

## Forces on Fluid Elements

Fluid Elements - Definition:

Fluid element can be defined as an infinitesimal region of the fluid continuum in isolation from its surroundings.

Two types of forces exist on fluid elements

- **Body Force:** distributed over the entire mass or volume of the element. It is usually expressed per unit mass of the element or medium upon which the forces act.  
Example: Gravitational Force, Electromagnetic force fields etc.
- **Surface Force:** Forces exerted on the fluid element by its surroundings through direct contact at the surface.  
Surface force has two components:
  - Normal Force: along the normal to the area
  - Shear Force: along the plane of the area.

The ratios of these forces and the elemental area in the limit of the area tending to zero are called the normal and shear stresses respectively.

*The shear force is zero for any fluid element at rest and hence the only surface force on a fluid element is the normal component.*

## Normal Stress in a Stationary Fluid

Consider a stationary fluid element of tetrahedral shape with three of its faces coinciding with the coordinate planes  $x$ ,  $y$  and  $z$ .

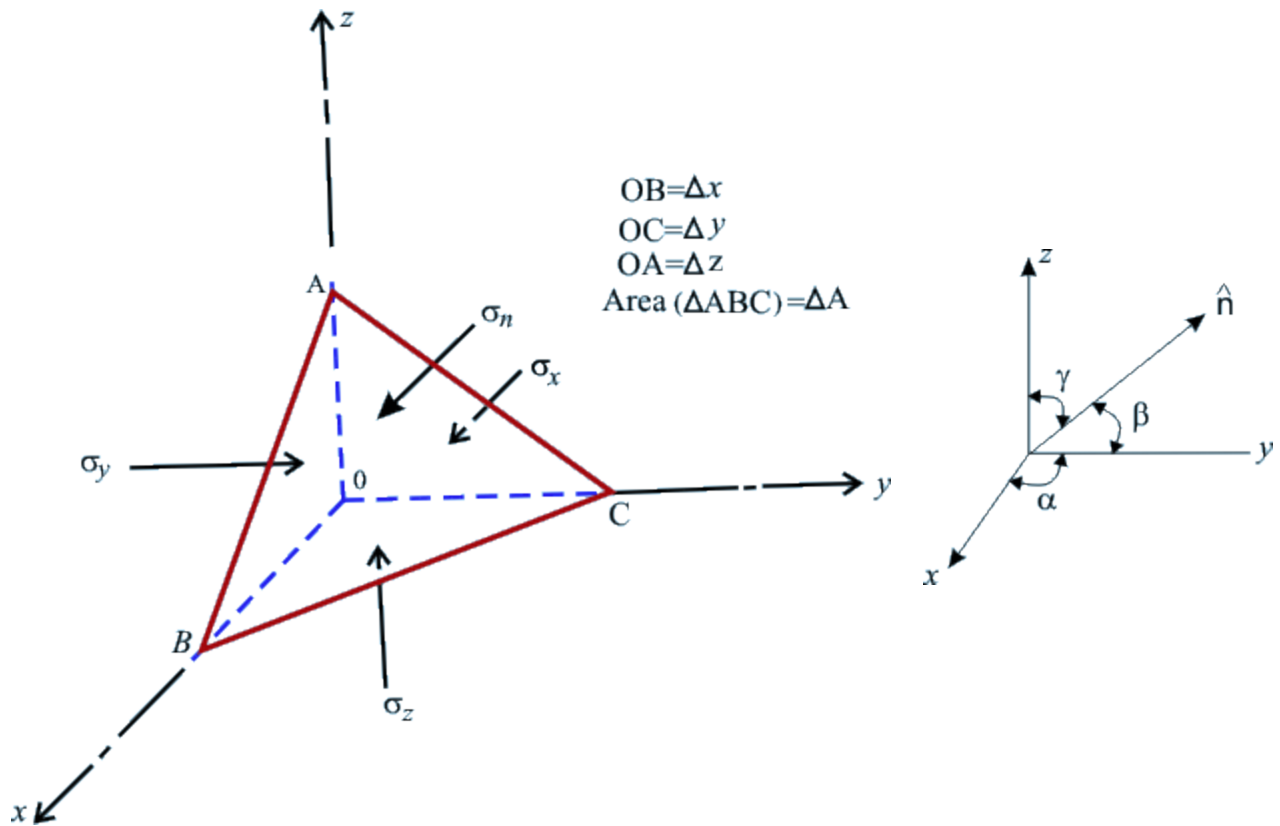


Fig 3.1 State of Stress in a Fluid Element at Rest

Since a fluid element at rest can develop neither shear stress nor tensile stress, the normal stresses acting on different faces are compressive in nature.

