Definition of Stress

Consider a small area δA on the surface of a body (Fig. 1.1). The force acting on this area is δF
This force can be resolved into **two perpendicular components**

- The component of force acting normal to the area called **normal** force and is denoted by δF_n
- The component of force acting along the plane of area is called **tangential** force and is denoted by δF_t

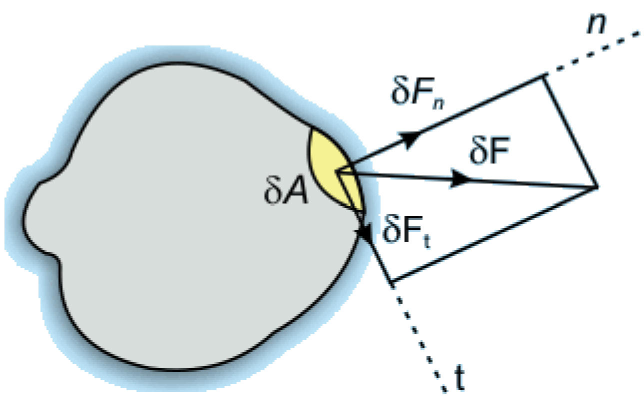


Fig 1.1 Normal and Tangential Forces on a surface

When they are expressed as force per unit area they are called as **normal stress** and **tangential stress** respectively**.** The tangential stress is also

called shear stress

The normal stress

$$\sigma = \lim_{\delta A \to 0} \left(\frac{\delta F_n}{\delta A} \right) \quad (1.1)$$

And shear stress

$$\tau = \lim_{\delta A \to 0} \left(\frac{\delta F_t}{\delta A} \right) \quad (1.2)$$

Definition of Fluid

- A fluid is a substance that **deforms continuously** in the face of tangential or shear stress, **irrespective of the magnitude of shear stress** .This continuous deformation under the application of shear stress constitutes a flow.

- In this connection fluid can also be defined as the **state of matter that cannot sustain any shear stress.**

Example : Consider Fig 1.2

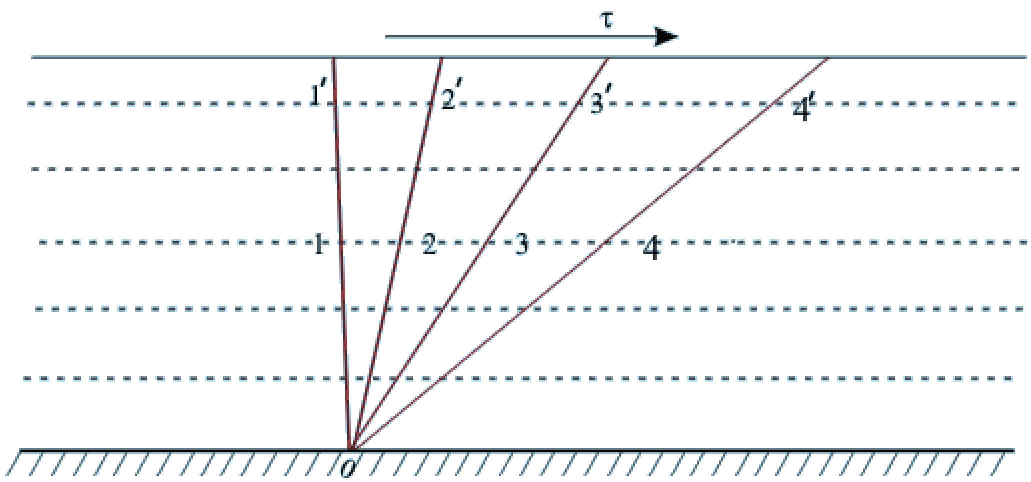


Fig 1.2 Shear stress on a fluid body

If a shear stress τ is applied at any location in a fluid, the element 011' which is initially at rest, will move to 022', then to 033'. Further, it moves to 044' and continues to move in a similar fashion.

In other words, the **tangential stress in a fluid body depends on velocity of deformation and vanishes as this velocity approaches zero. A good example is Newton's parallel plate experiment where dependence of shear force on the velocity of deformation was established.**

Concept of Continuum

- The concept of continuum is a kind of idealization of the continuous description of matter where the properties of the matter are considered as continuous functions of space variables. Although any matter is composed of several molecules, the concept of continuum assumes a continuous distribution of mass within the matter or system with no empty space, instead of the actual conglomeration of separate molecules.
- Describing a fluid flow quantitatively makes it necessary to assume that flow variables (pressure , velocity etc.) and fluid properties vary continuously from one point to another. Mathematical description of flow on this basis have proved to be reliable and treatment of fluid medium as a continuum has firmly become established. For example density at a point is normally defined as

$$\rho = \lim_{\Delta\forall \to 0}\left(\frac{m}{\Delta\forall}\right)$$

Here $\Delta\forall$ is the volume of the fluid element and m is the mass

- If $\Delta\forall$ is very large ρ is affected by the inhomogeneities in the fluid medium. Considering another extreme if $\Delta\forall$ is very small, random movement of atoms (or molecules) would change their number at different times. In the continuum approximation point density is defined at the smallest magnitude of $\Delta\forall$, before statistical fluctuations become significant. This is called continuum limit and is denoted by $\Delta\forall_c$.

$$\rho = \lim_{\Delta\forall \to \Delta\forall_C} \left(\frac{m}{\Delta\forall} \right)$$

Fluid Properties :

Characteristics of a continuous fluid which are independent of the motion of the fluid are called basic properties of the fluid. Some of the basic properties are as discussed below.

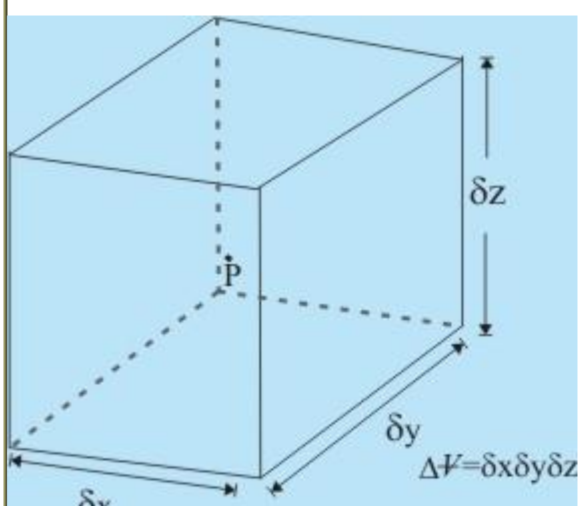
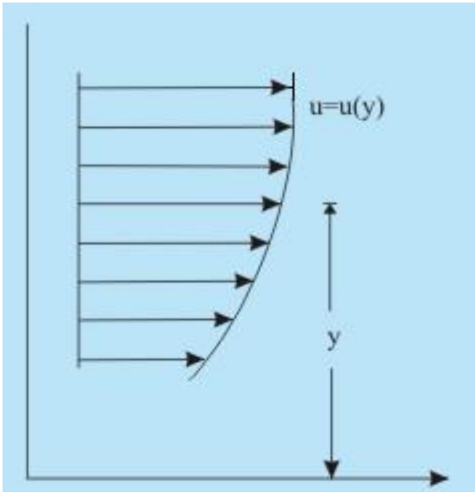
Property	Symbol	Definition	Unit
Density	ρ	The density *p* of a fluid is its mass per unit volume . If a fluid element enclosing a point *P* has a volume $\Delta\forall$ and mass Δm (Fig. 1.4), then density (ρ)at point *P* is written as $$\rho = \lim_{\Delta\forall \to \Delta\forall_C} \left(\frac{m}{\Delta\forall} \right)$$ However, in a medium where continuum model is valid one can write - $$\rho = \lim_{\Delta\forall \to 0} \left(\frac{m}{\Delta\forall} \right) = \left[\frac{dm}{d\forall} \right]_{\forall} \qquad (1.3)$$  	kg/m^3

		Fig 1.4 A fluid element enclosing point P	
Specific Weight	γ	The **specific weight** is the weight of fluid per unit volume. The specific weight is given by $\gamma = \rho g$ (1.4) Where *g* is the gravitational acceleration. Just as weight must be clearly distinguished from mass, so must the specific weight be distinguished from density.	N/m^3
Specific Volume	v	The **specific volume** of a fluid is the volume occupied by unit mass of fluid. Thus $v = \frac{1}{\rho}$ (1.5)	m^3 /kg
Specific Gravity	s	For liquids, it is the ratio of density of a liquid at actual conditions to the density of pure water at 101 kN/m^2 , and at 4°C. The specific gravity of a gas is the ratio of its density to that of either hydrogen or air at some specified temperature or pressure. **However, there is no general standard; so the conditions must be stated while referring to the specific gravity of a gas.**	-

Viscosity (μ) :

- Viscosity is a fluid property whose effect is understood when the fluid is in motion.
- In a flow of fluid, when the fluid elements move with different velocities, each element will feel some resistance due to fluid friction within the elements.
- Therefore, shear stresses can be identified between the fluid elements with different velocities.
- The relationship between the shear stress and the velocity field was given by Sir Isaac Newton.

Consider a flow (Fig. 1.5) in which all fluid particles are moving in the same direction in such a way that the fluid layers move parallel with different velocities.



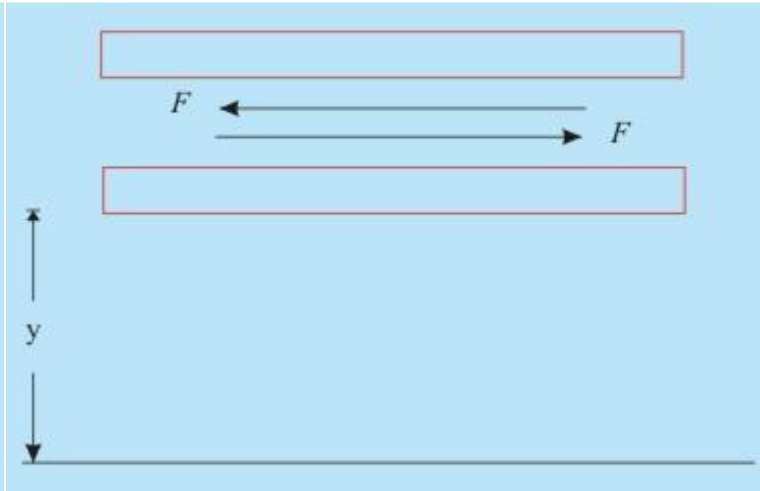


Fig 1.5 Parallel flow of a fluid

Fig 1.6 Two adjacent layers of a moving fluid.

- The upper layer, which is moving faster, tries to draw the lower slowly moving layer along with it by means of a force *F* along the direction of flow on this layer. Similarly, the lower layer tries to retard the upper one, according to Newton's third law, with an equal and opposite force *F* on it (Figure 1.6).
- Such a fluid flow where x-direction velocities, for example, change with y-coordinate is called **shear flow** of the fluid.
- Thus, the dragging effect of one layer on the other is experienced by a tangential force *F* on the respective layers. If *F* acts over an area of contact *A,* then the shear stress τ is defined as

$\tau = F/A$

Viscosity (μ) :

- Newton postulated that τ is proportional to the quantity Δu/ Δy where Δy is the distance of separation of the two layers and Δu is the difference in their velocities.
- In the limiting case of , Δu / Δy equals du/dy, the velocity gradient at a point in a direction perpendicular to the direction of the motion of the layer.
- According to Newton τ and du/dy bears the relation

$$\tau = \mu \frac{du}{dy} \qquad (1.7)$$

where, the constant of proportionality μ is known as the **coefficient of viscosity** or simply viscosity which is a property of the fluid and depends on its state. Sign of τ depends upon the sign of du/dy. For the profile shown in Fig. 1.5, du/dy is positive everywhere and hence, τ is positive. Both the velocity and

stress are considered positive in the positive direction of the coordinate parallel to them.
Equation

$$\tau = \mu \frac{du}{dy}$$

defining the viscosity of a fluid, is known as Newton's law of viscosity. Common fluids, viz. water, air, mercury obey Newton's law of viscosity and are known as *Newtonian fluids.*

Other classes of fluids, viz. paints, different polymer solution, blood do not obey the typical linear relationship, of τ and du/dy and are known as **non-Newtonian fluids**. In non-newtonian fluids viscosity itself may be a function of deformation rate as you will study in the next lecture.

Causes of Viscosity

- The causes of viscosity in a fluid are possibly attributed to two factors:

(i) intermolecular force of cohesion
(ii) molecular momentum exchange

- Due to strong cohesive forces between the molecules, any layer in a moving fluid tries to drag the adjacent layer to move with an equal speed and thus produces the effect of viscosity as discussed earlier. Since cohesion decreases with temperature, the liquid viscosity does likewise.

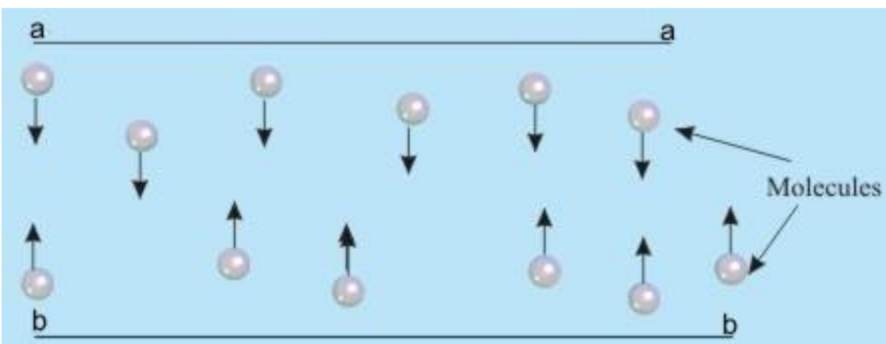


Fig 1.7 Movement of fluid molecules between two adjacent

moving layers

- Molecules from layer aa in course of continous thermal agitation migrate into layer bb
- Momentum from the migrant molecules from layer aa is stored by molecules of layer bb by way of collision
- Thus layer bb as a whole is speeded up

- Molecules from the lower layer bb arrive at aa and tend to retard the layer aa
- Every such migration of molecules causes forces of acceleration or deceleration to drag the layers so as to oppose the differences in velocity between the layers and produce the effect of viscosity.

