II unit

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# UNIT-II

## Differential Calculus

Topics:

Successive differentiation
leibnitz's theorem
Maclaurin's & Taylor's series
Curve tracing.

## Successive differentiation.

Given a function $y = f(x)$
the first derivative of $y$ w.r.t '$x$' is denoted by $\frac{dy}{dx}$ or $Dy$

Second derivative is denoted by $\frac{d^2y}{dx^2}$ or $D^2y$

Third derivative $\frac{d^3y}{dx^3}$ or $D^3y$

⋮

$n^{th}$ derivative is denoted by $\frac{d^ny}{dx^n}$ or $D^ny$.

Problems

① If $y = \frac{ax+b}{cx+d}$ show that $2y_1y_3 = 3y_2^2$.

Soln $y = \frac{ax+b}{cx+d}$

$y_1 = \frac{(cx+d)a - (ax+b)c}{(cx+d)^2} = \frac{ad-bc}{(cx+d)^2} = \frac{\alpha}{(cx+d)^2}$

$$y_2 = \frac{((x+d)^2 \cdot 0 - (ad-bc)\, 2((x+d)\cdot c}{((x+d)^4}$$

$$= \frac{-2\alpha c((x+d)}{(cx+d)^4} = \frac{-2\alpha c}{(cx+d)^3}$$

$$y_3 = -\left[\frac{(cx+d)^3 \cdot 0 - 2\alpha c\; 3((x+d)^2 c}{(cx+d)^6}\right]$$

$$= \frac{6\alpha c^2}{(cx+d)^4}$$

$$2y_1 y_3 = \frac{2\alpha}{(cx+d)^2}\,\frac{6\alpha c^2}{(cx+d)^4} = \frac{12\alpha^2 c^2}{(cx+d)^6} =$$

$$= 3\left[\frac{-2\alpha c}{(cx+d)^3}\right]^2 = \underline{3y_2^2}.$$

(2) If $x = a(t+\sin t)$, $y = a(1+\cos t)$ find $d^2y/dx^2$.

Sol^n^: Given $x = a(t+\sin t)$ $y = a(1+\cos t)$

$$\frac{dx}{dt} = a(1+\cos t) \qquad \frac{dy}{dt} = a(-\sin t)$$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{-a\sin t}{a(1+\cos t)} = \frac{-\sin t}{1+\cos t}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{-\sin t}{1+\cos t}\right)$$

$$\frac{(1+\cos t)(-\cos t)-(-\sin t).(-\sin t)}{(1+\cos t)^2}$$

$$= \frac{-\cos t-\cos^2 t-\sin^2 t}{(1+\cos t)^2}$$

$$= \frac{-\cos t-1}{(1+\cos t)^2} = -\frac{(1+\cos t)}{(1+\cos t)^2}.$$

$$= \frac{-1}{1+\cos t}.$$

$$\therefore \frac{d^2y}{dx^2} = \left(\frac{-1}{1+\cos t}\right)\cdot\frac{dt}{dx} =$$

$$\left(\frac{-1}{1+\cos t}\right)\cdot\frac{1}{a(1+\cos t)} = \frac{-1}{a(1+\cos t)^2}$$

$$= \frac{-1}{a(2\cos^2 t/2)^2} = \qquad \left[1+\cos t = 2\cos^2 t\right]$$

$$= \frac{-1}{a.4\cos^4 t/2} = \frac{-1}{4a}\sec^4 t/2$$

(3) If $x = 2\cos t - \cos 2t$, $y = 2\sin t - \sin 2t$ find the value of $d^2y/dx^2$ when $t = \pi/2$

Sol^n Given $x = 2\cos t - \cos 2t$

$y = 2\sin t - \sin 2t$

$$\frac{dx}{dt} = -2\sin t + 2\sin 2t$$

$$\frac{dy}{dt} = 2\cos t - 2\cos 2t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{2\cos t - 2\cos 2t}{-2\sin t + 2\sin 2t}$$

$$= \frac{\cos t - \cos 2t}{-\sin t + \sin 2t}$$

$$= \frac{-2\sin 3t/2 \; \sin(-t/2)}{2\cos 3t/2 \; \sin t/2}$$

$\left[ \cos C - \cos D = -2\sin \ldots \sin\frac{C-D}{2} \right.$

$\left. \sin C - \sin D = 2\cos\frac{C+D}{2} \sin\frac{C-D}{2} \right]$

$$= \frac{\sin 3t/2}{\cos 3t/2} = \tan 3t/2.$$

$$\therefore \frac{d^2y}{dx^2} = \tan 3t/2$$

At $t = \pi/2$

$$\frac{d^2y}{dx^2} = -3/2$$

(4) If $y = \sin^{-1}x$ show that $(1-x^2)y_5 - 7xy_4 - 9y\ldots$

Sol^n Given $y = \sin^{-1}x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$y_1 = \frac{1}{\sqrt{1-x^2}} \Rightarrow y_1\sqrt{1-x^2} = 1 \text{ or}$$

$$y_1^2(1-x^2) = 1 \quad \text{[squaring on both sides]}$$

Again diff. w.r.t '$x$'

$$y_1^2(-2x) + (1-x^2)\,2y_1y_2 = 0$$

$$-2y_1^2 x + 2(1-x^2)y_1 y_2 = 0$$

$$2y_1\left[(1-x^2)y_2 - xy_1\right] = 0$$

$$\rightarrow (1-x^2)y_2 - xy_1 = 0$$

Again diff w.r.t $x$'