Suppose,  $\Sigma F_x$ ,  $\Sigma F_y$  and  $\Sigma F_z$  are the net forces acting on the fluid element in positive x,y and z directions respectively. The direction cosines of the normal to the inclined plane of an area  $\Delta A$  are  $\cos \alpha$ ,  $\cos \beta$  and  $\cos \gamma$ . Considering gravity as the only source of external body force, acting in the -ve z direction, the equations of static equilibrium for the tetrahedral fluid element can be written as

$$\sum F_x = \sigma_x \left( \frac{\Delta y \Delta z}{2} \right) - \sigma_n \Delta A \cos \alpha = 0 \tag{3.1}$$

$$\sum F_y = \sigma_y \left( \frac{\Delta x \Delta z}{2} \right) - \sigma_n \Delta A \cos \beta = 0 \tag{3.2}$$

$$\sum F_z = \sigma_z \left( \frac{\Delta y \Delta x}{2} \right) - \sigma_n \Delta A \cos \gamma - \frac{\rho g}{6} (\Delta x \Delta y \Delta z) = 0 \tag{3.3}$$

where  $\left( \frac{\Delta x \Delta y \Delta z}{6} \right)$  = Volume of tetrahedral fluid element

## Pascal's Law of Hydrostatics

### Pascal's Law

The normal stresses at any point in a fluid element at rest are directed towards the point from all directions and they are of the equal magnitude.

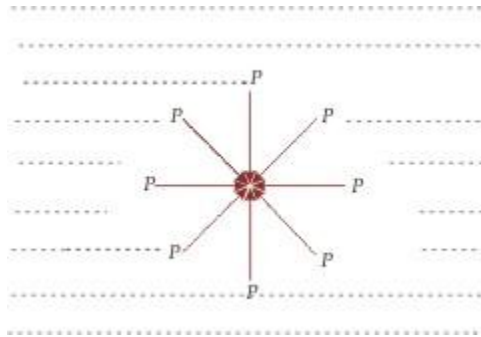


Fig 3.2 State of normal stress at a point in a fluid body at rest

#### Derivation:

The inclined plane area is related to the fluid elements (refer to Fig 3.1) as follows

$$\Delta A \cos \alpha = \left( \frac{\Delta y \Delta z}{2} \right) \quad (3.4)$$

$$\Delta A \cos \beta = \left( \frac{\Delta x \Delta z}{2} \right) \quad (3.5)$$

$$\Delta A \cos \gamma = \left( \frac{\Delta x \Delta y}{2} \right) \quad (3.6)$$

Substituting above values in equation 3.1- 3.3 we get

$$\sigma_x = \sigma_y = \sigma_z = \sigma_n \quad (3.7)$$

#### Conclusion:

The state of normal stress at any point in a fluid element at rest is same and directed towards the point from all directions. These stresses are denoted by a scalar quantity  $p$  defined as the hydrostatic or thermodynamic pressure.

Using "+" sign for the tensile stress the above equation can be written in terms of pressure as

$$\sigma_x = \sigma_y = \sigma_z = -p \tag{3.8}$$

**Fundamental Equation of Fluid Statics**

The fundamental equation of fluid statics describes the spatial variation of hydrostatic pressure  $p$  in the continuous mass of a fluid.

**Derivation:**

Consider a fluid element at rest of given mass with volume  $V$  and bounded by the surface  $S$ .