Causes of Viscosity - contd from previous slide...

- As the random molecular motion increases with a rise in temperature, the viscosity also increases accordingly. Except for very special cases (e.g., at very high pressure) the viscosity of both liquids and gases ceases to be a function of pressure.
- For Newtonian fluids, the coefficient of viscosity depends strongly on temperature but varies very little with pressure.
- For liquids, molecular motion is less significant than the forces of cohesion, thus **viscosity of liquids decrease with increase in temperature.**
- For gases,molecular motion is more significant than the cohesive forces, thus **viscosity of gases increase with increase in temperature.**

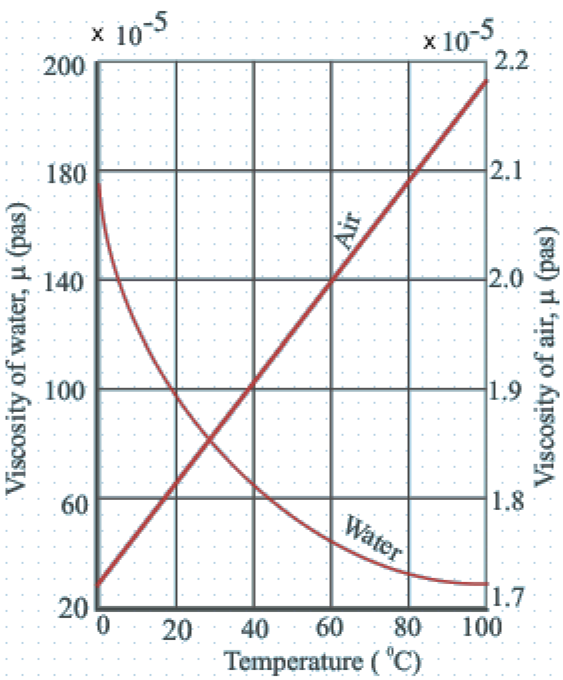


Fig 1.8: Change of Viscosity of Water and Air under 1 atm

No-slip Condition of Viscous Fluids

- It has been established through experimental observations that the relative velocity between the solid surface and the adjacent fluid particles is zero whenever a viscous fluid flows over a solid surface. This is known as no-slip condition.
- This behavior of no-slip at the solid surface is not same as the wetting of surfaces by the fluids. For example, mercury flowing in a stationary glass tube will not wet the surface, but will have zero velocity at the wall of the tube.
- The wetting property results from surface tension, whereas the no-slip condition is a consequence of fluid viscosity.

Ideal Fluid

- Consider a hypothetical fluid having a zero viscosity (μ = 0). Such a fluid is called an *ideal fluid* and the resulting motion is called as **ideal** or **inviscid flow**. **In an ideal flow, there is no existence of shear force because of vanishing viscosity.**

$$\tau = \mu \frac{du}{dy} = 0 \qquad \text{since } \mu = 0$$

- All the **fluids in reality have viscosity** (μ > 0) and hence they are termed as real fluid and their motion is known as viscous flow.
- Under certain situations of very high velocity flow of viscous fluids, an accurate analysis of flow field away from a solid surface can be made from the ideal flow theory.

Non-Newtonian Fluids

- There are certain fluids where the linear relationship between the shear stress and the deformation rate (velocity gradient in parallel flow) as expressed by the $\tau = \mu \frac{du}{dy}$ is not valid. For these fluids the viscosity varies with rate of deformation.
- Due to the deviation from Newton's law of viscosity they are commonly termed as **non-Newtonian fluids**. Figure 2.1 shows the class of fluid for which this relationship is nonlinear.

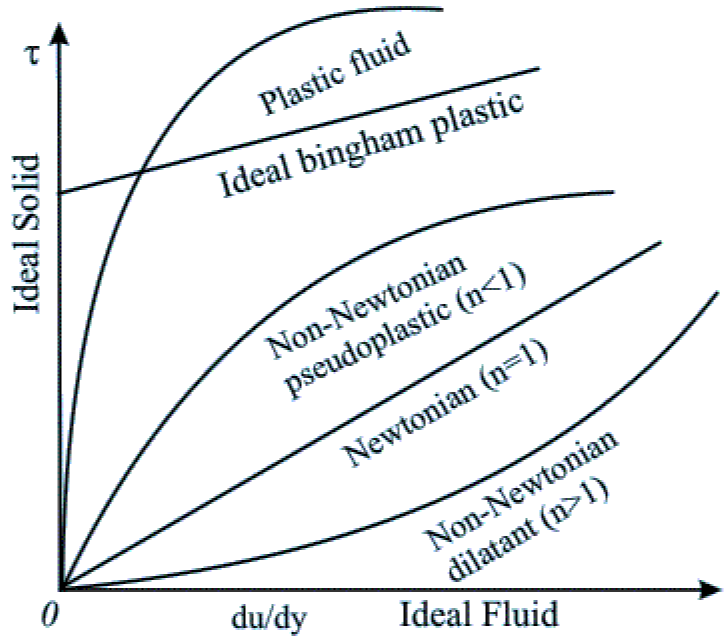


Figure 2.1 Shear stress and deformation rate relationship of different fluids

- The abscissa in Fig. 2.1 represents the behaviour of ideal fluids since for the ideal fluids the resistance to shearing deformation rate is always zero, and hence they exhibit zero shear stress under any condition of flow.
- The ordinate represents the ideal solid for there is no deformation rate under any loading condition.
- The Newtonian fluids behave according to the law that shear stress is linearly proportional to velocity gradient or rate of shear strain $\tau = \mu\, du/dy$. Thus for these fluids, the plot of shear stress against velocity gradient is a straight line through the origin. The slope of the line determines the viscosity.
- The non-Newtonian fluids are further classified as pseudo-plastic, dilatant and Bingham plastic.

Compressibility

- Compressibility of any substance is the measure of its change in volume under the action of external forces.
- The normal compressive stress on any fluid element at rest is known as hydrostatic pressure p and arises as a result of innumerable molecular collisions in the entire fluid.
- The degree of compressibility of a substance is characterized by the bulk modulus of elasticity E defined as

$$E = \lim_{\Delta\forall \to 0} \left(\frac{-\Delta p}{\Delta\forall \; / \forall} \right) \qquad (2.3)$$

- Where $\Delta\forall$ and Δp are the changes in the volume and pressure respectively, and $\forall$ is the initial volume. The negative sign (-sign) is included to make E positive, since increase in pressure would decrease the volume i.e for Δp>0 , $\Delta\forall$<0) in volume.

- For a given mass of a substance, the change in its volume and density satisfies the relation

Dm = 0, D($\rho\forall$) = 0

$$\frac{\Delta\forall}{\forall} = -\frac{\Delta\rho}{\rho} \qquad (2.4)$$

using $E = \lim_{\Delta\forall \to 0} \left(\frac{-\Delta p}{\Delta\forall \; / \forall} \right) \; \& \; \frac{\Delta\forall}{\forall} = -\frac{\Delta\rho}{\rho}$

we get

$$E = \lim_{\Delta\rho \to 0} \left(\frac{\Delta p}{\Delta\rho / \rho} \right) = \rho \frac{dp}{d\rho} \qquad (2.5)$$

- Values of *E* for liquids are very high as compared with those of gases (except at very high pressures). Therefore, liquids are usually termed as incompressible fluids though, in fact, no substance is theoretically incompressible with a value of *E* as ∞ .

- For example, the bulk modulus of elasticity for water and air at atmospheric pressure are approximately 2 x 10^6 kN/m ^{2} and 101 kN/m ^{2} respectively. It indicates that air is about 20,000 times more compressible than water. Hence water can be treated as incompressible.

- For gases another characteristic parameter, known as compressibility *K,* is usually defined , it is the reciprocal of *E*

$$K = \frac{1}{E} = \frac{1}{\rho}\left(\frac{d\rho}{dp}\right) = -\frac{1}{\forall}\left(\frac{d\forall}{dp}\right) \qquad (2.6)$$

- K is often expressed in terms of specific volume $\forall$.

- For any gaseous substance, a change in pressure is generally associated with a change in volume and a change in temperature simultaneously. A **functional relationship between the pressure,**

volume and temperature at any equilibrium state is known as thermodynamic equation of state for the gas.

For an ideal gas, the thermodynamic equation of state is given by

$$p = \rho RT \qquad (2.7)$$

- where T is the temperature in absolute thermodynamic or gas temperature scale (which are, in fact, identical), and R is known as the characteristic gas constant, the value of which depends upon a particular gas. However, this equation is also valid for the real gases which are thermodynamically far from their liquid phase. For air, the value of R is 287 J/kg K.
- K and E generally depend on the nature of process

Distinction between an Incompressible and a Compressible Flow

- In order to know, if it is necessary to take into account the compressibility of gases in fluid flow problems, we need to consider whether the change in pressure brought about by the fluid motion causes large change in volume or density.

  Using Bernoulli's equation

  $p + (1/2)\rho V^2$= constant (*V* being the velocity of flow), change in pressure, Δp, in a flow field, is of the order of $(1/2)\rho V^2$ (dynamic head).
  Invoking this relationship into

$$E = \lim_{\delta\rho \to 0}\left(\frac{\Delta p}{\Delta\rho / \rho}\right) = \rho \frac{dp}{d\rho}$$

- we get ,

$$\frac{\Delta\rho}{\rho} \approx \frac{1}{2}\frac{\rho V^2}{E} \qquad (2.12)$$

So if $\Delta\rho/\rho$ is very small, the flow of gases can be treated as incompressible with a good degree of approximation.

- According to Laplace's equation, the velocity of sound is given by

$$a = \sqrt{\frac{E}{\rho}}$$

- Hence

$$\frac{\Delta\rho}{\rho} \approx \frac{1}{2}\frac{V^2}{a^2} \approx \frac{1}{2}Ma^2$$

- where, Ma is the ratio of the velocity of flow to the acoustic velocity in the flowing medium at the condition and is known as **Mach number**. So we can conclude that the compressibility of gas in a flow can be neglected if Δρ/ρ is considerably smaller than unity, i.e. $(1/2)Ma^2 << 1$.

- In other words, if the flow velocity is small as compared to the local acoustic velocity, compressibility of gases can be neglected. **Considering a maximum relative change in density of 5 per cent as the criterion of an incompressible flow, the upper limit of Mach number becomes approximately 0.33**. In the case of air at standard pressure and temperature, the acoustic velocity is about 335.28 m/s. Hence a Mach number of 0.33 corresponds to a velocity of about 110 m/s. Therefore flow of air up to a velocity of 110 m/s under standard condition can be considered as incompressible flow.

Surface Tension of Liquids

- The phenomenon of surface tension arises due to the two kinds of intermolecular forces

  (i) **Cohesion :** The force of attraction between the molecules of a liquid by virtue of which they are bound to each other to remain as one assemblage of particles is known as the force of cohesion. This property enables the liquid to resist tensile stress.

(ii) **Adhesion :** The force of attraction between unlike molecules, i.e. between the molecules of different liquids or between the molecules of a liquid and those of a solid body when they are in contact with each other, is known as the force of adhesion. This force enables two different liquids to adhere to each other or a liquid to adhere to a solid body or surface.

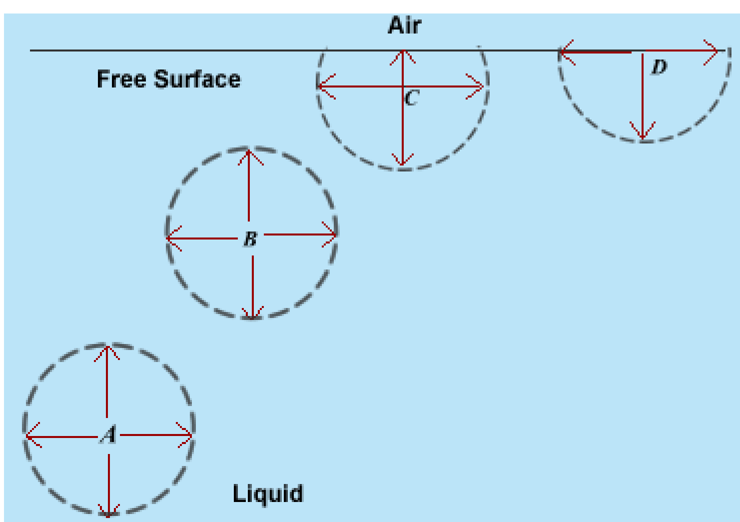


Figure 2.3 The intermolecular cohesive force field in a bulk of liquid with a free surface

A and B experience equal force of cohesion in all directions, C experiences a net force interior of the liquid The net force is maximum for D since it is at surface

- Work is done on each molecule arriving at surface against the action of an inward force. Thus mechanical work is performed in creating a free surface or in increasing the area of the surface. Therefore, a surface requires mechanical energy for its formation and the existence of a free surface implies the presence of stored mechanical energy known as free surface energy. Any system tries to attain the condition of stable equilibrium with its potential energy as minimum. Thus a quantity of liquid will adjust its shape until its surface area and consequently its free surface energy is a minimum.

- The magnitude of surface tension is defined as the tensile force acting across imaginary short and straight elemental line divided by the length of the line.
- The dimensional formula is *F/L* or MT^{-2} . It is usually expressed in N/m in SI units.
- Surface tension is a binary property of the liquid and gas or two liquids which are in contact with each other and defines the interface. It decreases slightly with increasing temperature. The surface tension of water in contact with air at 20°C is about 0.073 N/m.
- It is due to surface tension that a curved liquid interface in equilibrium results in a greater pressure at the concave side of the surface than that at its convex side.