$$(1-x^2)y_3 + y_2(-2x) - (xy_2 + y_1) = 0$$

$$(1-x^2)y_3 - 2xy_2 - xy_2 - y_1 = 0$$

$$(1-x^2)y_3 - 3xy_2 - y_1 = 0$$

$$(1-x^2)y_4 + y_3(-2x) - 3\left[xy_3 + y_2\right] - y_2 = 0$$

$$(1-x^2)y_4 - 5xy_3 - 4y_2 = 0$$

$$(1-x^2)y_5 + y_4(-2x) - 5(xy_4 + y_3) - 4y_3 = 0$$

$$\Rightarrow (1-x^2)y_5 - 7xy_4 - 9y_3 = 0$$

## standard results

1) $D^n(ax+b)^m = m(m-1)(m-2)\ldots(m-n+1)a^n(ax+b)^{m-n}$

In particular

$$D^n(x^n) = n!$$

2) $D^n\left(\dfrac{1}{ax+b}\right) = \dfrac{(-1)^n (n!) a^n}{(ax+b)^{n+1}}$

$$\frac{\quad}{(ax+b)^n}.$$

4) $D^n(a^{mx}) = m^n(\log a)^n \cdot a^{mx}$

5) $D^n(e^{mx}) = m^n e^{mx}$

6) $D^n \sin(ax+b) = a^n \sin(ax+b+n\pi/2)$

7) $D^n \cos(ax+b) = a^n \cos(ax+b+n\pi/2)$

8) $D^n\left[e^{ax}\sin(bx+c)\right] =$

$(a^2+b^2)^{n/2} e^{ax} \sin(bx+c+n\tan^{-1} b/a)$

9) $D^n\left[e^{ax}\cos(bx+c)\right] =$

$(a^2+b^2)^{n/2} e^{ax} \cos(bx+c+n\tan^{-1} b/a)$.

problems on $n^{th}$ derivatives.

① Find the $n^{th}$ derivative of

$e^x \cos x \cos 2x$

Soln $\quad y = e^x \cos x \cos 2x = e^x\left[\frac{\cos 3x}{2} + \frac{\cos x}{2}\right]$

$y = e^x \cos 3x/2 + e^x \cos x/2 \quad \left[\because \cos A \cos B = \frac{\cos(A+B)}{2} + \frac{\cos(A-B)}{2}\right]$.

$\therefore y_n = \frac{1}{2}\left[(1+9)^{n/2} e^x \cos(3x+n\tan^{-1}3)\right] +$

$\frac{1}{2}\left[(1+1)^{n/2} e^x \cos(x+n\tan^{-1} 1/1)\right]$

$\left[D^n e^{ax}\cos(bx+c) = (a^2+b^2)^{n/2} e^{ax}\cos(bx+c+n\tan^{-1}\right.$

$= \frac{1}{2}\left[10^{n/2} e^x \cos(3x+n\tan^{-1}3) + 2^{n/2} e^x \cos(x+n\pi/4)\right]$

(2) find the $n^{th}$ derivative of $\frac{x}{x^2+a^2}$

Sol$^n$ Given $y = \frac{x}{x^2+a^2}$.

By using partial fractions

$$\frac{x}{x^2+a^2} = \frac{x}{(x+ai)(x-ai)}$$

$$= \frac{1}{2}\left[\frac{1}{x+ai} + \frac{1}{x-ai}\right]$$

$$\therefore y_n = \frac{1}{2}\left[\frac{(-1)^n n!}{(x+ai)^{n+1}} + \frac{(-1)^n n!}{(x-ai)^{n+1}}\right]$$

$$\left[D^n\left(\frac{1}{ax+b}\right) = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}\right]$$

$$= \frac{1}{2}(-1)^n n! \left[(x+ai)^{-n-1} + (x-ai)^{-n-1}\right]$$

put $x = r\cos\theta$, $a = r\sin\theta$

$r = \sqrt{x^2+a^2}$  $\theta = \tan^{-1} a/x$

+

$$\therefore y_n = \frac{(-1)^n n!}{2}\left[(r\cos\theta + ri\sin\theta)^{-n-1} + (r\cos\theta - ir\sin\theta)^{-n-1}\right]$$

$$= \frac{(-1)^n n!}{2}\left[r^{-n-1}\left[(\cos\theta + i\sin\theta)^{-n-1} + (\cos\theta - i\sin\theta)^{-n-1}\right]\right]$$

$\tan^{-1} b/a$

$$= \frac{(-1)^n n!}{2} r^{-(n+1)}\left[\cos(n+1)\theta - i\sin(n+1)\theta + \cos(n+1)\theta + i\sin(n+1)\theta\right]$$

[By using De-moviers theorem]

$$= \frac{(-1)^n n!}{2} r^{-(n+1)} \left[ 2\cos(n+1)\theta \right]$$

$$= (-1)^n n! \frac{1}{(x^2+a^2)^{(n+1)/2}} \cos(n+1)\tan^{-1} a/x$$

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③ find the $n^{th}$ derivative of $\tan^{-1}\frac{2x}{1-x^2}$ in terms of $r$ and $\theta$.

Sol$^n$ Given $y = \tan^{-1}\frac{2x}{1-x^2} = 2\tan^{-1}x$

$$y_1 = \frac{2}{1+x^2} =$$

By using partial fraction

$$\frac{2}{(1+x^2)} = \frac{2}{(x+i)(x-i)} = \frac{1}{i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right]$$

$$= \frac{1}{i}\left[(x-i)^{-1} - (x+i)^{-1}\right]$$

$$\therefore y_n = \frac{(-1)^{n-1}(n-1)!}{\cancel{2}i}\left[(x-i)^{-n} - (x+i)^{-n}\right]$$

putting $x = r\cos\theta$ $1 = r\sin\theta$.