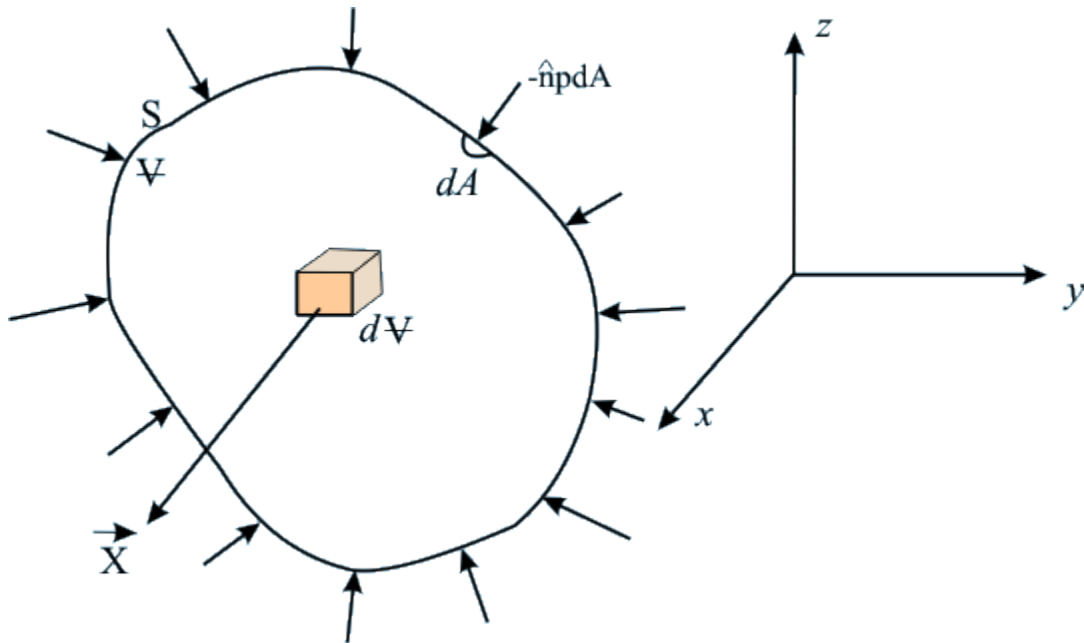


Fig 3.3 External Forces on a Fluid Element at Rest

The fluid element stays at equilibrium under the action of the following two forces

- **The Resultant Body Force**

$\vec{F}_B = \iiint_V \vec{X} \rho dV \tag{3.9}$	<p><math>dV</math> : element of volume</p> <p><math>\rho dV</math> : mass of the element</p> <p><math>\vec{X}</math> : body Force per unit mass acting on the elementary volume</p>
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▪ **The Resultant Surface Force**

$\vec{F}_s = - \iint_s \hat{n} p dA \quad (3.10)$	<p><math>dA</math> : area of an element of surface</p> <p><math>\hat{n}</math> : the unit vector normal to the elemental surface, taken positive when directed outwards</p>
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Using Gauss divergence theorem, Eq (3.10) can be written as

$$\vec{F}_s = - \iint_s \hat{n} p dA = - \iint_s p (\hat{n} dA)$$

$$\vec{F}_s = - \iiint_V (\nabla p) dV \quad (3.11)$$

[Click here to see the derivation](#)

For the fluid element to be in equilibrium , we have

$$\vec{F}_B + \vec{F}_s = \iiint_V (\vec{X}\rho - \nabla p) dV = 0 \quad (3.12)$$

The equation is valid for any volume of the fluid element, no matter how small, thus we get

$$\vec{X}\rho - \nabla p = 0$$

$$\nabla p = \vec{X}\rho \quad (3.13)$$

This is the fundamental equation of fluid statics.

**Fundamental Fluid Static Equations in Scalar Form**

Considering gravity as the only external body force acting on the fluid element, Eq. (3.13) can be expressed in its scalar components with respect to a cartesian coordinate system (see Fig. 3.3) as

$\frac{\partial p}{\partial x} = 0$ <p style="text-align: center;">(in x direction)</p>	<p>(3.13a)</p>	<p><math>X_z</math>: the external body force per unit mass in the positive direction of z (vertically upward),</p>
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$\frac{\partial p}{\partial y} = 0$ <p>(in y direction) (3.13b)</p>	equals to the negative value of g (the acceleration due to gravity).
$\frac{\partial p}{\partial z} = X_z \rho = -\rho g$ <p>(in z direction) (3.13c)</p>	

From Eqs (3.13a)-(3.13c), it can be concluded that the pressure  $p$  is a function of  $z$  only.

Thus, Eq. (3.13c) can be re-written as,

$$\frac{\partial p}{\partial z} = -\rho g \quad (3.14)$$

### Constant and Variable Density Solution

#### Constant Density Solution

The explicit functional relationship of hydrostatic pressure  $p$  with  $z$  can be obtained by integrating the Eq. (3.14).

For an incompressible fluid, the density  $\rho$  is constant throughout. Hence the Eq. (3.14) can be integrated and expressed

as

$$p = -\rho g z + C \quad (3.15)$$

where  $C$  is the integration constant.

If we consider an expanse of fluid with a free surface, where the pressure is defined as  $p = p_0$ , which is equal to atmospheric pressure.