Capillarity

- The interplay of the forces of cohesion and adhesion explains the phenomenon of capillarity. When a liquid is in contact with a solid, if the forces of adhesion between the molecules of the liquid and the solid are greater than the forces of cohesion among the liquid molecules themselves, the liquid molecules crowd towards the solid surface. The area of contact between the liquid and solid increases and the liquid thus wets the solid surface.
- The reverse phenomenon takes place when the force of cohesion is greater than the force of adhesion. These adhesion and cohesion properties result in the phenomenon of capillarity by which a liquid either rises or falls in a tube dipped into the liquid depending upon whether the force of adhesion is more than that of cohesion or not (Fig.2.4).
- The angle ϑ as shown in Fig. 2.4, is the area wetting contact angle made by the interface with the solid surface.

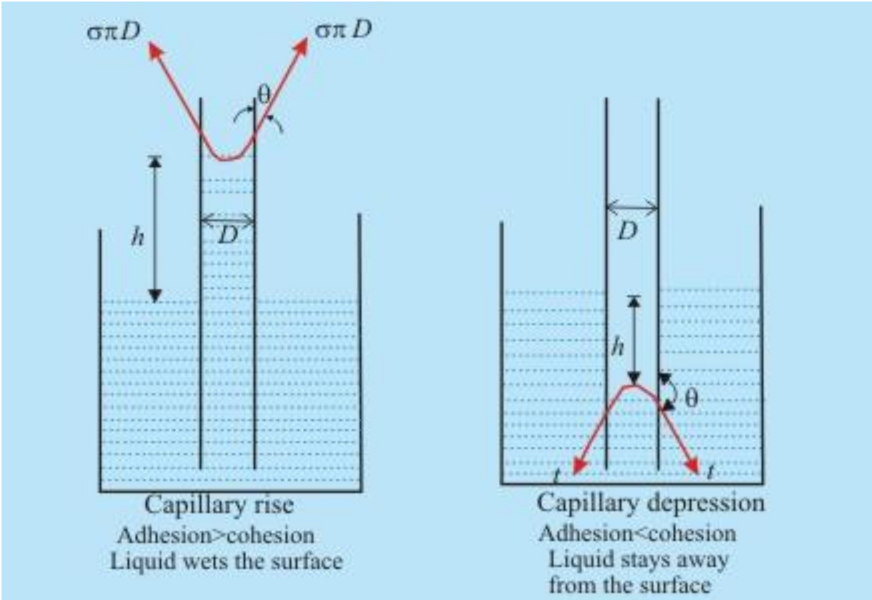


Fig 2.4 Phenomenon of Capillarity

- For pure water in contact with air in a clean glass tube, the capillary rise takes place with θ = 0 . Mercury causes capillary depression with an angle of contact of about 130^0 in a clean glass in contact with air. Since *h* varies inversely with D as found from Eq. ($h = \frac{4\sigma\cos\theta}{\rho g D}$), an appreciable capillary rise or depression is observed in tubes of small diameter only.

Vapour pressure

All liquids have a tendency to evaporate when exposed to a gaseous atmosphere. The rate of evaporation depends upon the molecular energy of the liquid which in turn depends upon the type of liquid and its temperature. The vapour molecules exert a partial pressure in the space above the liquid, known as vapour pressure. If the space above the liquid is confined (Fig. 2.5) and the liquid is maintained at constant temperature, after sufficient time, the confined space above the liquid will contain vapour molecules to the extent that some of them will be forced to enter the liquid. Eventually an equilibrium condition will evolve when the rate at which the number of vapour molecules striking back the liquid surface and condensing is just equal to the rate at which they leave from the surface. The space above the liquid then becomes saturated with vapour. The vapour pressure of a given liquid is a function of temperature only and is equal to the saturation pressure for boiling corresponding to that temperature. Hence, the vapour pressure increases with the increase in temperature. Therefore the phenomenon of boiling of a liquid is closely related to the vapour pressure. In fact, when the vapour pressure of a liquid becomes equal to the total pressure impressed on its surface, the liquid starts boiling. This concludes that boiling can be achieved either by raising the temperature of the liquid, so that its vapour pressure is elevated to the ambient pressure, or by lowering the pressure of the ambience (surrounding gas) to the liquid's vapour pressure at the existing temperature.

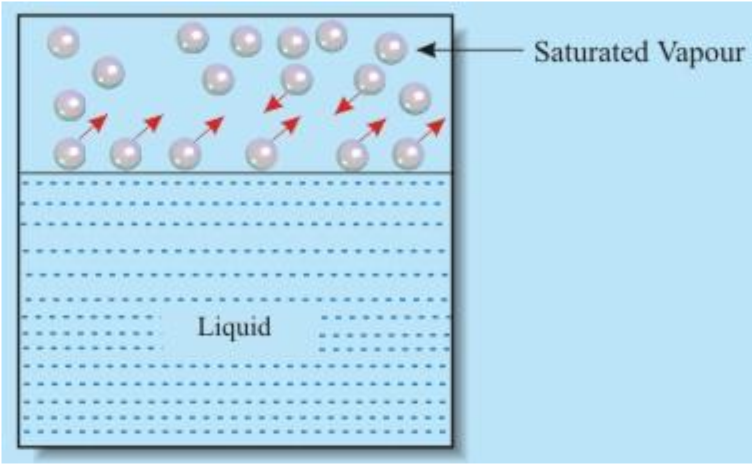


Figure 2.5 To and fro movement of liquid molecules from an interface in a confined space as a closed surrounding

Exercise Problems - Chapter 1

1. A thin film of liquid flows down an inclined channel. The velocity distribution in the flow is given by

$$u = \frac{1}{2\mu}\left(h^2 - y^2\right)\rho g \sin\alpha$$

where, h = depth of flow, α = angle of inclination of the channel to the horizontal, u = velocity at a depth h below the free surface, ρ = density of liquid, μ = dynamic viscosity of the fluid. Calculate the shear stress: (a) at the bottom of the channel (b) at mid-depth and (c) at the free surface. The coordinate y is measured from the free surface along its normal

[(a) $\rho gh \sin\alpha$, (b) $\frac{\rho gh}{2}\sin\alpha$, (c) 0]

2. Two discs of 250 mm diameter are placed 1.5 mm apart and the gap is filled with an oil. A power of 500 W is required to rotate the upper disc at 500 rpm while keeping the lower one stationary. Determine the viscosity of the oil.

[0. 71 kg/ms]

3. Eight kilometers below the surface of the ocean the pressure is 100 MPa. Determine the specific weight of sea water at this depth if the specific weight at the surface is 10 kN/m^3 and the average bulk modulus of elasticity of water is 2.30 GPa. Neglect the variation of g.

[10. 44 kN/m^3]

4. The space between two large flat and parallel walls 20 mm apart is filled with a liquid of absolute viscosity 0.8 Pas. Within this space a thin flat plate 200 mm × 200mm is towed at a velocity of 200 mm/s at a distance of 5 mm from one wall. The plate and its movement are parallel to the walls. Assuming a linear velocity distribution between the plate and the walls, determine the force exerted by the liquid on the plate.

[1. 71 N]

5. What is the approximate capillary rise of water in contact with air (surface tension 0.073 N/m) in a clean glass tube of 5mm in diameter?

[5.95]

Forces on Fluid Elements

Fluid Elements - Definition:

Fluid element can be defined as an infinitesimal region of the fluid continuum in isolation from its surroundings.

Two types of forces exist on fluid elements

- **Body Force**: distributed over the entire mass or volume of the element. It is usually expressed per unit mass of the element or medium upon which the forces act.
  Example: Gravitational Force, Electromagnetic force fields etc.

- **Surface Force**: Forces exerted on the fluid element by its surroundings through direct contact at the surface.
  Surface force has two components:

  - Normal Force: along the normal to the area

  - Shear Force: along the plane of the area.

The ratios of these forces and the elemental area in the limit of the area tending to zero are called the normal and shear stresses respectively.

The shear force is zero for any fluid element at rest and hence the only surface force on a fluid element is the normal component.

Normal Stress in a Stationary Fluid

Consider a stationary fluid element of tetrahedral shape with three of its faces coinciding with the coordinate planes x, y and z.

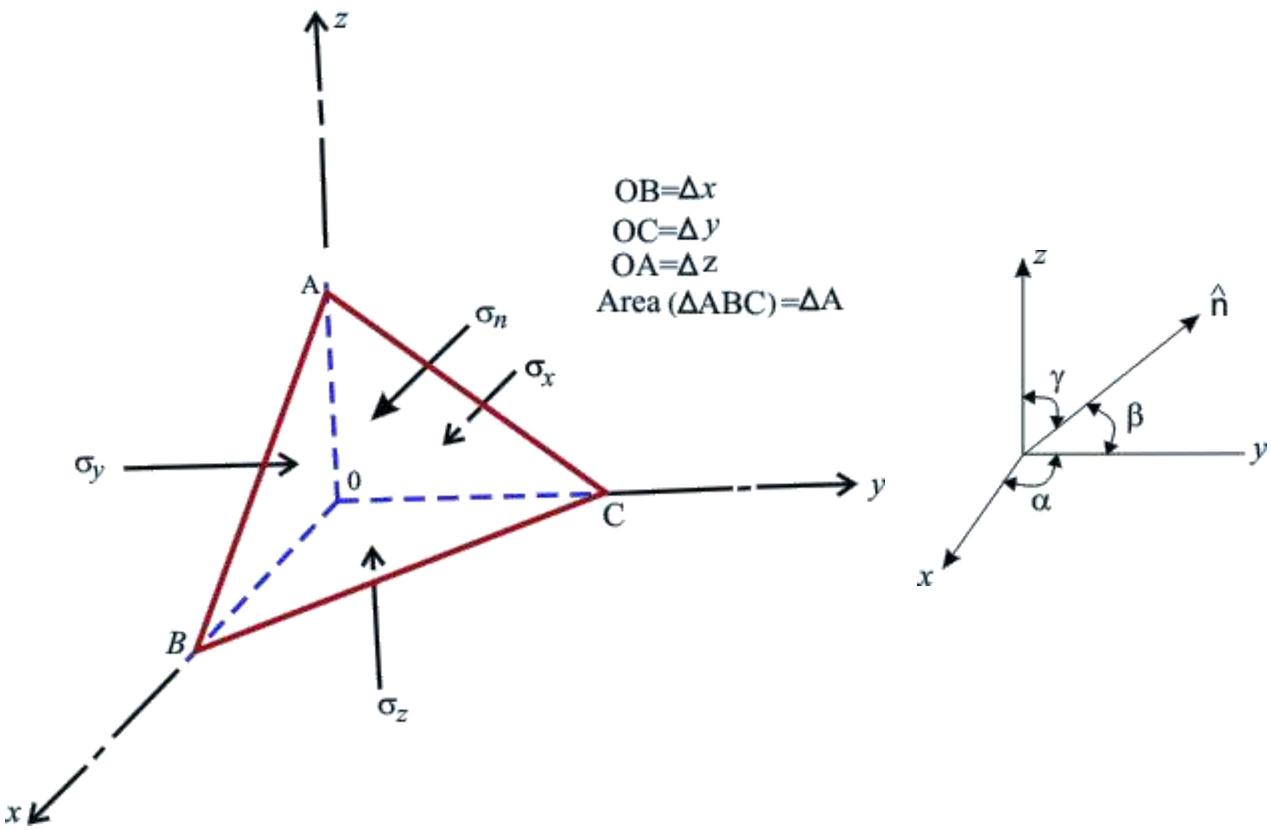


Fig 3.1 State of Stress in a Fluid Element at Rest

Since a fluid element at rest can develop neither shear stress nor tensile stress, the normal stresses acting on different faces are compressive in nature.

Suppose, ΣF_x, ΣF_y and ΣF_z are the net forces acting on the fluid element in positive x,y and z directions respectively. The direction cosines of the normal to the inclined plane of an area ΔA are cos α, cos β and cos g .Considering gravity as the only source of external body force, acting in the -ve z direction, the equations of static equlibrium for the tetrahedronal fluid element can be written as

$$\sum F_x = \sigma_x\left(\frac{\Delta y \Delta z}{2}\right) - \sigma_n \Delta A \cos\alpha = 0 \quad (3.1)$$

$$\sum F_y = \sigma_y\left(\frac{\Delta x \Delta z}{2}\right) - \sigma_n \Delta A \cos\beta = 0 \quad (3.2)$$

$$\sum F_z = \sigma_z\left(\frac{\Delta y \Delta x}{2}\right) - \sigma_n \Delta A \cos\gamma - \frac{\rho g}{6}(\Delta x \Delta y \Delta z) = 0 \quad (3.3)$$

where $\left(\frac{\Delta x\, \Delta y\, \Delta z}{6}\right)$ = Volume of tetrahedral fluid element

Pascal's Law of Hydrostatics

Pascal's Law

The normal stresses at any point in a fluid element at rest are directed towards the point from all directions and they are of the equal magnitude.

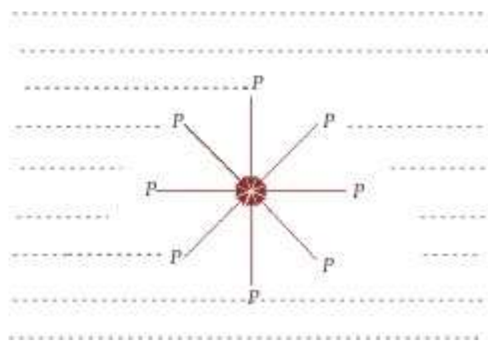


Fig 3.2 State of normal stress at a point in a fluid body at rest

Derivation:
The inclined plane area is related to the fluid elements (refer to Fig 3.1) as follows

$$\Delta A\cos\alpha = \left(\frac{\Delta y\Delta z}{2}\right) \quad (3.4)$$

$$\Delta A\cos\beta = \left(\frac{\Delta x\Delta z}{2}\right) \quad (3.5)$$

$$\Delta A\cos\gamma = \left(\frac{\Delta x\Delta y}{2}\right) \quad (3.6)$$

Substituting above values in equation 3.1- 3.3 we get

$$\sigma_x = \sigma_y = \sigma_z = \sigma_n \quad (3.7)$$

Conclusion:

The state of normal stress at any point in a fluid element at rest is same and directed towards the point from all directions. These stresses are denoted by a scalar quantity p defined as the hydrostatic or thermodynamic pressure.
Using "+" sign for the tensile stress the above equation can be written in terms of pressure as

$$\sigma_x = \sigma_y = \sigma_z = -p \qquad (3.8)$$

Fundamental Equation of Fluid Statics

The fundamental equation of fluid statics describes the spatial variation of hydrostatic pressure p in the continuous mass of a fluid.