$r = \sqrt{x^2+1^2}$ $\theta = \tan^{-1} x/1$

$$y_n = \frac{(-1)^{n-1}(n-1)!}{\cancel{2}i}\left[(r\cos\theta - ir\sin\theta)^{-n} - (r\cos\theta + ir\sin\theta)^{-n}\right]$$

$$=\frac{(-1)^{n-1}(n-1)!}{2i} r^{-n}\left[(\cos\theta - i\sin\theta)^{-n} - (\cos\theta + i\sin\theta)^{-n}\right]$$

$$=\frac{(-1)^{n-1}(n-1)!}{i} r^{-n}\left[\cos n\theta + i\sin n\theta - (\cos n\theta - i\sin n\theta)\right]$$

[using Demovier's theorem]

$$=\frac{(-1)^{n-1}(n-1)!}{i} r^{-n}\, 2i\sin n\theta$$

$$=(-1)^{n-1}(n-1)!\,(x^2+1^2)^{-n/2}\sin n\theta.$$

$$\therefore y_n = (-1)^{n-1}(n-1)!\left(\frac{1}{\sin\theta}\right)^{-n} 2\sin n\theta$$

$$=2(-1)^{n-1}(n-1)!\,\sin^n\theta\,\sin n\theta$$

## Leibnitz's theorem

Definition: If $u, v$ be two functions of $x$ possessing derivatives of $n^{th}$ order then

$$D^n(uv) = D^n u \cdot v + nc_1 D^{n-1}u\,Dv + nc_2 D^{n-2}u\,D^2v + \cdots + nc_r D^{n-r}u\,D^r v + \cdots + uD^n v.$$

(or)

$$(uv)_n = u_n v + nc_1 u_{n-1}v_1 + nc_2 u_{n-2}v_2 + \cdots + nc_r u_{n-r}v_r + \cdots + nc_n u v_n.$$

problems

(1) . If $\cos^{-1}(y/b) = \log(x/n)^n$, prove that

$$x^2 y_{n+2} + (2n+1) x y_{n+1} + 2n^2 y_n = 0.$$

Soln. Given $\cos^{-1}(y/b) = \log(x/n)^n$

$$\frac{y}{b} = \cos \log (x/n)^n$$

$$y = b \cos \log (x/n)^n \quad \text{—} (1)$$

Differentiating (1) w.r.t 'x'

$$y_1 = -b \sin \log (x/n)^n \, n . \frac{1}{x/n} . \frac{1}{n}$$

or

$$x y_1 = -bn \sin \log (x/n)^n$$

Again differentiating

$$x y_2 + y_1 = -bn \cos \log (x/n)^n . n . \frac{1}{x/n} . \frac{1}{n}$$

$$x^2 y_2 + x y_1 = -bn^2 \cos \log (x/n)^n$$

$$= -n^2 y$$

$$\Rightarrow \quad x^2 y_2 + x y_1 + n^2 y = 0.$$

Differentiating 'n' times by using leibnitz's theorem.

$$\left( (y_{n+2}) x^2 + n . y_{n+1} . 2x + \frac{n(n-1)}{2} y_n . 2 \right) +$$

$$\left(y_{n+1}x + n.y_n.1\right) + n^2 y_n = 0$$

$$\Rightarrow x^2 y_{n+2} + (2n+1)xy_{n+1} + 2n^2 y_n = 0$$

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(2). If $y = [\log(x) + \sqrt{1+x2}]^2$, prove that $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$. Hence find $(y_n)_0$.

Sol$^n$.

Given $y = [\log x + \sqrt{1+x2}]^2$.

$$y_1 = 2\log(x+\sqrt{1+x2}).\frac{1}{x+\sqrt{1+x2}} \times \left[1+\frac{1}{2\sqrt{1+x2}}.2x\right]$$

[Diff. y w.r.t 'x'].

$$\text{or } y_1 = \frac{2\log(x+\sqrt{1+x2})}{\sqrt{1+x2}}$$

squaring on both sides we get

$$y_1^2(1+x^2) = 4[\log(x+\sqrt{1+x2})]^2.$$

$$= 4y.$$

$$y_1^2(1+x^2) = 4y.$$

Again differentiating.

$$(1+x^2)\,2y_1y_2 + y_1^2(2x) = 4y_1$$

$\Rightarrow y_2(1+x^2) + xy_1 = 2$.

Differentiating 'n' times by leibnitz's theo

$\left(y_{n+2}(1+x^2) + n \cdot y_{n+1} \cdot 2x + \frac{n(n-1)}{2} \cdot y_n \cdot 2\right) +$

$y_{n+1} x + n \cdot y_n \cdot 1 = 0$.

$\Rightarrow (1+x^2) y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$.

putting $x = 0$.

$(y_{n+2})_0 = -n^2 (y_n)_0$.

Again when $x = 0$ $y_1 = 0$, $y_2 = 2$.

putting $n = 1$ in $(y_{n+2})_0 = -n^2 (y_n)_0$.

$(y_3)_0 = -(y_1)_0 = 0$.

At $n = 3$

$(y_5)_0 = -9(y_3)_0 = 0$.

$\therefore$ when $n$ is odd $(y_n)_0 = 0$.

At $n = 2$

$(y_4)_0 = -4(y_2)_0 = -4 \cdot 2 = -2^2 \cdot 2$

At $n = 4$

$(y_6)_0 = -4^2 (y_4)_0 = -4^2(-2^2) \cdot 2$

$= (-1)^2 \cdot 4^2 \cdot 2^2 \cdot 2$.

$y$

$$(y_8)_0 = -6^2 (y_6)_0 = -6^2 (-1)^2 \cdot 4^2 \cdot 2^2 \cdot 2$$

$$= (-1)^3 \cdot 6^2 \cdot 4^2 \cdot 2^2 \cdot 2$$

$\therefore$ when $n$ is even

$$(y_n)_0 = (-1)^{n/2 - 1} (n-2)^2 (n-4)^2 \cdots 6^2 \cdot 4^2 \cdot 2^2 \cdot 2$$

3) If $y = e^{m\cos^{-1}x}$ prove that

i) $(1-x^2)y_2 - xy_1 = m^2 y$

ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$

Also find $(y_n)_0$.