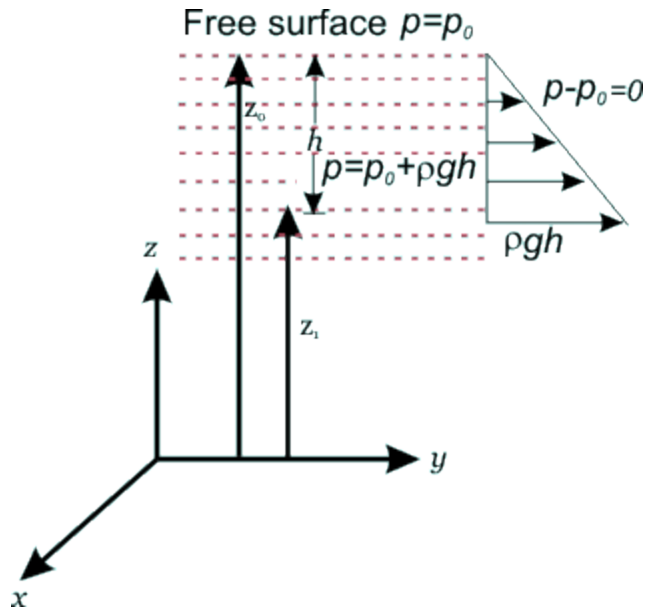


Fig 3.4 Pressure Variation in an Incompressible Fluid at rest with a Free Surface

Eq. (3.15) can be written as,

$$p - p_0 = \rho g (z_0 - z_1) = \rho g h \quad (3.16a)$$

Therefore, Eq. (3.16a) gives the expression of hydrostatic pressure  $p$  at a point whose vertical depression from the free surface is  $h$ .

Similarly,

$$p_1 - p_2 = \rho g (z_1 - z_2) = \rho g h \quad (3.16b)$$

Thus, the difference in pressure between two points in an incompressible fluid at rest can be expressed in terms of the vertical distance between the points. This result is known as **Torricelli's principle**, which is the basis for differential pressure measuring devices. The pressure  $p_0$  at free surface is the local atmospheric pressure.

Therefore, it can be stated from Eq. (3.16a), that the pressure at any point in an expanse of a fluid at rest, with a free surface exceeds that of the local atmosphere by an amount  $\rho g h$ , where  $h$  is the vertical depth of the point from the free surface.

**Variable Density Solution:** As a more generalised case, for compressible fluids at rest, the pressure variation at rest depends on how the fluid density changes with height  $z$  and pressure  $p$ . For example this can be done for special cases of ["isothermal and non-isothermal fluids"](#)

### Units and scales of Pressure Measurement

Pascal ( $\text{N/m}^2$ ) is the unit of pressure .

Pressure is usually expressed with reference to either absolute zero pressure (a complete vacuum) or local atmospheric pressure.

- The absolute pressure: It is the difference between the value of the pressure and the absolute zero pressure.

$$p_{abs} = p - 0 = p$$

- Gauge pressure: It is the difference between the value of the pressure and the local atmospheric pressure ( $p_{atm}$ )

$$p_{gauge} = p - p_{atm}$$

- Vacuum Pressure: If  $p < p_{atm}$  then the gauge pressure ( $p_{gauge}$ ) becomes negative and is called the vacuum pressure. But one should always remember that hydrostatic pressure is always compressive in nature

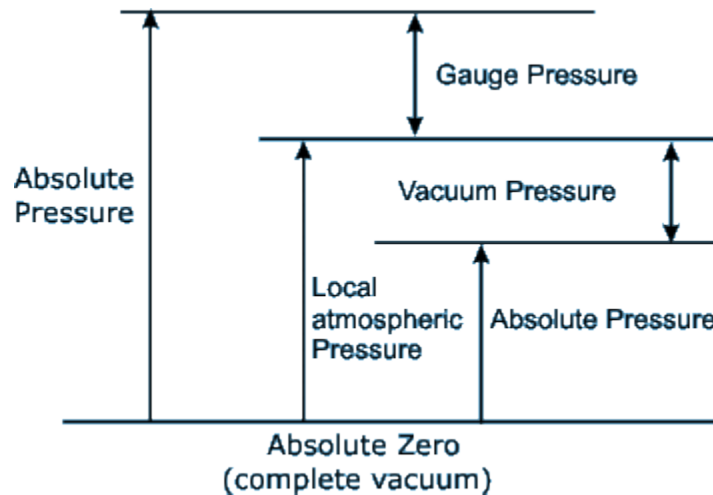


Fig 4.1 The Scale of Pressure

At sea-level, the international standard atmosphere has been chosen as  $P_{atm} = 101.32 \text{ kN/m}^2$

### Piezometer Tube

The direct proportional relation between gauge pressure and the height  $h$  for a fluid of constant density enables the pressure to be simply visualized in terms of the vertical height,  $h = p / \rho g$ .