Derivation:

Consider a fluid element at rest of given mass with volume *V* and bounded by the surface *S*.

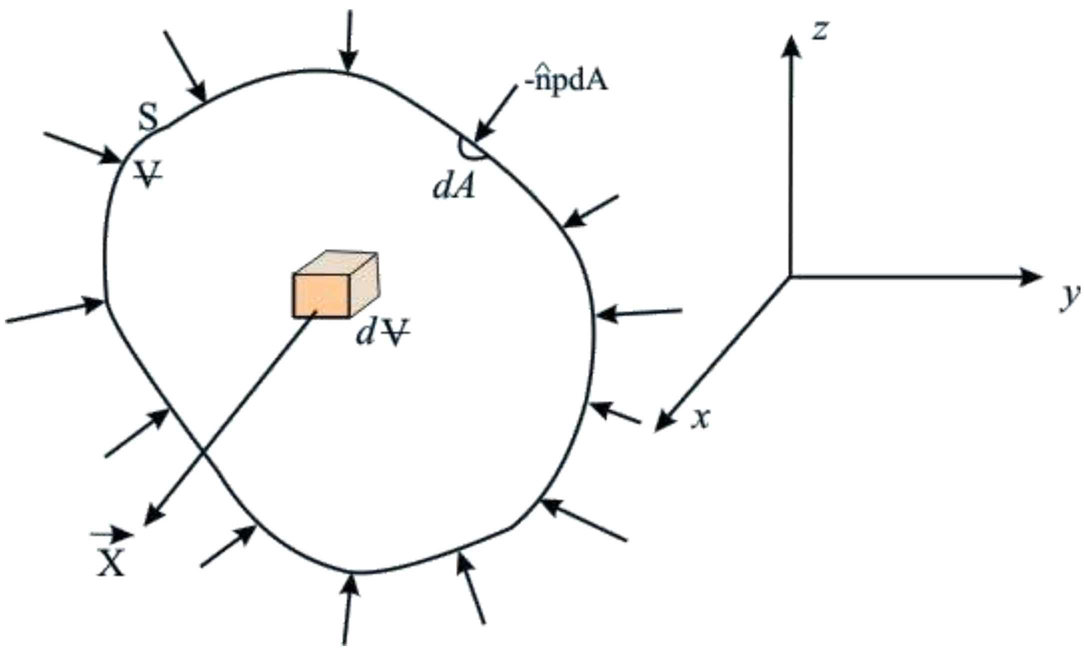


Fig 3.3 External Forces on a Fluid Element at Rest

The fluid element stays at equilibrium under the action of the following two forces

- **The Resultant Body Force**

$\vec{F}_B = \iiint\limits_{\forall} \vec{X}\,\rho\, d\forall$ (3.9)	$d\forall$: element of volume $\rho\, d\forall$: mass of the element $\vec{X}$: body Force per unit mass acting on the elementary volume

- **The Resultant Surface Force**

$\vec{F}_s = -\iint_s \hat{n} \mathrm{p} dA$ (3.10)	dA : area of an element of surface $\hat{n}$: the unit vector normal to the elemental surface,taken positive when directed outwards

Using Gauss divergence theorem, Eq (3.10) can be written as

$$\vec{F}_s = -\iint_s \hat{n} \mathrm{p} dA = -\iint_s \mathrm{p}(\hat{n}\, dA)$$

$$\vec{F}_s = -\iiint_{\forall} (\nabla \mathrm{p}) d\forall \qquad (3.11)$$

Click here to see the derivation

For the fluid element to be in equilibrium , we have

$$\vec{F}_B + \vec{F}_s = \iiint_{\forall} \left(\vec{X}\rho - \nabla \mathrm{p}\right) d\forall = 0 \qquad (3.12)$$

The equation is valid for any volume of the fluid element, no matter how small, thus we get

$$\vec{X}\rho - \nabla p = 0$$

$$\nabla p = \vec{X}\rho \qquad (3.13)$$

This is the fundamental equation of fluid statics.

Fundamental Fluid Static Equations in Scalar Form

Considering gravity as the only external body force acting on the fluid element, Eq. (3.13) can be expressed in its scalar components with respect to a cartesian coordinate system (see Fig. 3.3) as

$\frac{\partial p}{\partial x} = 0$ (in x direction) (3.13a)	X_z: the external body force per unit mass in the positive direction of z (vertically upward),

$\frac{\partial p}{\partial y} = 0$ (in y direction) (3.13b)	equals to the negative value of g (the acceleration due to gravity).
$\frac{\partial p}{\partial z} = X_z \rho = -\rho g$ (in z direction) (3.13c)	

From Eqs (3.13a)-(3.13c), it can be concluded that the pressure p is a function of z only.

Thus, Eq. (3.13c) can be re-written as,

$$\frac{\partial p}{\partial z} = \rho g \qquad (3.14)$$

Constant and Variable Density Solution

Constant Density Solution

The explicit functional relationship of hydrostatic pressure *p* with *z* can be obtained by integrating the Eq. (3.14).

For an incompressible fluid, the density r is constant throughout. Hence the Eq. (3.14) can be integrated and expressed

as

$$p = -\rho g z + C \qquad (3.15)$$

where C is the integration constant.

If we consider an expanse of fluid with a free surface, where the pressure is defined as $p = p_0$,which is equal to atmospheric pressure.

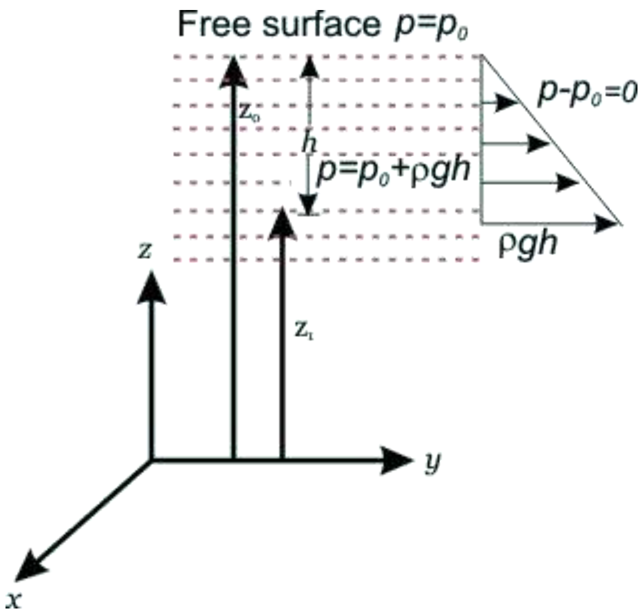


Fig 3.4 Pressure Variation in an Incompressible Fluid at rest with a Free Surface

Eq. (3.15) can be written as,

$$p - p_0 = \rho g (z_0 - z_1) = \rho g h \quad (3.16a)$$

Therefore, Eq. (3.16a) gives the expression of hydrostatic pressure p at a point whose vertical depression from the free surface is h.

Similarly,

$$p_1 - p_2 = \rho g (z_1 - z_2) = \rho g h \quad (3.16b)$$

Thus, the difference in pressure between two points in an incompressible fluid at rest can be expressed in terms of the vertical distance between the points. This result is known as **Torricelli's principle**, which is the basis for differential pressure measuring` devices. The pressure p_0 at free surface is the local atmospheric pressure.

Therefore, it can be stated from Eq. (3.16a), that the pressure at any point in an expanse of a fluid at rest, with a free surface exceeds that of the local atmosphere by an amount r*gh*, where *h* is the vertical depth of the point from the free surface.

Variable Density Solution: As a more generalised case, for compressible fluids at rest, the pressure variation at rest depends on how the fluid density changes with height *z* and pressure p. For example this can be done for special cases of "isothermal and non-isothermal fluids"

Units and scales of Pressure Measurement

Pascal (N/m^2) is the unit of pressure .

Pressure is usually expressed with reference to either absolute zero pressure (a complete vacuum)or local atmospheric pressure.

- The absolute pressure: It is the difference between the value of the pressure and the absolute zero pressure.

$$p_{abs} = p - 0 = p$$

- Gauge pressure: It is the diference between the value of the pressure and the local atmospheric pressure(p_{atm})

$$p_{gauge} = p - p_{atm}$$

- Vacuum Pressure: If $p<p_{atm}$ then the gauge pressure (p_{gauge}) becomes negative and is called the vacuum pressure.But one should always remember that hydrostatic pressure is always compressive in nature

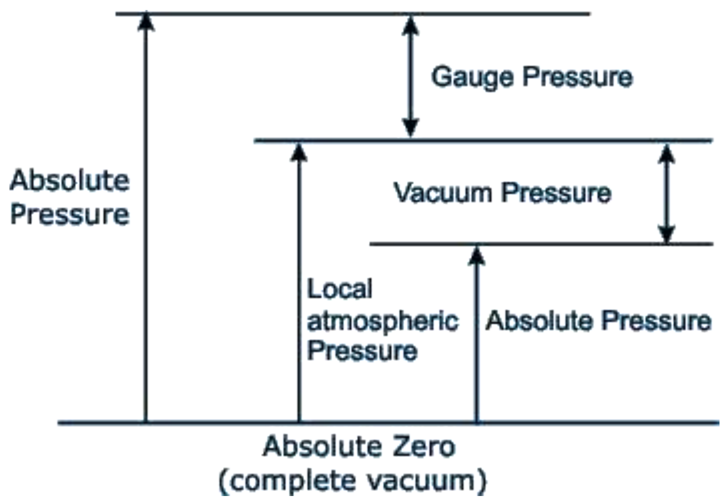


Fig 4.1 The Scale of Pressure

At sea-level, the international standard atmosphere has been chosen as P_{atm} = 101.32 kN/m^2

Piezometer Tube

The direct proportional relation between gauge pressure and the height h for a fluid of constant density enables the pressure to be simply visualized in terms of the vertical height, $h = p/\rho g$.

The height h is termed as pressure head corresponding to pressure p. For a liquid without a free surface in a closed pipe, the pressure head $p/\rho g$ at a point corresponds to the vertical height above the point to which a free surface would rise, if a small tube of sufficient length and open to atmosphere is connected to the pipe

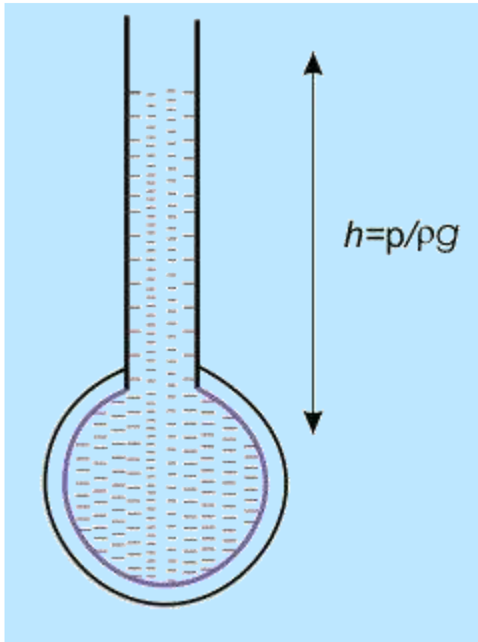
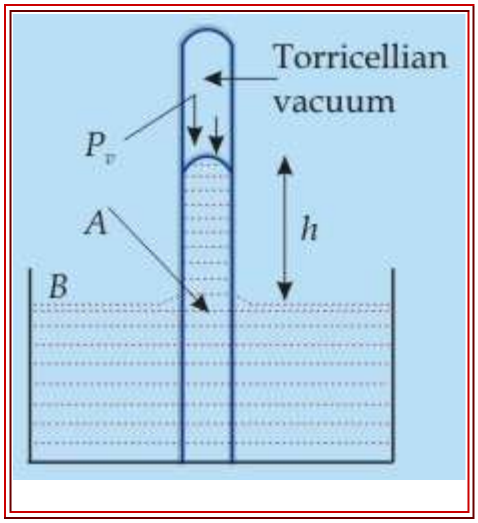


Fig 4.2 A piezometer Tube

Such a tube is called a piezometer tube, and the height h is the measure of the gauge pressure of the fluid in the pipe. If such a piezometer tube of sufficient length were closed at the top and the space above the liquid surface were a perfect vacuum, the height of the column would then correspond to the absolute pressure of the liquid at the base. This principle is used in the well known mercury barometer to determine the local atmospheric pressure.

The Barometer

Barometer is used to determine the local atmospheric pressure. Mercury is employed in the barometer because its density is sufficiently high for a relative short column to be obtained. and also because it has very small vapour pressure at normal temperature. High density scales down the pressure head(h) to repesent same magnitude of pressure in a tube of smaller height.



Click to play the Demonstration

Fig 4.3 A Simple Barometer

Even if the air is completely absent, a perfect vacuum at the top of the tube is never possible. The space would be occupied by the mercury vapour and the pressure would equal to the vapour pressure of mercury at its existing temperature. This almost vacuum condition above the mercury in the barometer is known as Torricellian vacuum.