Soln i) Given $y = e^{m\cos^{-1}x}$

Differentiating

$$y_1 = \frac{-1}{\sqrt{1-x^2}} e^{m\cos^{-1}x} \cdot (-m)$$

$$y_1\sqrt{1-x^2} = -e^{m\cos^{-1}x} m = -my$$

squaring on both sides

$$y_1^2(1-x^2) = m^2 y^2$$

Again differentiation

$$(1-x^2)\, 2y_1 y_2 + y_1^2(-2x) = 2m^2 y y_1$$

ii) $(1-x^2)y_2 - xy_1 = m^2 y$

Differentiating each term $n$ times by Leibnitz's theorem.

$\left(y_{n+2}(1-x^2) + nc_1 y_{n+1}(-2x) + nc_2 \cdot y_n(-2)\right) -$

$\left[y_{n+2} \cdot x + nc_1 y_n \cdot 1\right] = m^2 y_n$

$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2+m^2) y_n =$

At $x=0$.

$(y_{n+2})_0 = (n^2+m^2)(y_n)_0$.

when $n$ is even.

$(y_n)_0 = [m^2+(n-2)^2][m^2+(n-4)^2] \text{---}$

$\text{---} (m^2+4^2)(m^2+2^2)m^2 e^{m\pi/2}$.

when $n$ is odd.

$(y_n)_0 = -[m^2+(n-2)^2][m^2+(n-4)^2] \text{---}$

$\text{---}(m^2+3^2)(m^2+1^2) m e^{m\pi/2}$

If $x = \sin t$, $y = \cos pt$ show that

$(1-x^2)y_2 - xy_1 + p^2y = 0$ Hence prove that

$(1-x^2)y_{n+2} - (2n+1)y_{n+1}x - (n^2-p^2)y_n = 0$.

Sol$^n$ Given $x = \sin t$ $y = \cos pt$

$\frac{dy}{dt} = -\sin pt \cdot p$ $\quad \frac{dx}{dt} = \cos t$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = -\frac{p\sin pt}{\cos t}$

$\Rightarrow y_1 = \frac{-p\sin pt}{\cos t}$

$\frac{d^2y}{dx^2} = y_2 = \frac{d}{dt}\left[\frac{dy}{dx}\right]\frac{dt}{dx} =$

$\frac{\cos t[-p\cos pt \cdot p] - [-p\sin pt] \times -\sin t}{\cos^2 t} \times \frac{1}{\cos t}$

$y_2 = \frac{-p^2\cos t\cos pt - p\sin t\sin pt}{\cos^2 t} \times \frac{1}{\cos t}$

$= \frac{-p[\sin t\sin pt + p\cos t\cos pt]}{\cos^3 t}$

$= -\frac{p\sin t\sin pt}{\cos^3 t} - \frac{p^2\cos t\cos pt}{\cos^3 t}$

$$\frac{\quad}{\cos^2 t} \quad \frac{\quad}{\cos^2 t} \quad \frac{\quad}{\cos^2 t}$$

$$= \frac{xy_1 - p^2 y}{1-\sin^2 t} = \frac{xy_1 - p^2 y}{1-x^2}.$$

$$y_2(1-x^2) = xy_1 - p^2 y$$

or $y_2(1-x^2) - xy_1 + p^2 y = 0$.

Diff. n times by Leibnitz theorem.

$$y_{n+2}(1-x^2) + n\cdot y_{n+1}(-2x) + \frac{n(n-1)}{2}\cdot y_n(-2)$$
$$-[y_{n+1}x + n\cdot y_n\cdot 1] + p^2 y_n = 0.$$

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-p^2)y_n = 0$$

## Maclaurin's Series

Definition:

If f(x) can be expanded as an infinite series then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots \infty.$$

(1) using Maclaurin's series, expand $\log(1+x)$.

Soln: Given $f(x) = \log(1+x)$

By putting $x=0$ $f(0) = \log(1+0) = 1$.

$f'(x) = -(1+x)^{-1}$ $f'(0) = 1$.
$f''(x) = -(1+x)^{-2}$ $f''(0) = -1$
$f'''(x) = 2(1+x)^{-3}$ $f'''(0) = 2(1)^{-3} = 2!$.

$f^{iv}(x) = -2\times 3(1+x)^{-4}$ $f^{iv}(0) = -(3)!$.

$\vdots$

$f^{n}(x) = (-1)^{n-1}(n-1)!\,(1+x)^{-n}$ $f^{n}(0) = (-1)^{n-1}(n-1)!$.

$\therefore$ By Maclaurin's series we have.

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots$$

$$\log(1+x) = x,x - (x^2/2) + (x^3/3) - (x^4/4) + \cdots$$

$$\cdots + (-1)^{n-1}(x^n/n) + \cdots$$

(2) Using Maclaurin's theorem prove that

$$\log(\sec x) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \cdots$$

∴ Differentiating successively and putting $x=0$

$f'(x) = \frac{\sec x . \tan x}{\sec x} = \tan x$ $\quad f'(0) = 0$.

$f''(x) = \sec^2 x$, $\quad f''(0) = 1$

$f'''(x) = 2\sec^2 x \tan x$ $\quad f'''(0) = 0$

$f^{iv}(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$ $\quad f^{iv}(0) = 2$

$f^{v}(x) = 8\sec^2 x \tan^3 x + 16\sec^4 x \tan x$ $\quad f^{v}(0) = 0$

$f^{vi}(x) = 16\sec^2 x \tan^4 x + 88\sec^4 x \tan^2 x$
$+ 16\sec^6 x$ $\quad f^{iv}(0) = 16$

∴ By Maclaurin's theorem we have

$\log\sec x = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \text{---}$

$= 0 + x.0 + \frac{x^2}{2!}.1 + \frac{x^3}{3!}.0 + \frac{x^4}{4!}.2 + \frac{x^5}{5!}.0 + \frac{x^6}{6!}.16 + \text{---}$

$= \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \text{---}$

(3) Prove that

$e^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{2^2 x^4}{4!} \text{---}$

Proof Given $f(x) = e^x \cos x = y$.

$x=0.$

putting $x=0$ $f(0)=1$.

Differentiating $f(x)$ successively and putting $x=0$

$y_1 = f'(x) = e^x \cos x - e^x \sin x$ $f'(0) = 1$.