The height  $h$  is termed as pressure head corresponding to pressure  $p$ . For a liquid without a free surface in a closed pipe, the pressure head  $p / \rho g$  at a point corresponds to the vertical height above the point



to which a free surface would rise, if a small tube of sufficient length and open to atmosphere is connected to the pipe

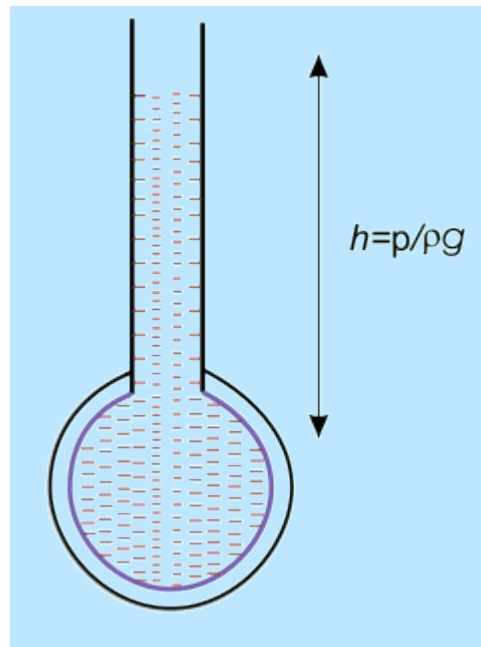


Fig 4.2 A piezometer Tube

Such a tube is called a piezometer tube, and the height  $h$  is the measure of the gauge pressure of the fluid in the pipe. If such a piezometer tube of sufficient length were closed at the top and the space above the liquid surface were a perfect vacuum, the height of the column would then correspond to the absolute pressure of the liquid at the base. This principle is used in the well known mercury barometer to determine the local atmospheric pressure.

### The Barometer

Barometer is used to determine the local atmospheric pressure. Mercury is employed in the barometer because its density is sufficiently high for a relative short column to be obtained. and also because it has very small vapour pressure at normal temperature. High density scales down the pressure head( $h$ ) to represent same magnitude of pressure in a tube of smaller height.

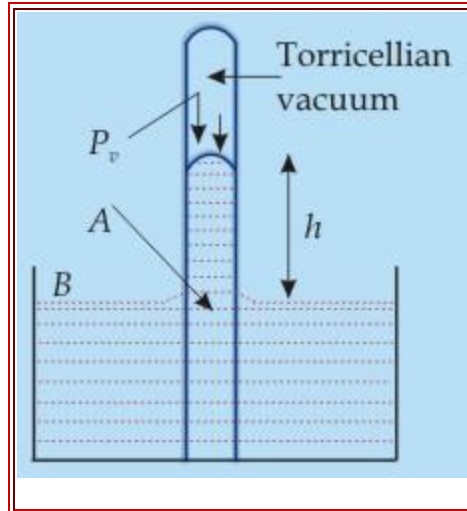


Fig 4.3 A Simple Barometer

Even if the air is completely absent, a perfect vacuum at the top of the tube is never possible. The space would be occupied by the mercury vapour and the pressure would equal to the vapour pressure of mercury at its existing temperature. This almost vacuum condition above the mercury in the barometer is known as Torricellian vacuum.

The pressure at A equal to that at B (Fig. 4.3) which is the atmospheric pressure  $p_{atm}$  since A and B lie on the same horizontal plane. Therefore, we can write

$$p_B = p_{atm} = p_v + \rho g h \quad (4.1)$$

The vapour pressure of mercury  $p_v$ , can normally be neglected in comparison to  $p_{atm}$ .

At 20°C,  $p_v$  is only 0.16  $p_{atm}$ , where  $p_{atm} = 1.0132 \times 10^5$  Pa at sea level. Then we get from Eq. (4.1)

$$h = p_{atm} / \rho g = \frac{1.0132 \times 10^5 \text{ N/m}^2}{(13560 \text{ kg/m}^3)(9.81 \text{ N/Kg})} = 0.752 \text{ m of Hg}$$

For accuracy, small corrections are necessary to allow for the variation of  $r$  with temperature, the thermal expansion of the scale (usually made of brass). and surface tension effects. If water was used instead of mercury, the corresponding height of the column would be about 10.4 m provided that a perfect vacuum could be achieved above the water. However, the vapour pressure of water at ordinary temperature is appreciable and so the actual height at, say, 15°C would be about 180 mm less than this value. Moreover, with a tube smaller in diameter than about 15 mm, surface tension effects become significant.

### Manometers for measuring Gauge and Vacuum Pressure

Manometers are devices in which columns of a suitable liquid are used to measure the difference in pressure between two points or between a certain point and the atmosphere.

Manometer is needed for measuring large gauge pressures. It is basically the modified form of the piezometric tube. A common type manometer is like a transparent "U-tube" as shown in Fig. 4.4.