The pressure at A equal to that at B (Fig. 4.3) which is the atmospheric pressure p_{atm} since A and B lie on the same horizontal plane. Therefore, we can write

$$p_B = p_{atm} = p_v + \rho g h \qquad (4.1)$$

The vapour pressure of mercury p_v, can normally be neglected in comparison to p_{atm}.
At 20^0C,P_v is only 0.16 p_{atm}, where p_{atm} =1.0132 X10^5 Pa at sea level. Then we get from Eq. (4.1)

$$h = p_{atm} / \rho g = \frac{1.0132 \times 10^5 \ N/m^2}{(13560\, kg/m^3)(9.81 N/Kg)} = 0.752 m \ of \ Hg$$

For accuracy, small corrections are necessary to allow for the variation of r with temperature, the thermal expansion of the scale (usually made of brass). and surface tension effects. If water was used instead of mercury, the corresponding height of the column would be about 10.4 m provided that a perfect vacuum could be achieved above the water. However, the vapour pressure of water at ordinary temperature is appreciable and so the actual height at, say, 15°C would be about 180 mm less than this value. Moreover. with a tube smaller in diameter than about 15 mm, surface tension effects become significant.

Manometers for measuring Gauge and Vacuum Pressure

Manometers are devices in which columns of a suitable liquid are used to measure the difference in pressure between two points or between a certain point and the atmosphere.

Manometer is needed for measuring large gauge pressures. It is basically the modified form of the piezometric tube. A common type manometer is like a transparent "U-tube" as shown in Fig. 4.4.

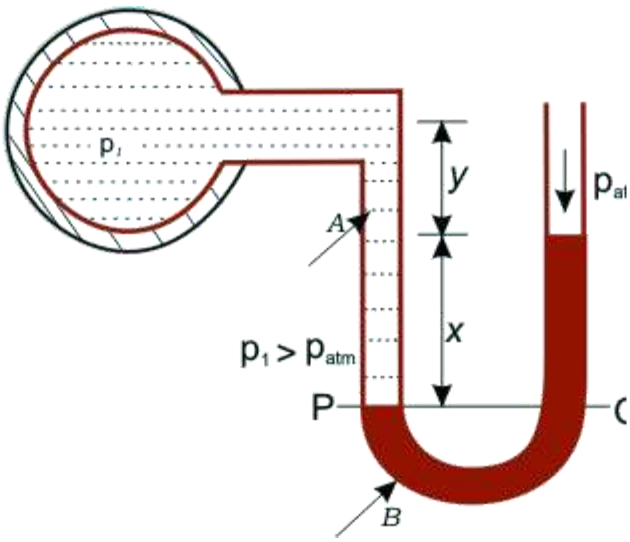


Fig 4.4 A simple manometer to measure gauge pressure

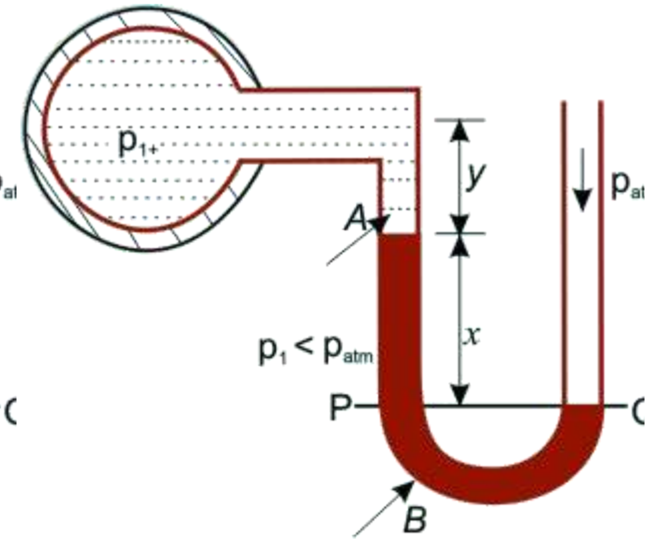


Fig 4.5 A simple manometer to measure vacuum pressure

One of the ends is connected to a pipe or a container having a fluid (A) whose pressure is to be measured while the other end is open to atmosphere. The lower part of the U-tube contains a liquid immiscible with the fluid A and is of greater density than that of A. This fluid is called the manometric fluid.

The pressures at two points *P* and *Q* (Fig. 4.4) in a horizontal plane within the continuous expanse of same fluid (the liquid B in this case) must be equal. Then equating the pressures at *P* and *Q* in terms of the heights of the fluids above those points, with the aid of the fundamental equation of hydrostatics (Eq 3.16), we have

$$p_1 + \rho_A g(y + x) = p_{atm} + \rho_B g x$$

Hence,

$$p_1 - p_{atm} = (\rho_B - \rho_A) g x - \rho_A g y$$

where p_1 is the absolute pressure of the fluid A in the pipe or container at its centre line, and p_{atm} is the local atmospheric pressure. When the pressure of the fluid in the container is lower than the atmospheric pressure, the liquid levels in the manometer would be adjusted as shown in Fig. 4.5. Hence it becomes,

$$p_1 + \rho_A g y + \rho_B g x = p_{atm}$$

$$p_{atm} - p_1 = (\rho_A y + \rho_B x)g \qquad (4.2)$$

Manometers to measure Pressure Difference

A manometer is also frequently used to measure the pressure difference, in course of flow, across a restriction in a horizontal pipe.

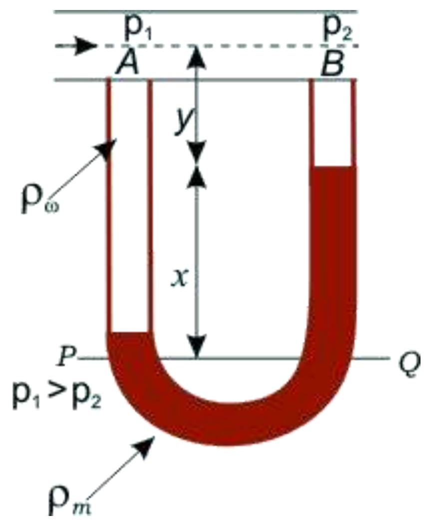


Fig 4.6 Manometer measuring pressure difference

The axis of each connecting tube at A and B should be perpendicular to the direction of flow and also for the edges of the connections to be smooth. Applying the principle of hydrostatics at P and Q we have,

$$p_1 + (y + x)\rho_w g = p_2 + y\rho_w g + \rho_m g x$$

$$p_1 - p_2 = (\rho_m - \rho_w)gx \qquad (4.3)$$

where, ρ_m is the density of manometric fluid and ρ_w is the density of the working fluid flowing through the pipe.

We can express the difference of pressure in terms of the difference of heads (height of the working fluid at equilibrium).

$$h_1 - h_2 = \frac{p_1 - p_2}{\rho_w g} = \left(\frac{\rho_m}{\rho_w} - 1\right)x \qquad (4.4)$$

Inclined Tube Manometer

- For accurate measurement of small pressure differences by an ordinary u-tube manometer, it is essential that the ratio r_m/r_w should be close to unity. This is not possible if the working fluid is a gas; also having a manometric liquid of density very close to that of the working liquid and giving at the same time a well defined meniscus at the interface is not always possible. For this purpose, an inclined tube manometer is used.
- If the transparent tube of a manometer, instead of being vertical, is set at an angle θ to the horizontal (Fig. 4.7), then a pressure difference corresponding to a vertical difference of levels x gives a movement of the meniscus s = x/sinq along the slope.

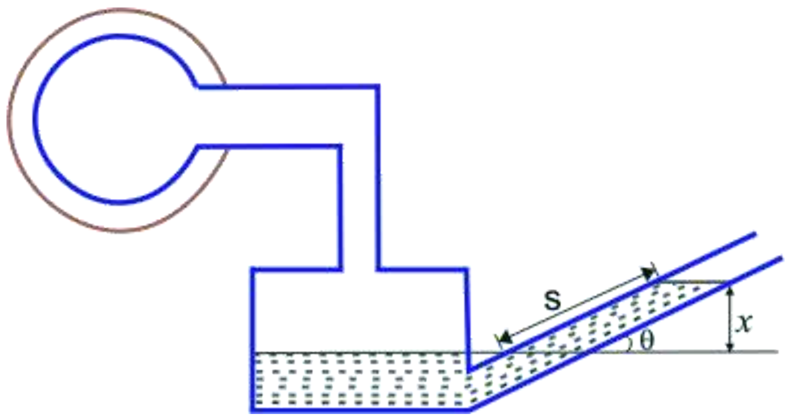


Fig 4.7 An Inclined Tube Manometer

- If θ is small, a considerable mangnification of the movement of the meniscus may be achieved.
- Angles less than 5^0 are not usually satisfactory, because it becomes difficult to determine the exact position of the meniscus.
- One limb is usually made very much greater in cross-section than the other. When a pressure difference is applied across the manometer, the movement of the liquid surface in the wider limb is practically negligible compared to that occurring in the narrower limb. If the level of the surface in the wider limb is assumed constant, the displacement of the meniscus in the narrower limb needs only to be measured, and therefore only this limb is required to be transparent.

Inverted Tube Manometer

For the measurement of small pressure differences in liquids, an inverted U-tube manometer is used.

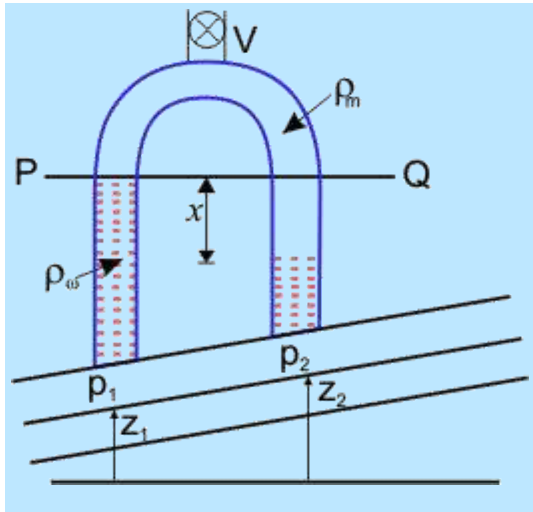


Fig 4.8 An Inverted Tube Manometer

Here $\rho_m < \rho_w$ and the line PQ is taken at the level of the higher meniscus to equate the pressures at P and Q from the principle of hydrostatics. It may be written that

$$p_1^* - p_2^* = (\rho_w - \rho_m)gx$$

where p^* represents the **piezometric pressure**, $p + \rho gz$ (z being the vertical height of the point concerned from any reference datum). In case of a horizontal pipe ($z_1 = z_2$) the difference in piezometric pressure becomes equal to the difference in the static pressure. If $(\rho_w - \rho_m)$ is sufficiently small, a large value of x may be obtained for a small value of $p_1^* - p_2^*$. Air is used as the manometric fluid. Therefore, ρ_m is negligible compared with ρ_w and hence,

$$p_1^* - p_2^* \approx \rho_w gx \qquad (4.5)$$

Air may be pumped through a valve V at the top of the manometer until the liquid menisci are at a suitable level.

Micromanometer

When an additional gauge liquid is used in a U-tube manometer, a large difference in meniscus levels may be obtained for a very small pressure difference.

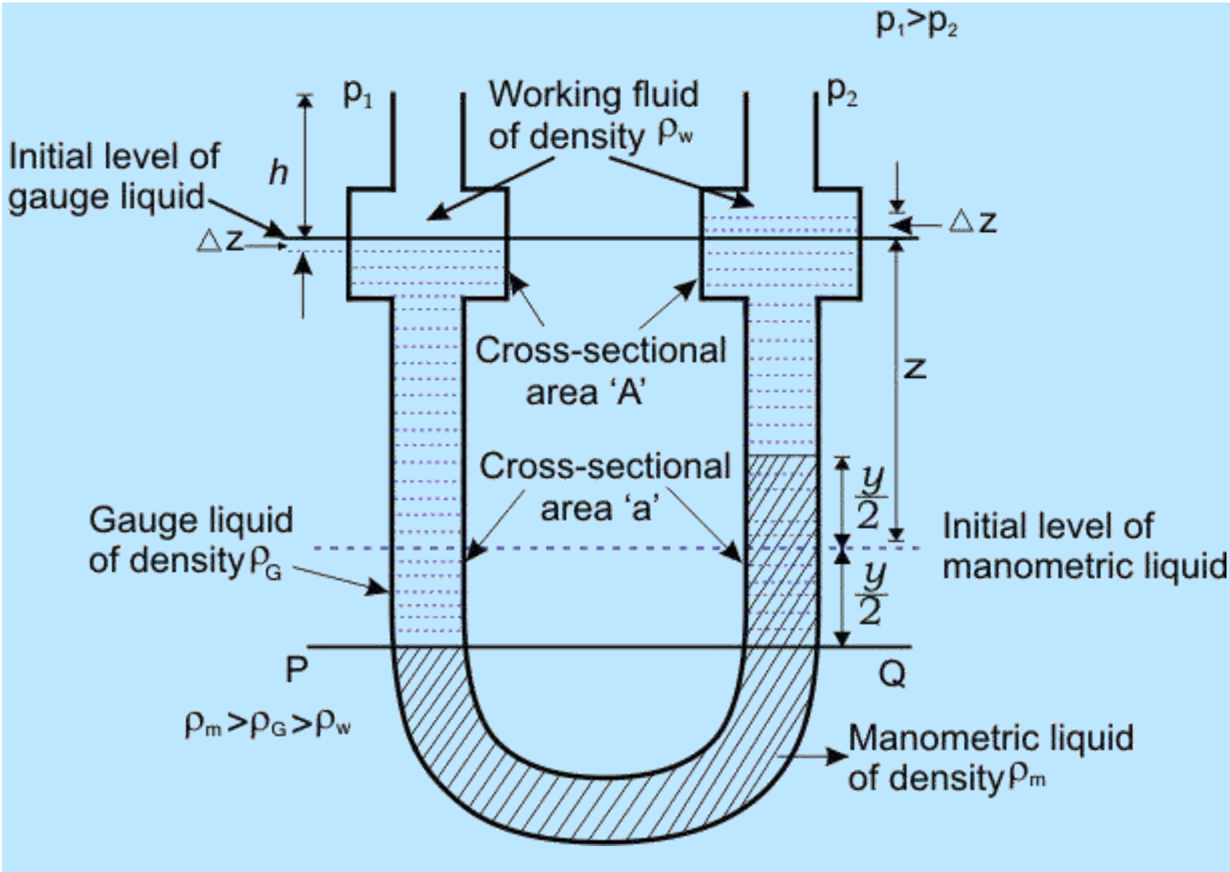


Fig 4.9 A Micromanometer

The equation of hydrostatic equilibrium at PQ can be written as

$$p_1 + \rho_w g(h+\Delta z) + \rho_G g\left(z - \Delta z + \frac{y}{2}\right) = p_2 + \rho_w g(h - \Delta z) + \rho_G g\left(z + \Delta z - \frac{y}{2}\right) + \rho_m g y$$

where ρ_w, ρ_G and ρ_m are the densities of working fluid, gauge liquid and manometric liquid respectively.