$= y - e^x \sin x$ $(y_1)$

$f''(x) = y_2 = y_1 - y - e^x \sin x$ $(y_2)_0 = 1-1 = 0$.

$y_3 = y_2 - y_1 - y - e^x \sin x$

$= y_2 - 2y$ $(y_3)_0 = -2$

$=16$ Again $y_4 = y_3 - 2y_1$ $(y_4)_0 = -2^2$

$y_5 = y_4 - 2y_2$ $(y_5)_0 = -2^2$

$y_6 = y_5 - 2y_3$ $(y_6)_0 = 0$.

$\therefore$ By Maclaurin's theorem

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$

or

$y = y_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \cdots$

$e^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{2^2 x^4}{4!} - \frac{2^2 x^5}{5!} + \frac{2^3 x^7}{7!} + \cdots$

(4). Prove that
$$\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cancel{\frac{x^4}{\ldots}}$$

Sol$^n$ Given $f(x) = y = \log(1+\sin x)$.

$(y)_0 = \log(1+0) = 0$.

Taking logs on both sides

$e^y = 1+\sin x$ —— (1)

Differentiating we get

$e^y \cdot y_1 = \cos x$ putting $x=0$

$e^0 \cdot (y_1)_0 = 1 \Rightarrow (y_1)_0 = 1$.

Differentiating (1) again

$e^y \cdot y_2 + e^y \cdot y_1^2 = -\sin x$ or

$e^y \cdot y_2 + y_1 \cos x = -\sin x$

putting $x=0$

$(y_2)_0 = -1$.

Again diff. (1)

$e^y y_3 + e^y y_1 y_2 + y_2 \cos x - y_1 \sin x = -\cos x$

or $e^y y_3 + y_2 \cos x + y_2 \cos x - y_1 \sin x = -\cos x$.

or $e^y y_3 + 2y_2 \cos x - y_1 \sin x = -\cos x$

putting $x=0$

$1 \cdot (y_3)_0 + 2(-1) \cdot 1 - 0 = -1 \Rightarrow (y_3)_0 = 1.$

Differentiating again

$e^y y_4 + e^y y_1 y_3 + 2y_3 \cos x - 2y_2 \sin x - y_2 \sin x$
$\quad - y_1 \cos x = \sin x$

or $e^y y_4 + 3y_3 \cos x - 3y_2 \sin x - y_1 \cos x = \sin x.$

Putting $x=0$

$e^0 (y_4)_0 + 3 \cdot 1 \cdot 1 - 0 - 1 \cdot 1 = 0 \Rightarrow (y_4)_0 = -2.$

Diff. again

$e^y y_5 + e^y y_1 y_4 + 3y_4 \cos x - 3y_3 - \sin x - 3y_3 \sin x$

$-3y_2 \cos x - y_2 \cos x + y_1 \sin x = \cos x$

Putting $x=0$

$1 \cdot (y_5)_0 + 1 \cdot 1 \cdot (-2) + 3(-2) \cdot 1 - 0$
$\quad - 3(-1) - (-1) + 0 = 1.$

$\Rightarrow (y_5)_0 = 5$

Substituting these values in Maclaurin's theorem we get

$\log(1+\sin x) = 0 + x \cdot 1 \frac{x^2}{2!}(-1) + \frac{x^3}{3!} \cdot 1 +$

$\frac{x^4}{4!}(-2) + \frac{x^5}{5!} \cdot 5 + \cdots$

$= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots$

# Taylor's Series

**Definition:** If $f(x+h)$ can be expanded as an infinite series, then

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$

Replacing $x$ by $a$ and $h$ by $(x-a)$ we get

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) +$$

$$\frac{(x-a)^3}{3!}f'''(a) + \cdots \infty.$$

Problems:

(1) - Expand $e^x$ in powers of $(x-1)$ upto four terms.

Soln. Given $f(x) = e^x$

$e^x = f(x) = f(1+(x-1)) = f(x+a)$ $x =$.

By taylor's theorem.

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots$$

$$e^x = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}$$

$$f'''(1) + \cdots$$

$f(x) = e^x$ $\quad$ $f(1) = e$

$f'(x) = e^x$ $\quad$ $f'(1) = e$

$f''(x) = e^x$ $\quad$ $f''(1) = e$ and so on.

∴ substituting these values in the above eqn.

$$e^x = e + (x-1)e + \frac{(x-1)^2}{2}e + \frac{(x-1)^3}{6}e + \frac{(x-1)^4}{24}e + \cdots$$

or $$e^x = e\left[1 + \frac{(x-1)}{1!} + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \cdots\right].$$

(2) - Expand $\sin x$ in powers of $(x - \pi/2)$. Hence find the value of $\sin 91$ correct to 4 decimal places.

Soln. Given $f(x) = \sin x$

$$f(x) = f\left[\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right)\right]$$

By taylor's theorem

$$f(x) = f\left[\frac{1}{2}\pi + \left(x - \frac{\pi}{2}\right)\right]$$

$$= f\left(\frac{\pi}{2}\right) + (x - \pi/2)\, f'(\pi/2) + \frac{(x-\pi/2)^2}{2!} f''(\pi/2) + \cdots \quad (1)$$

$f(x) = \sin x$      $f(\pi/2) = \sin \pi/2 = 1.$

$f'(x) = \cos x$      $f'(\pi/2) = 0$

$f''(x) = -\sin x$      $f''(\pi/2) = -1$

$f'''(x) = -\cos x$      $f'''(\pi/2) = 0.$

$f^{iv}(x) = \sin x$      $f^{iv}(\pi/2) = 1.$

putting these values in (1) we get

$$\sin x = 1 + (x-\pi/2)\cdot 0 + \frac{(x-\pi/2)^2}{2!}(-1) + \frac{(x-\pi/2)^3}{3!}\cdot 0 + \frac{(x-\pi/2)^4}{4!}\cdot 1 + \ldots$$

or $\sin x = 1 - (1/2)!\,(x-\pi/2)^2 - (1/4)!\,(x-\pi/2)^4 \ldots$

Putting $x = 9i$

(2) $\sin 9i = 1 + \cancel{9i}(0.5\pi - \pi/2)\cdot 0 + (-1)\frac{(0.5\pi - \pi/2)^2}{2!} + \frac{(0.5\pi-\pi/2)^3}{3!}\cdot 0 + \frac{(0.5\pi - \pi/2)^4}{4!}\cdot 1 + \ldots$

$= 0.9998$

(3) Compute to 4 decimal places, the value of $\cos 32^\circ$, by the use of Taylor's series.