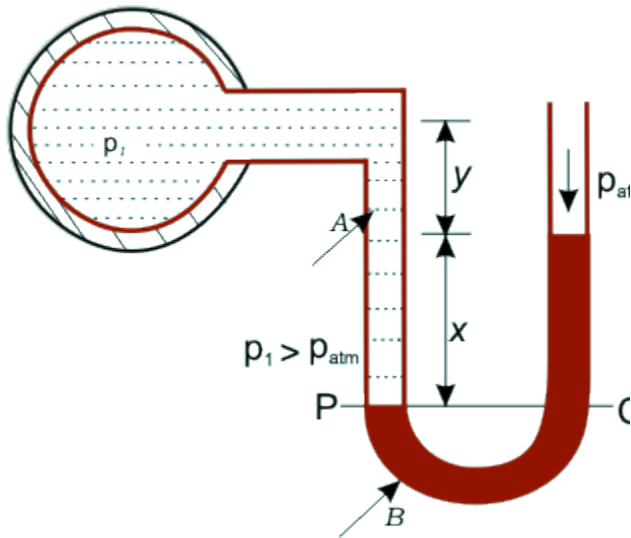


Fig 4.4 A simple manometer to measure gauge pressure

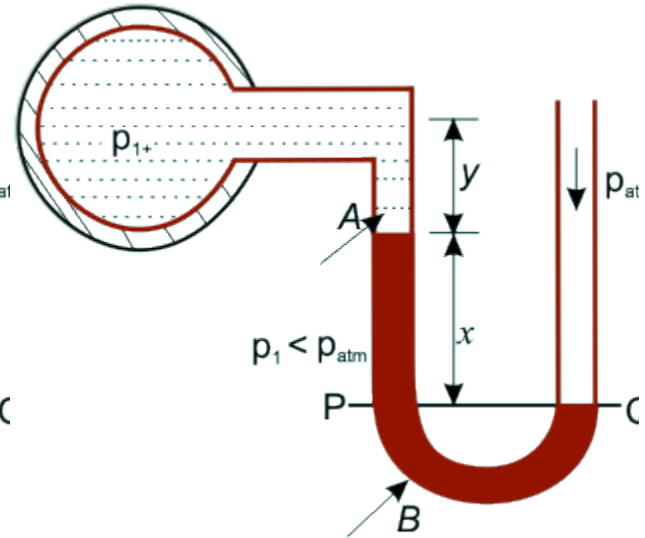


Fig 4.5 A simple manometer to measure vacuum pressure

One of the ends is connected to a pipe or a container having a fluid (A) whose pressure is to be measured while the other end is open to atmosphere. The lower part of the U-tube contains a liquid immiscible with the fluid A and is of greater density than that of A. This fluid is called the manometric fluid.

The pressures at two points P and Q (Fig. 4.4) in a horizontal plane within the continuous expanse of same fluid (the liquid B in this case) must be equal. Then equating the pressures at P and Q in terms of the heights of the fluids above those points, with the aid of the fundamental equation of hydrostatics (Eq 3.16), we have

$$p_1 + \rho_A g(y + x) = p_{atm} + \rho_B g x$$

Hence,

$$p_1 - p_{atm} = (\rho_B - \rho_A) g x - \rho_A g y$$

where  $p_1$  is the absolute pressure of the fluid A in the pipe or container at its centre line, and  $p_{atm}$  is the local atmospheric pressure. When the pressure of the fluid in the container is lower than the atmospheric pressure, the liquid levels in the manometer would be adjusted as shown in Fig. 4.5. Hence it becomes,

$$p_1 + \rho_A g y + \rho_B g x = p_{atm}$$

$$p_{atm} - p_1 = (\rho_A y + \rho_B x)g \quad (4.2)$$

### Manometers to measure Pressure Difference

A manometer is also frequently used to measure the pressure difference, in course of flow, across a restriction in a horizontal pipe.

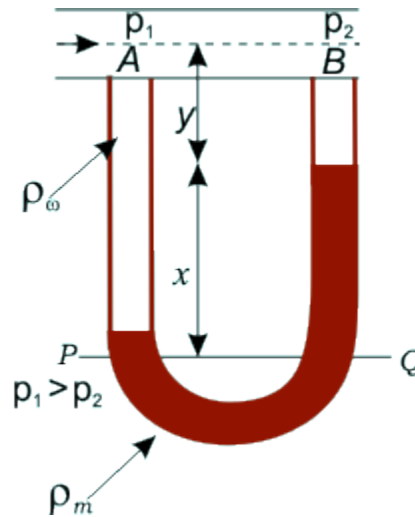


Fig 4.6 Manometer measuring pressure difference

The axis of each connecting tube at A and B should be perpendicular to the direction of flow and also for the edges of the connections to be smooth. Applying the principle of hydrostatics at P and Q we have,

$$p_1 + (y + x)\rho_w g = p_2 + y\rho_w g + \rho_m g x$$

$$p_1 - p_2 = (\rho_m - \rho_w)g x \quad (4.3)$$

where,  $\rho_m$  is the density of manometric fluid and  $\rho_w$  is the density of the working fluid flowing through the pipe.