From continuity of gauge liquid,

$$A\Delta z = \alpha \frac{y}{2} \qquad (4.6)$$

$$p_1 - p_2 = gy\left[\rho_m - \rho_G\left(1 - \frac{\alpha}{A}\right) - \rho_w \frac{\alpha}{A}\right] \qquad (4.7)$$

If a is very small compared to A

$$p_1 - p_2 \approx (\rho_m - \rho_G) g y \qquad (4.8)$$

With a suitable choice for the manometric and gauge liquids so that their densities are close $(\rho_m \approx \rho_G)$ a reasonable value of y may be achieved for a small pressure difference.

Hydrostatic Thrusts on Submerged Plane Surface

Due to the existence of hydrostatic pressure in a fluid mass, a normal force is exerted on any part of a solid surface which is in contact with a fluid. The individual forces distributed over an area give rise to a resultant force.

Plane Surfaces

Consider a plane surface of arbitrary shape wholly submerged in a liquid so that the plane of the surface makes an angle θ with the free surface of the liquid. We will assume the case where the surface shown in the figure below is subjected to hydrostatic pressure on one side and atmospheric pressure on the other side.

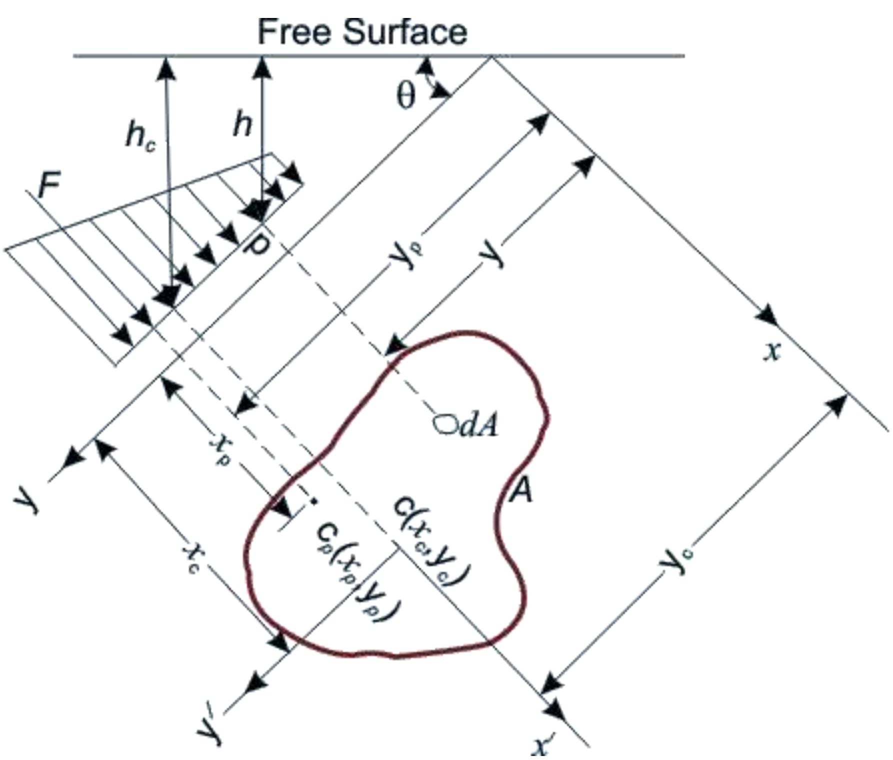


Fig 5.1 Hydrostatic Thrust on Submerged Inclined Plane Surface

Let p denotes the gauge pressure on an elemental area dA. The resultant force F on the area A is therefore

$$F = \iint_A p\, dA \tag{5.1}$$

According to Eq (3.16a) Eq (5.1) reduces to

$$F = \rho g \int\int h dA = \rho g \sin\theta \int\int y dA \qquad (5.2)$$

Where **h is the vertical depth of the elemental area dA from the free surface** and the distance y is measured from the x-axis, the line of intersection between the extension of the inclined plane and the free surface (Fig. 5.1). The ordinate of the centre of area of the plane surface A is defined as

$$y_c = \frac{1}{A}\int\int y dA \qquad (5.3)$$

Hence from Eqs (5.2) and (5.3), we get

$$F = \rho g y_c \sin\theta A = \rho g h_c A \qquad (5.4)$$

where $h_c (= y_c \sin\theta)$ is the vertical depth (from free surface) of centre c of area .

Equation (5.4) implies that the hydrostatic thrust on an inclined plane is equal to the pressure at its centroid times the total area of the surface, i.e., the force that would have been experienced by the surface if placed horizontally at a depth h_c from the free surface (Fig. 5.2).

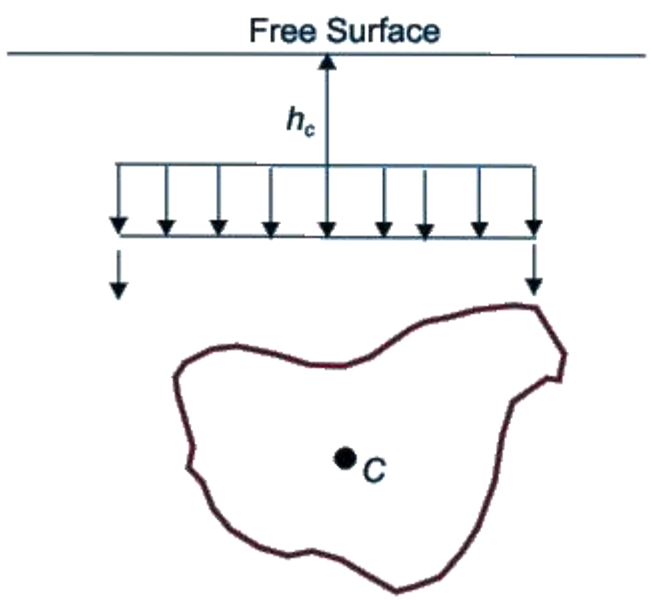


Fig 5.2 Hydrostatic Thrust on Submerged Horizontal Plane Surface

The point of action of the resultant force on the plane surface is called the centre of pressure c_p. Let x_p and y_p be the distances of the centre of pressure from the y and x axes respectively. Equating the moment of the resultant force about the x axis to the summation of the moments of the component forces, we have

$$y_p F = \int y dF = \rho g \sin\theta \iint y^2 dA \tag{5.5}$$

Solving for y_p from Eq. (5.5) and replacing F from Eq. (5.2), we can write

$$y_p = \frac{\iint_A y^2 dA}{\iint_A y dA} \tag{5.6}$$

In the same manner, the x coordinate of the centre of pressure can be obtained by taking moment about the y-axis. Therefore,

$$x_p F = \int x dF = \rho g \sin\theta \iint xy dA$$

From which,

$$x_p = \frac{\iint_A xy dA}{\iint_A y\, dA} \tag{5.7}$$

The two double integrals in the numerators of Eqs (5.6) and (5.7) are the moment of inertia about the x-axis $I_{xx}$and the product of inertia I_{xy} about x and y axis of the plane area respectively. By applying the theorem of parallel axis

$$I_{xx} = \iint y^2\, dA = I_{x'x'} + Ay_c^2 \tag{5.8}$$

$$I_{xy} = \iint xy dA = I_{x'y'} + Ax_c y_c \tag{5.9}$$

where, $I_{x'x'}$ and $I_{x'y'}$ are the moment of inertia and the product of inertia of the surface about the centroidal axes $(x' - y')$, x_c , and y_c are the coordinates of the center c of the area with respect to x-y axes.

With the help of Eqs (5.8), (5.9) and (5.3), Eqs (5.6) and (5.7) can be written as

$$y_p = \frac{I_{x'x'}}{Ay_c} + y_c \qquad (5.10a)$$

$$x_p = \frac{I_{x'y'}}{Ay_c} + x_c \qquad (5.10b)$$

The first term on the right hand side of the Eq. (5.10a) is always positive. Hence, the centre of pressure is always at a higher depth from the free surface than that at which the centre of area lies. This is obvious because of the typical variation of hydrostatic pressure with the depth from the free surface. When the plane area is symmetrical about the y' axis, $I_{x'y'} = 0$, and $x_p = x_c$.

Hydrostatic Thrusts on Submerged Curved Surfaces

On a curved surface, the direction of the normal changes from point to point, and hence the pressure forces on individual elemental surfaces differ in their directions. Therefore, a scalar summation of them cannot be made. Instead, the resultant thrusts in certain directions are to be determined and these forces may then be combined vectorially. An arbitrary submerged curved surface is shown in Fig. 5.3. A rectangular Cartesian coordinate system is introduced whose xy plane coincides with the free surface of the liquid and z-axis is directed downward below the x - y plane.

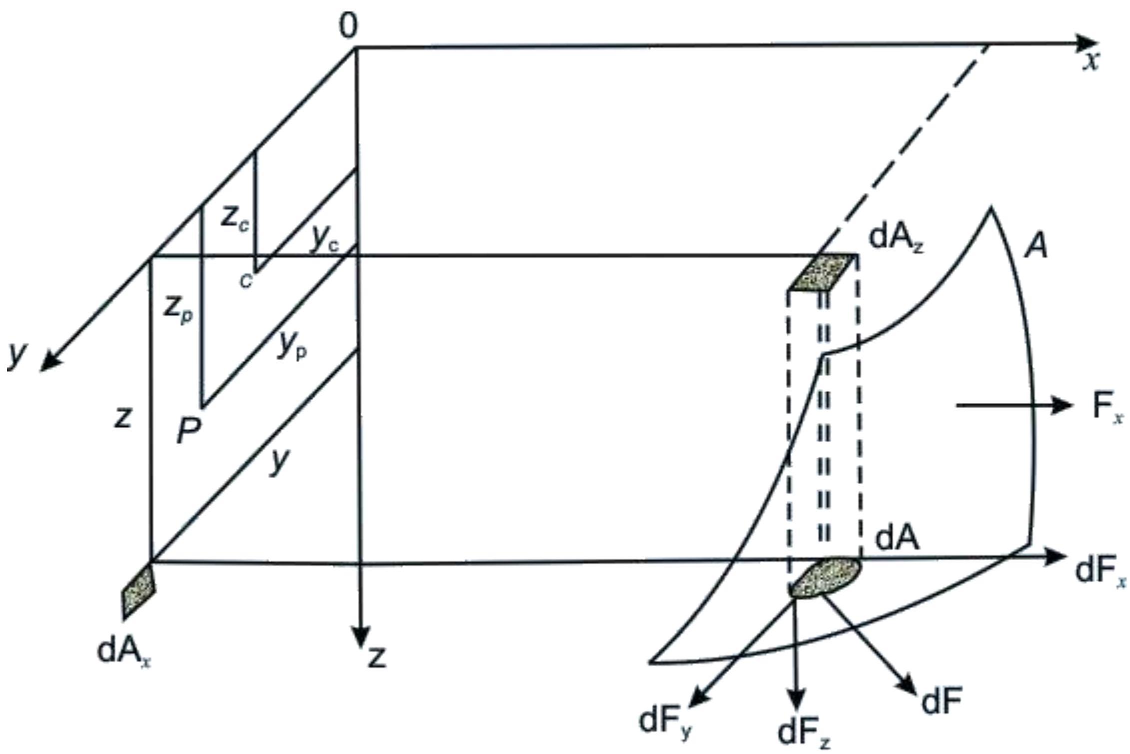


Fig 5.3 Hydrostatic thrust on a Submerged Curved Surface

Consider an elemental area dA at a depth z from the surface of the liquid. The hydrostatic force on the elemental area dA is

$$d\,F = \rho\, g\, z\, dA \qquad (5.11)$$

and the force acts in a direction normal to the area dA. The components of the force dF in x, y and z directions are

$$dF_x = ldF = l\rho gzdA \qquad (5.12a)$$

$$dF_y = mdF = m\rho gzdA \qquad (5.12b)$$

$$dF_z = ndF = n\rho gzdA \qquad (5.13c)$$

Where l, m and n are the direction cosines of the normal to dA. The components of the surface element dA projected on yz, xz and xy planes are, respectively

$$dA_x = l\,dA \qquad (5.13a)$$

$$dA_y = m dA \qquad (5.13b)$$

$$dA_z = n\,dA \qquad (5.13c)$$

Substituting Eqs (5.13a-5.13c) into (5.12) we can write

$$dF_x = \rho g\, z\, dA_x \qquad (5.14a)$$

$$dF_y = \rho g\, z\, dA_y \qquad (5.14b)$$

$$dF_z = \rho g\, z\, dA_z \qquad (5.14c)$$

Therefore, the components of the total hydrostatic force along the coordinate axes are

$$F_x = \int\int \rho gzdA_x = \rho g z_c A_x \qquad (5.15a)$$

$$F_y = \int\int \rho gzdA_y = \rho g z_c A_y \qquad (5.15b)$$

$$F_z = \int\int \rho gzdA_z \qquad (5.15c)$$

where z_c is the z coordinate of the centroid of area A_x and A_y (the projected areas of curved surface on yz and xz plane respectively). If z_p and y_p are taken to be the coordinates of the point of action of F_x on the projected area A_x on yz plane, , we can write

$$z_p = \frac{1}{A_x z_c} \iint z^2 dA_x = \frac{I_{yy}}{A_x z_c} \quad (5.16a)$$

$$y_p = \frac{1}{A_x z_c} \iint yz dA_x = \frac{I_{yz}}{A_x z_c} \quad (5.16b)$$

where I_{yy} is the moment of inertia of area A_x about y-axis and I_{yz} is the product of inertia of A_x with respect to axes y and z. In the similar fashion, z_p' and x_p' the coordinates of the point of action of the force F_y on area A_y, can be written as

$$z'_p = \frac{1}{A_y z_c} \iint z^2 dA_y = \frac{I_{xx}}{A_y z_c} \quad (5.17a)$$

$$x'_p = \frac{1}{A_y z_c} \iint xz dA_y = \frac{I_{xz}}{A_y z_c} \quad (5.17b)$$

where I_{xx} is the moment of inertia of area A_y about x axis and I_{xz} is the product of inertia of A_y about the axes x and z.