Sol$^n$ Given ~~f(x) = cos x~~

Let $f(x+h) = \cos(x+h)$ $\therefore f(x) = \cos x$

Since $f(x+h) = f(x) + h f'(x) + \frac{h^2}{\lfloor 2} f''(x) + \frac{h^3}{\lfloor 3} f'''(x) + \ldots \infty$

$\therefore \cos(x+h) = \cos x + h(-\sin x) + \frac{h^2}{\lfloor 2}(-\cos x) + \frac{h^3}{\lfloor 3}\sin x + \ldots \infty$

put $x = 30^\circ$, $h = 2^\circ$

$$\therefore \cos 32^\circ = \cos 30^\circ + 2^\circ(-\sin 30^\circ) + \frac{2^{\circ 2}}{2}(-\cos 30^\circ) + \frac{2^{\circ 3}}{\lfloor 3}\sin 30^\circ + \text{-----} \infty$$

$$= 0.8660 + (0.0348)(-0.5) + (0.0006)(-0.8660) + \text{-----} \infty$$

$$= 0.8660 - 0.0174 - 0.0005 + \text{--} \infty$$

$$= 0.8481$$

---

Q. 1 Given $\log_{10} 4 = 0.6021$, calculate approximately $\log_{10} 404$

Sol$^n$ since $\log_{10} 404 = \frac{1}{2.3026} \log_e 404$

$$= \frac{1}{2.3026} \log_e (101 \times 4)$$

$$= \frac{1}{2.3026} (\log_e 101 + \log_e 4)$$

$$= \frac{1}{2.3026} \log_e 101 + \frac{\log_e 4}{2.3026}$$

$$= \frac{1}{2.3026} \log_e (100 + 1) + \log_{10} 4 \quad \text{---} (i)$$

By ~~tay~~ Taylor's series we get

$$\log(x+h) = \log x + h \cdot \frac{1}{x} + \frac{h^2}{2}\frac{(-1)}{x^2} + \frac{h^3}{6}\frac{2}{x^3}$$

$$+ \cdots\cdots$$

put $x = 100$, $h = 1$

$$\log_e 101 = \log_e 100 + \frac{1}{100} + \frac{1}{2}\frac{(-1)}{(100)^2} + \frac{1}{3} \times \frac{1}{(100)^3}$$

$$+ - - -$$

$$= 4.6051 + 0.01 - 0.00005 + - - - -$$

$$= 4.6150$$

Substitute these value in (i) we get

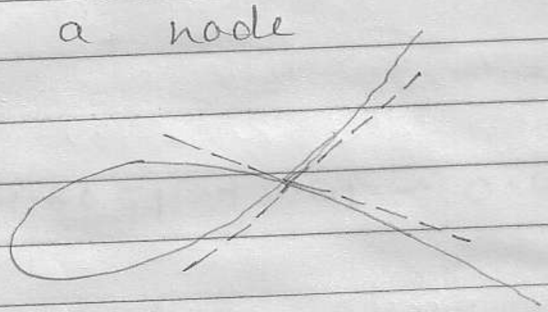
$$\log_{10} 404 = \frac{4.6150}{2.3026} + 0.6021$$

$$= 2.6063$$

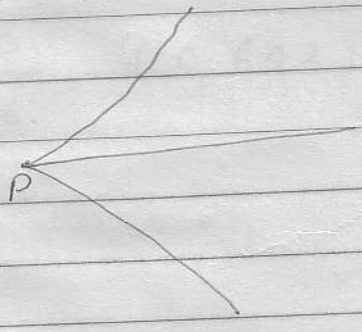
# Curve Tracing

**Double point** — A point through which ~~has~~ two branches of a curve pass is called a double point. At such a point P, the curve has two tangents, one for each branch.

(i) **Node** — If the tangents are real and distinct, the double point is called a node



(ii) **Cusp** — If the tangents are real and coincident, the double point is called a cusp.



(iii) **Conjugate point** — If the tangents are imaginary, the double point is called a conjugate point (or an isolated point) Such a point can not be shown in the figure.

## Procedure for tracing Cartesian Curves :-

1. **Symmetry** :- (i) A curve is symmetrical about the $x$ axis, if only even power of $y$ occur in its eqn
For example $y^2 = 4ax$ is symmetrical about the $x$ axis.

(ii) A curve is symmetrical about the $y$ axis, if only even power of $x$ occur in its equation
For example $x^2 = 4ay$ is symmetrical about $y$ axis.

(iii) A curve is symmetrical about the line $y = x$, if on interchanging $x$ and $y$ its eqn remains unchanged
For example $x^3 + y^3 = 3axy$ is symmetrical about the line $y = x$.

2. **Origin** — (i) A curve passes through the origin if there is no constant term in its equation.

(ii) If it does, find the eqn of the tangent thereat, by ~~changing~~ equating to zero the lowest degree term.

(iii) If the origin is a double point, find whether the origin is a node cusp or conjugate point.

3. **Asymptotes** — (i) To find the asymptotes parallel to $x$ axis, equate to zero the coefficient of the highest power of $x$ in the equation, provided this is not merely

a constant.

(ii) To find the asymptotes parallel to y axis, equate to zero the coefficient of the highest power of y in the equation, provided this is not merely a constant.

