$-4\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=0$$

multiply eqn (i) by A^{-1}, we get

$$A^2-6A+9I-4A^{-1}=0$$

$$A^{-1}=\frac{1}{4}\left[A^2-6A+9I\right]$$

$$\Rightarrow A^{-1}=\frac{1}{4}\left\{\begin{bmatrix}6&-5&5\\-5&6&-5\\5&-5&6\end{bmatrix}-\right.$$

$$6\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}+9\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\Bigg\}$$

Teacher's Signature

PAGE NO. :
DATE : / /

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

x

Teacher's Signature