We can conclude from Eqs (5.15), (5.16) and (5.17) that for a curved surface, the component of hydrostatic force in a horizontal direction is equal to the hydrostatic force on the projected plane surface perpendicular to that direction and acts through the centre of pressure of the projected area. From Eq. (5.15c), the vertical component of the hydrostatic force on the curved surface can be written as

$$F_z = \rho g \iint z \, dA_z = \rho g \forall \quad (5.18)$$

where $\forall$ is the volume of the body of liquid within the region extending vertically above the submerged surface to the free surfgace of the liquid. Therefore, the vertical component of hydrostatic force on a submerged curved surface is equal to the weight of the liquid volume vertically above the solid surface of the liquid and acts through the center of gravity of the liquid in that volume.

Buoyancy

- When a body is either wholly or partially immersed in a fluid, a lift is generated due to the net vertical component of hydrostatic pressure forces experienced by the body.
- This lift is called the buoyant force and the phenomenon is called buoyancy
- Consider a solid body of arbitrary shape completely submerged in a homogeneous liquid as shown in Fig. 5.4. **Hydrostatic pressure forces act on the entire surface of the body.**

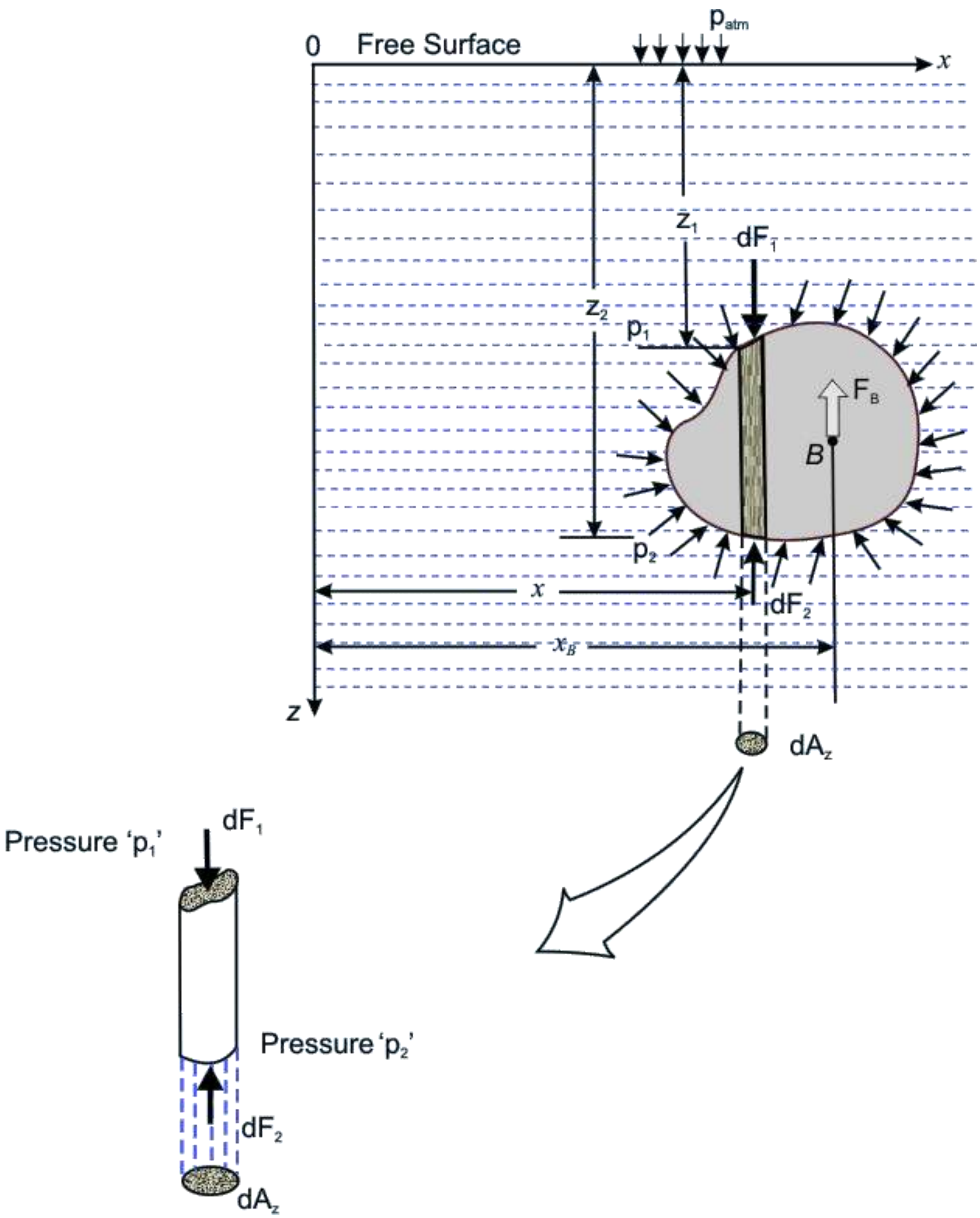


Fig 5.4 Buoyant Force on a Submerged Body

To calculate the vertical component of the resultant hydrostatic force, the body is considered to be divided into a number of elementary vertical prisms. The vertical forces acting on the two ends of such a prism of cross-section dA_z (Fig. 5.4) are respectively

$$dF_1 = (p_{atm} + p_1)dA_z = (p_{atm} + \rho g z_1)dA_z \quad (5.19a)$$

$$dF_2 = (p_{atm} + p_2)dA_z = (p_{atm} + \rho g z_2)dA_z \quad (5.19b)$$

Therefore, the buoyant force (the net vertically upward force) acting on the elemental prism of volume $d\forall$ is -

$$dF_B = dF_2 - dF_1 = \rho g(z_2 - z_1)dA_z = \rho g d\forall \quad (5.19c)$$

Hence the buoyant force F_B on the entire submerged body is obtained as

$$F_B = \iiint_{\forall} \rho g d\forall = \rho g \forall \quad (5.20)$$

Where $\forall$ **is the total volume of the submerged body.** The line of action of the force F_B can be found by taking moment of the force with respect to z-axis. Thus

$$x_B F_B = \int x dF_B \quad (5.21)$$

Substituting for dF_B and F_B from Eqs (5.19c) and (5.20) respectively into Eq. (5.21), the **x coordinate of the center of the buoyancy is** obtained as

$$x_B = \frac{1}{\forall}\iiint_{\forall} x d\forall \quad (5.22)$$

which is **the centroid of the displaced volume**. It is found from Eq. (5.20) that the buoyant force F_B equals to the weight of liquid displaced by the submerged body of volume $\forall$. This phenomenon was discovered by Archimedes and is known as the Archimedes principle.

ARCHIMEDES PRINCIPLE

The buoyant force on a submerged body

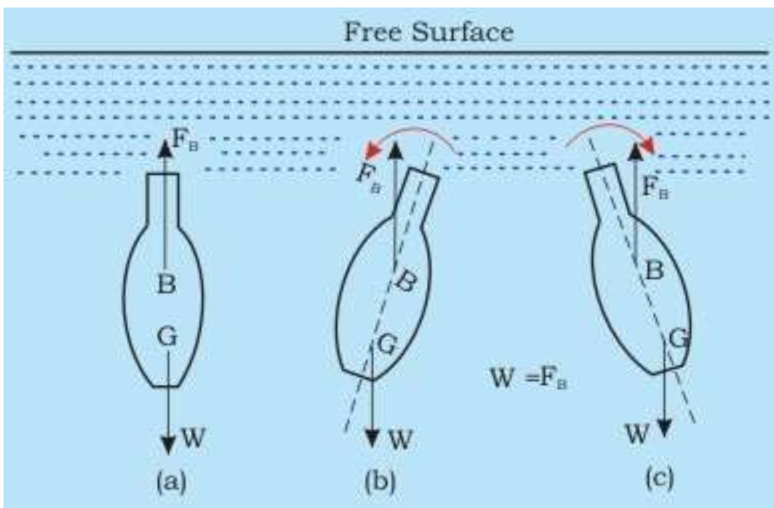
- The Archimedes principle states that the buoyant force on a submerged body is equal to the weight of liquid displaced by the body, and acts vertically upward through the centroid of the displaced volume.
- Thus the net weight of the submerged body, (the net vertical downward force experienced by it) is reduced from its actual weight by an amount that equals the buoyant force.

The buoyant force on a partially immersed body

- According to Archimedes principle, the buoyant force of a partially immersed body is equal to the weight of the displaced liquid.
- Therefore the buoyant force depends upon the density of the fluid and the submerged volume of the body.
- For a floating body in static equilibrium and in the absence of any other external force, the buoyant force must balance the weight of the body.

Stable Equilibrium

Consider a submerged body in equilibrium whose centre of gravity is located below the centre of buoyancy (Fig. 5.5a). If the body is tilted slightly in any direction, the buoyant force and the weight always produce a restoring couple trying to return the body to its original position (Fig. 5.5b, 5.5c).

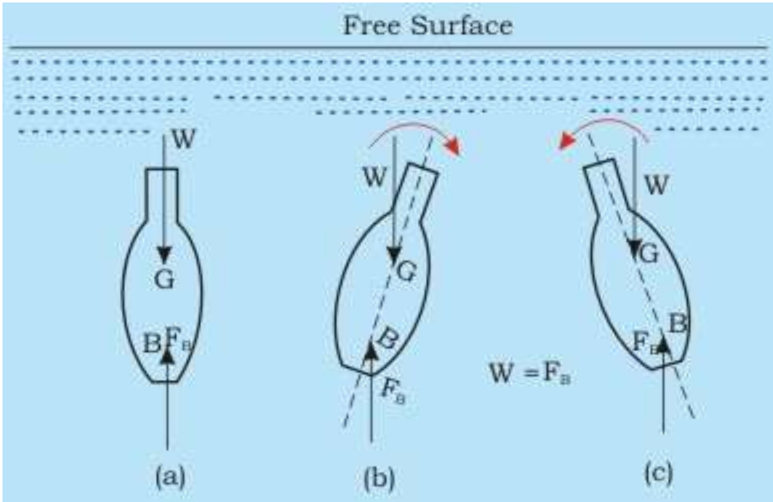


Click to play the Demonstration

Fig 5.5 A Submerged body in Stable Equilibrium

Unstable Equilibrium

On the other hand, if point G is above point B (Fig. 5.6a), any disturbance from the equilibrium position will create a destroying couple which will turn the body away from its original position (5.6b, 5.6c).

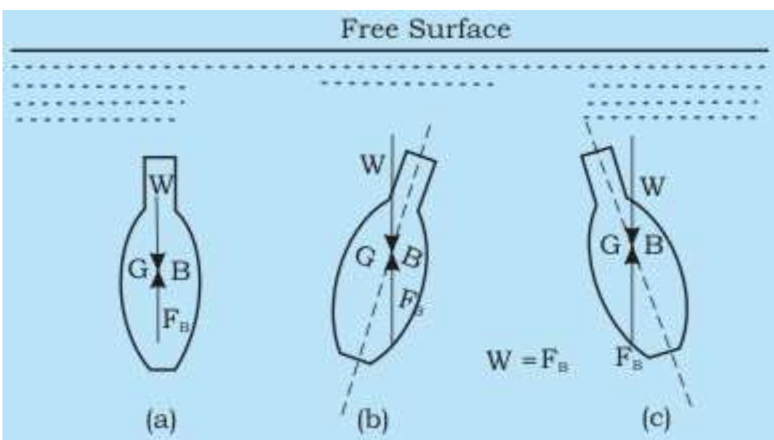


Click to play the Demonstration

Fig 5.6 A Submerged body in Unstable Equilibrium

Neutral Equilibrium

When the centre of gravity G and centre of buoyancy B coincides, the body will always assume the same position in which it is placed (Fig 5.7) and hence it is in neutral equilibrium.



Click to play the Demonstration

Fig 5.7 A Submerged body in Neutral Equilibrium

Therefore, it can be concluded that **a submerged body will be in stable, unstable or neutral equilibrium if its centre of gravity is below, above or coincident with the center of buoyancy respectively (Fig. 5.8).**

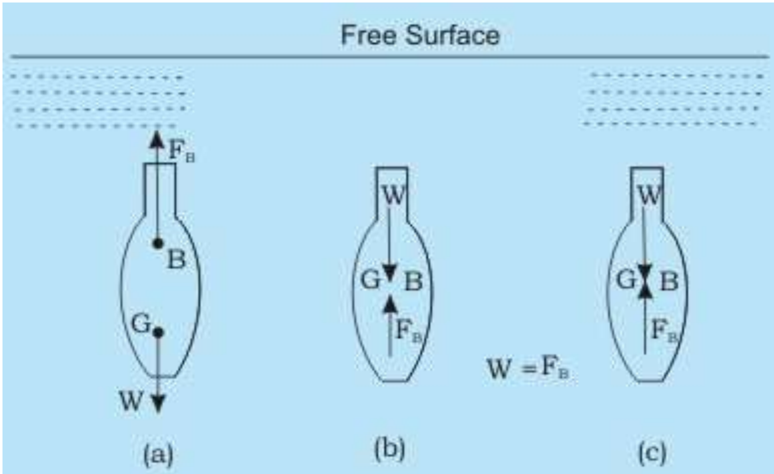


Fig 5.8   States of Equilibrium of a Submerged Body

(a) STABLE EQUILIBRIUM   (B) UNSTABLE EQUILIBRIUM   (C) NEUTRAL EQUILIBRIUM

Stability of Floating Bodies in Fluid

- When the body undergoes an angular displacement about a horizontal axis, the shape of the immersed volume changes and so the centre of buoyancy moves relative to the body.
- As a result of above observation stable equlibrium can be achieved, under certain condition, even when G is above B.
  Figure 5.9a illustrates a floating body -a boat, for example, in its equilibrium position.