We can express the difference of pressure in terms of the difference of heads (height of the working fluid at equilibrium).

$$h_1 - h_2 = \frac{p_1 - p_2}{\rho_w g} = \left( \frac{\rho_m}{\rho_w} - 1 \right) x \quad (4.4)$$

### Inclined Tube Manometer

- For accurate measurement of small pressure differences by an ordinary u-tube manometer, it is essential that the ratio  $r_m/r_w$  should be close to unity. This is not possible if the working fluid is a gas; also having a manometric liquid of density very close to that of the working liquid and giving at the same time a well defined meniscus at the interface is not always possible. For this purpose, an inclined tube manometer is used.
- If the transparent tube of a manometer, instead of being vertical, is set at an angle  $\theta$  to the horizontal (Fig. 4.7), then a pressure difference corresponding to a vertical difference of levels  $x$  gives a movement of the meniscus  $s = x/\sin\theta$  along the slope.

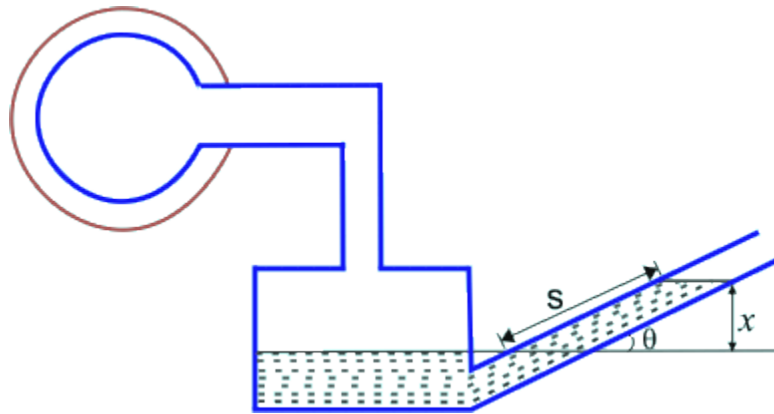


Fig 4.7 An Inclined Tube Manometer

- If  $\theta$  is small, a considerable magnification of the movement of the meniscus may be achieved.
- Angles less than  $5^\circ$  are not usually satisfactory, because it becomes difficult to determine the exact position of the meniscus.
- One limb is usually made very much greater in cross-section than the other. When a pressure difference is applied across the manometer, the movement of the liquid surface in the wider limb is practically negligible compared to that occurring in the narrower limb. If the level of the surface in the wider limb is assumed constant, the displacement of the meniscus in the narrower limb needs only to be measured, and therefore only this limb is required to be transparent.

### Inverted Tube Manometer

For the measurement of small pressure differences in liquids, an inverted U-tube manometer is used.

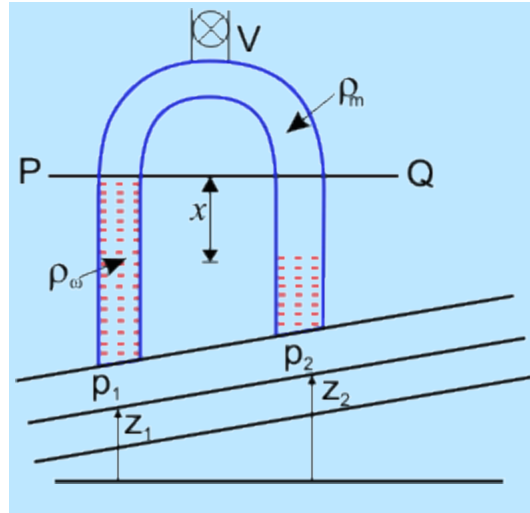


Fig 4.8 An Inverted Tube Manometer

Here  $\rho_m < \rho_w$  and the line PQ is taken at the level of the higher meniscus to equate the pressures at P and Q from the principle of hydrostatics. It may be written that

$$p_1^* - p_2^* = (\rho_w - \rho_m)gx$$

where  $p^*$  represents the **piezometric pressure**,  $p + \rho gz$  ( $z$  being the vertical height of the point concerned from any reference datum). In case of a horizontal pipe ( $z_1 = z_2$ ) the difference in piezometric pressure becomes equal to the difference in the static pressure. If  $(\rho_w - \rho_m)$  is sufficiently small, a large value of  $x$  may be obtained for a small value of  $p_1^* - p_2^*$ . Air is used as the manometric fluid.