Points — (i) Find the points where the curve crosses the axes and the asymptotes

(ii) Find the points where the tangent is parallel or perpendicular to the x axis (ie the points where $\frac{dy}{dx} = 0$ or $\infty$)

Q. (1) Trace the curve

$$y^2(2a-x) = x^3$$

Soln (i) symmetry → The curve is symmetrical about the x axis

[∵ only even power of y occur in the equation]

(ii) Origin → The curve passes through the origin

[∵ there is no constant term in its equation]

The tangents at the origin are

$y=0, \quad y=0$

$\therefore$ Origin is a Cusp

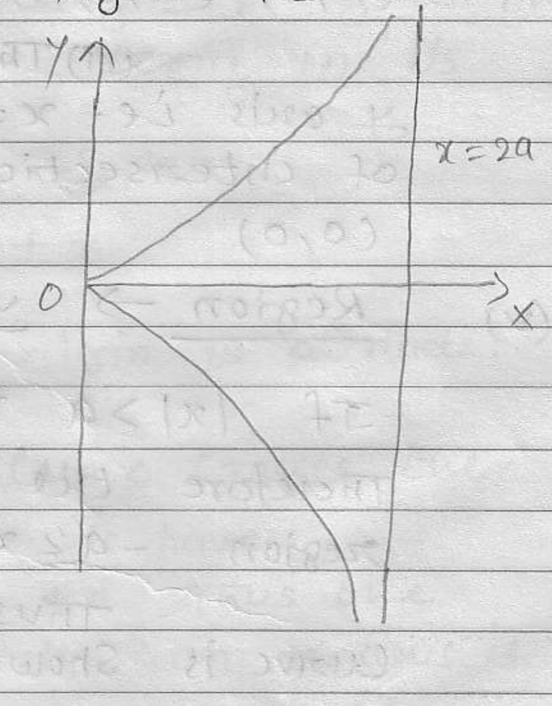
(iii) Asymptotes — The curve has ~~no~~ an asymptotes $x=2a$

($\because$ coefficient of $y^3$ is absent, coefficient of $y^2$ is an asymptote)

(iv) Points — (i) Curve meet the axes at $(0,0)$ only (ii) $\cancel{y^2/x^2}$ $y^2 = \dfrac{x^3}{2a-x}$

when $x$ is $-$ve, $y^2$ is $-$ve (i.e. $y$ is imaginary) so that no portion of the curve lies to the left of the $y$ axis. Also when $x > 2a$, $y^2$ is again $-$ve, so that no portion of the curve lies to the ~~left~~ right of the line $3x=2a$.

Hence the Shape of the curve is as shown in figure. The curve is known as Cissoid.



Trace the following Curves

1. $y^2 (a+x) = x^2 (a-x)$

i) Symmetry — (i) The curve is symmetrical about the axis since even power of $y$ occur in its equation.

(ii) Origin → The curve passes through the origin since there is no constant term in its equation.

(iii) Asymptotes → There is no asymptotes parallel to $x$ axis. There is asymptotes Parallel to $y$ axis is $a+x=0 \Rightarrow x=-a$

(iv) Points → (i) The curve crosses the $x$ ~~axis~~ axis i.e. $y=0$ Then the Points of intersection on $x$ axis is $(0,0)$, $(a,0)$

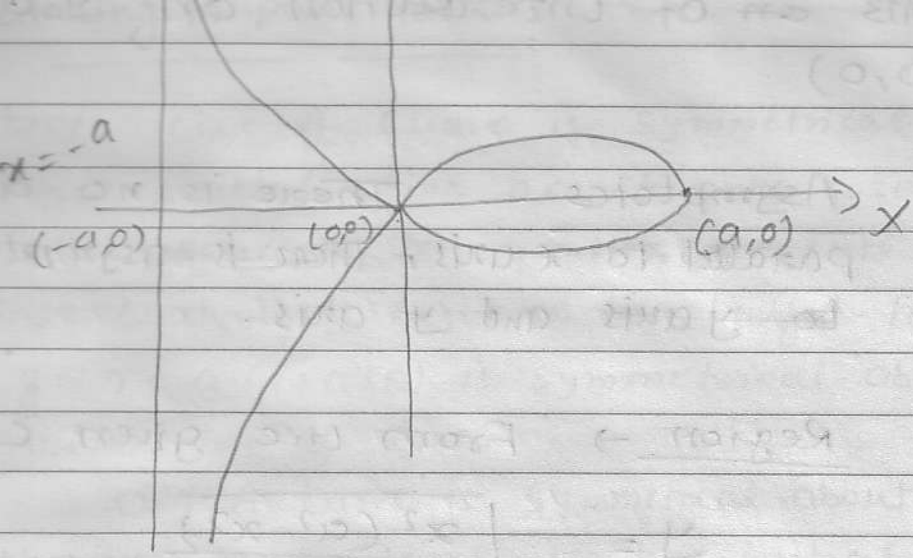
(ii) The curve crosses the $y$ axis i.e. $x=0$ - Then the points of intersection on $y$ axis is $(0,0)$

(v) Region → $y = \sqrt{x^2 (a-x)/(a+x)}$

If $|x|>a$ Then $y$ is imaginary

Therefore the curve lies in the region $-a < x \le a$

Thus the shape of the curve is shown in the figure




2. $y^2(a^2+x^2) = x^2(a^2-x^2)$

1. Symmetry → The given curve is symmetrical about the $x$ axis and $y$ axis. Since the even power of $x$ and $y$ occur in its equation.

2. Origin → The curve passes through the origin, since there is no constant term in its equation.
Now tangent at the origin is given by equating to zero the lowest degree term in its eqn

i.e. $y^2a^2 = x^2a^2$

$\Rightarrow y = \pm x$

Thus origin is a node.