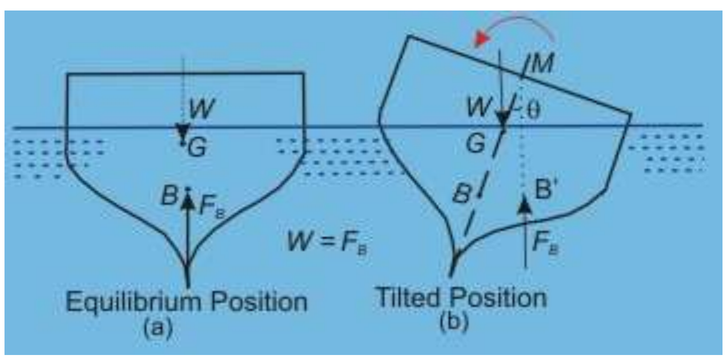


Fig 5.9   A Floating body in Stable equilibrium

Important points to note here are

a. The force of buoyancy F_B is equal to the weight of the body W

b. Centre of gravity G is above the centre of buoyancy in the same vertical line.

c. Figure 5.9b shows the situation after the body has undergone a small angular displacement q with respect to the vertical axis.

d. The centre of gravity G remains unchanged relative to the body (This is not always true for ships where some of the cargo may shift during an angular displacement).

e. During the movement, the volume immersed on the right hand side increases while that on the left hand side decreases. Therefore the centre of buoyancy moves towards the right to its new position B'.

Let the new **line of action of the buoyant force** (which **is always vertical**) through B' intersects the axis BG (the old vertical line containing the centre of gravity G and the old centre of buoyancy B) at M. For small values of q the **point M is** practically constant in position and is **known as metacentre**. For the body shown in Fig. 5.9, M is above G, and the couple acting on the body in its displaced position is a restoring couple which tends to turn the body to its original position. If M were below G, the couple would be an overturning couple and the original equilibrium would have been unstable. When M coincides with G, the body will assume its new position without any further movement and thus will be in neutral equilibrium. T**herefore, for a floating body, the stability is determined not simply by the relative position of B and G, rather by the relative position of M and G**. The distance of metacentre above G along the line BG is known as metacentric height GM which can be written as

GM = BM -BG

Hence the **condition of stable equilibrium for a floating body** can be expressed **in terms of metacentric height** as follows:

GM > 0 (M is above G)	**Stable equilibrium**
GM = 0 (M coinciding with G)	**Neutral equilibrium**
GM < 0 (M is below G)	**Unstable equilibrium**

The angular displacement of a boat or ship about its longitudinal axis is known as 'rolling' while that about its transverse axis is known as "pitching".

Floating Bodies Containing Liquid

If a floating body carrying liquid with a free surface undergoes an angular displacement, the liquid will also move to keep its free surface horizontal. Thus not only does the centre of buoyancy B move, but also the centre of gravity G of the floating body and its contents move in the same direction as the movement of B. Hence the stability of the body is reduced. For this reason, liquid which has to be carried in a ship is put into a number of separate compartments so as to minimize its movement within the ship.

Period of Oscillation

The restoring couple caused by the buoyant force and gravity force acting on a floating body displaced from its equilibrium placed from its equilibrium position is $W.GM \sin\theta$ (Fig. 5.9). Since the torque equals to mass moment of inertia (i.e., second moment of mass) multiplied by angular acceleration, it can be written

$$W(GM)\sin\theta = -I_M \frac{d^2\theta}{dr^2} \qquad (5.23)$$

Where I_M represents the mass moment of inertia of the body about its axis of rotation. The minus sign in the RHS of Eq. (5.23) arises since the torque is a retarding one and decreases the angular acceleration. If θ is small, sin θ=θ and hence Eq. (5.23) can be written as

$$\frac{d^2\theta}{dt^2} + \frac{W.GM}{I_M}\theta = 0 \qquad (5.24)$$

Equation (5.24) represents a simple harmonic motion. The time period (i.e., the time of a complete oscillation from one side to the other and back again) equals to $2\pi\left(I_M/W.GM\right)^{1/2}$. The oscillation of the body results in a flow of the liquid around it and this flow has been disregarded here. In practice, of course, viscosity in the liquid introduces a damping action which quickly suppresses the oscillation unless further disturbances such as waves cause new angular displacements.

Exercise Problems - Chapter 2

1. For the system shown in Fig 5.10,determine the air pressure p_A which will make the pressure at N one

[3.33 kPa]

fourth of that at M.

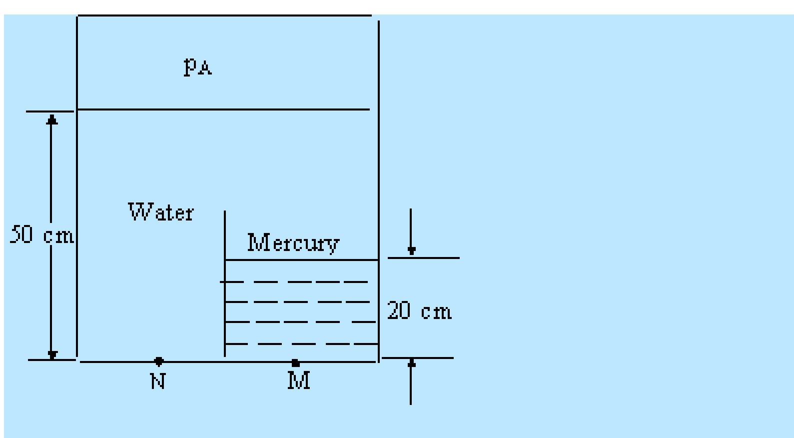


Fig 5.10

2. Consider the pipe and manometer system as shown in Fig 5.11. The pipe contains water. Find the value of manometer reading h, and the difference in pressure between A and B if there is no flow. If there is a flow from A towards B and the manometer reading is h = 60 mm, then determine the static pressure difference p_A - p_B

[0, 2.94 kPa; 3.53 kPa]

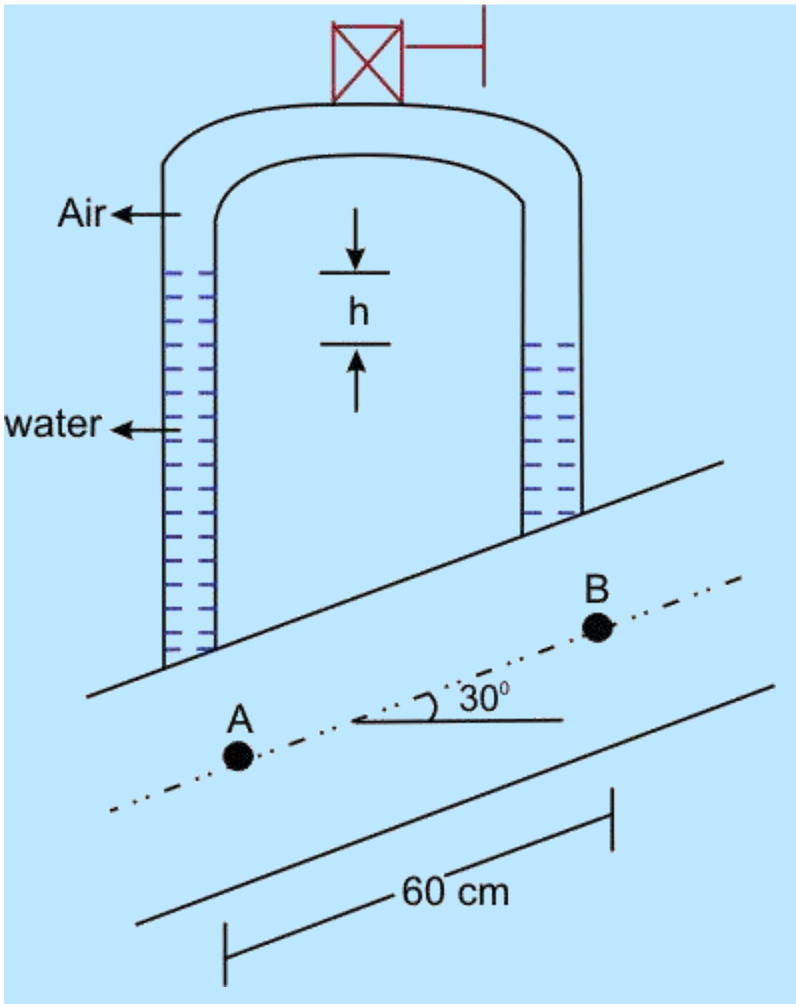


Fig 5.11

3. Determine the air pressure above the water surface in the tank if a force of 8 kN is required to hold the hinged door in position as
shown in Fig 5.12.

[10.76 kPa]

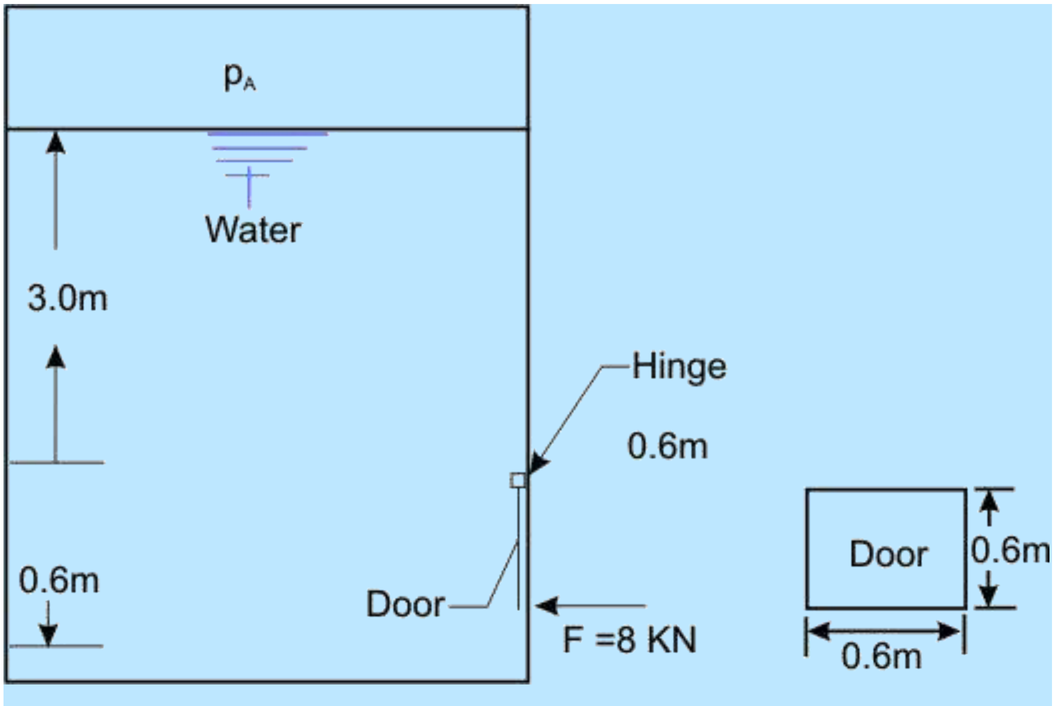


Fig 5.12

4. The profile of the inner face of a dam takes the form of a parabola with the equation 18y = x^2 , where y is the height above the base and x is the horizontal distance of the face from the vertical reference line. The water level is 27m above the base. Determine the thrust on the dam (per meter with) due to the water pressure, its inclination to the vertical and the point where the line of action of this force

[5.28 MN/m, 42° 33', 30.29 m from face]

intersects the free water surface

5. A solid uniform cylinder of length 150 mm and diameter 75 mm is to float upright in water. Determine

[0.641 kg and 0.663 kg]

the limits within which its mass should lie.

6. A long prism, the cross-section of which is an equilateral traingle of side **a**, floats in water with one side horizontal and submerged to a depth **h**. Find

(a) **h/a** as a function of the specific gravity, **S** of the prism.
(b) The metacentric height in terms of side **a**, for small angle of rotation if specific gravity, **S=0.8**.

7. A metal sphere of volume $\forall_m = 0 \cdot 1\ m^3$, specific gravity $s_m = 2$ and fully immersed in water is attached by a flexible wire to a buoy of volume $\forall_B = 1\ m^3$ and specific gravity $s_B = 0 \cdot 1$. Calculate the tension T in the wire and volume of the buoy that is submerged. Refer to Fig 5.13.

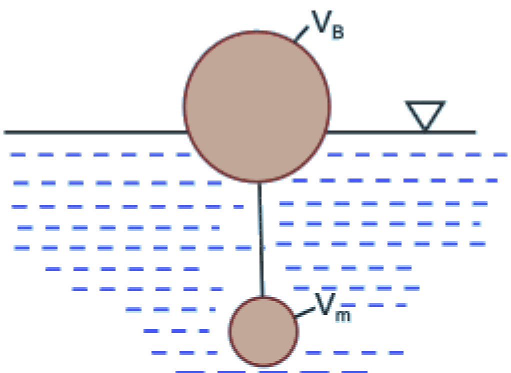


Fig 5.13