3. Points → The curve crosses the $x$ axis i.e. $y=0$. We have

$x=0,\ x=\pm a$ Thus the points of intersection on $x$ axis is $(0,0), (a,0), (-a,0)$

The curve crosses the $y$ axis i.e. $x=0$. We have $y=0$ The

Points ~~on~~ of intersection on y axis is
(0,0)

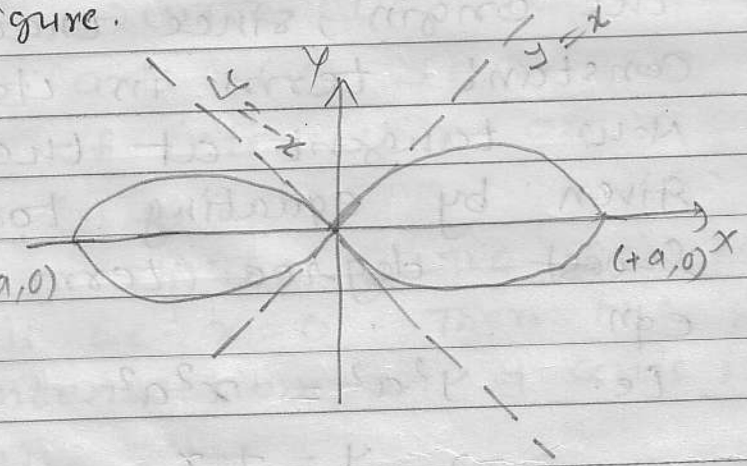
4. Asymptotes — There is no asymptotes parallel to x axis. ~~There is asymptotes parallel to y axis~~ and y axis.

5. Region → From the given curve

$$y = \sqrt{\frac{x^2(a^2-x^2)}{(a^2+x^2)}}$$

If $|x| > a$ Then y is imaginary. Thus the curve lies in the region $-a \le x \le a$.

Now the shape of the curve is shown in the figure.



# Curve tracing in polar form

1. Symmetry (i) A curve is symmetrical about the initial line OX, if only $\cos\theta$ (or $\sec\theta$) occur in its equation i.e. (it remains unchanged when $\theta$ is changed to $-\theta$) e.g. $r = a(1+\cos\theta)$ is symmetrical about the ~~initia~~ initial line.

(ii) A curve is symmetrical about the line through the pole $\perp$ to the initial line (i.e OY), if only $\sin\theta$ (or $\operatorname{cosec}\theta$) occur in its equation. (i.e. it remains unchanged when $\theta$ is changed to $\pi-\theta$) e.g. $r = a\sin 3\theta$ is symmetrical about OY.

(iii) A curve is symmetrical about the pole, if only even power of $r$ occur in the equation (i.e. it remains unchanged when $r$ is changed to $-r$) e.g. $r^2 = a^2\cos 2\theta$ is symmetrical about the pole.

2. Points — (i) Giving successive ~~possesses an infinite branch~~ value to $\theta$, find the corresponding value of $r$

(ii) Determine the points where the tangent coincides with the radius vector or is perpendicular to it (i.e. the points where $\tan\phi = r\frac{d\theta}{dr} = 0$ or $\infty$)

Trace the following Curve :-

(1) $r = a(1-\cos\theta)$

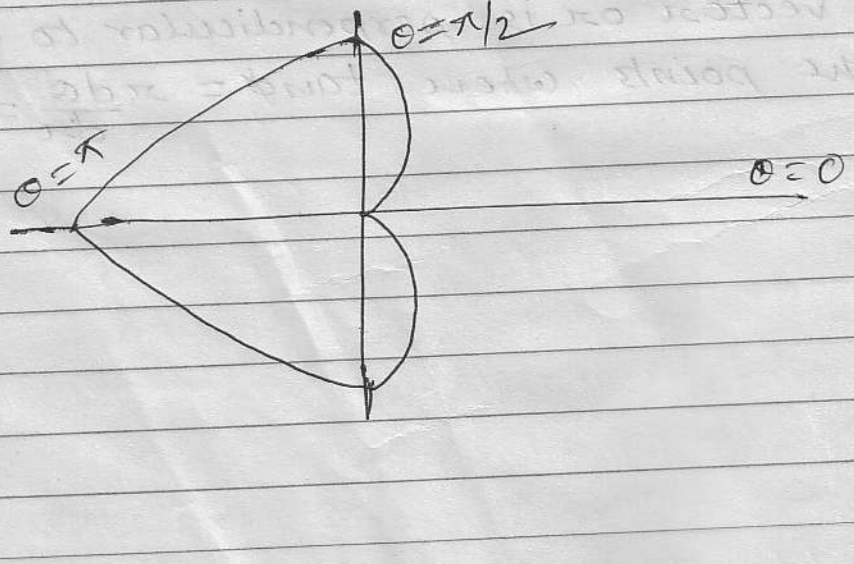
(i) Symmetry → The Curve is Symmetrical about the initial line i.e. $\theta = 0$

(ii) Pole → The Curve passes through the pole when $\theta = 0$
Therefore tangent at the pole is $\theta = 0$.

(iii) Now we have to ~~form~~ table for various value of $r$ corresponding to different value of $\theta$.

| $\theta$ | $0$ | $\pi/2$ | $\pi$ | $3\pi/2$ | $2\pi$ |
|---|---|---|---|---|---|
| $r$ | $0$ | $a$ | $2a$ | $a$ | $0$ |

The shape of the curve is shown in the figure



(2) Trace the Curve $r = 2a\cos\theta$.

Sol$^n$ (i) Symmetry → The given Curve is symmetrical about the initial line $\theta = 0$.

(ii) pole → The Curve passes through the pole when $\theta = \pi/2$ Thus tangent at the pole is $\theta = \pi/2$

(iii) Now we have to form a table for ~~vertue~~ various value of $r$ corresponding to the value of $\theta$

| $\theta$ | $0$ | $\pi/2$ | $\pi$ | $3\pi/2$ | $2\pi$ |
|---|---|---|---|---|---|
| $r$ | $0$ | $a$ | $2a$ | $a$ | $0$ |

The shape of the Curve is shown in the figure