Therefore,  $\rho_m$  is negligible compared with  $\rho_w$  and hence,

$$p_1^* - p_2^* \approx \rho_w gx \quad (4.5)$$

Air may be pumped through a valve V at the top of the manometer until the liquid menisci are at a suitable level.

### Micromanometer

When an additional gauge liquid is used in a U-tube manometer, a large difference in meniscus levels may be obtained for a very small pressure difference.

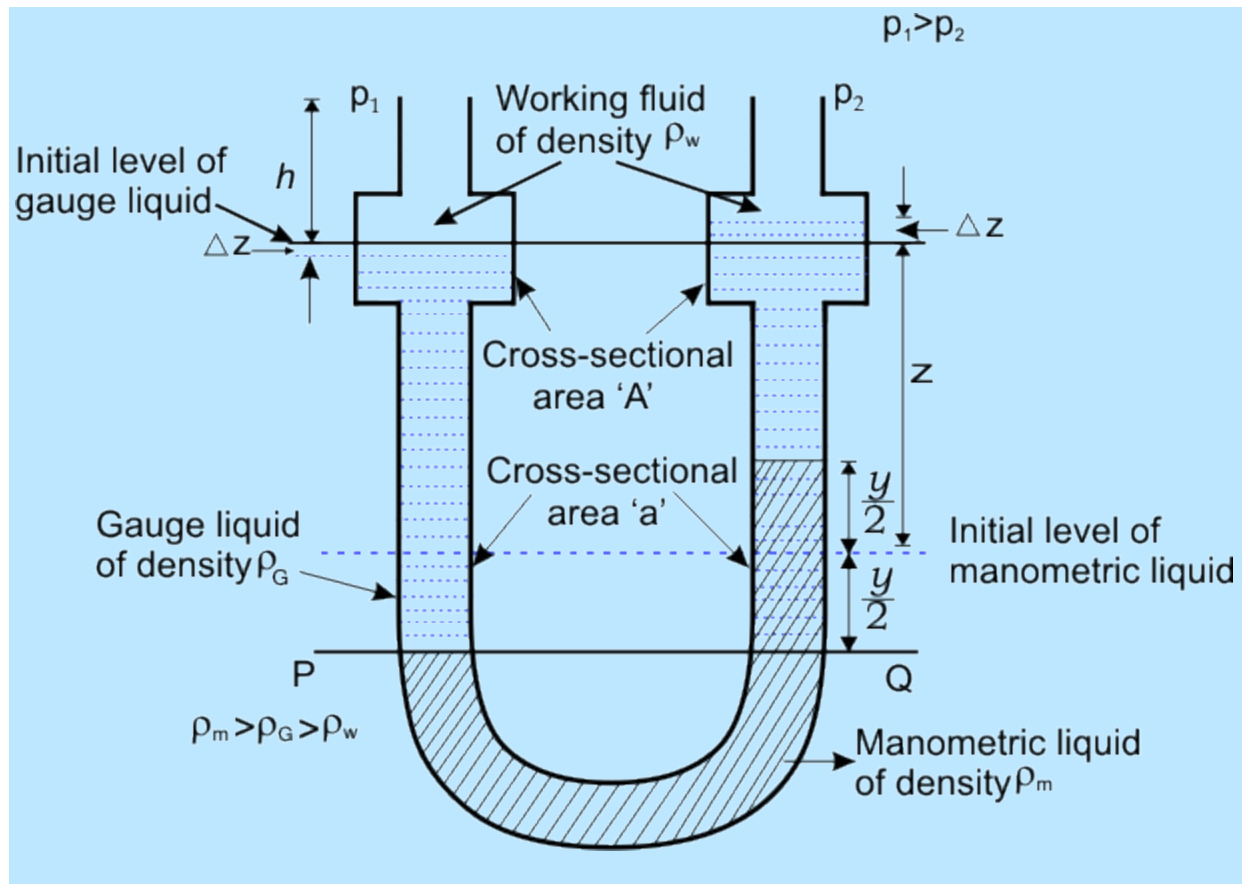


Fig 4.9 A Micromanometer

The equation of hydrostatic equilibrium at PQ can be written as

$$p_1 + \rho_w g(h + \Delta z) + \rho_G g\left(z - \Delta z + \frac{y}{2}\right) = p_2 + \rho_w g(h - \Delta z) + \rho_G g\left(z + \Delta z - \frac{y}{2}\right) + \rho_m g y$$

where  $\rho_w$ ,  $\rho_G$  and  $\rho_m$  are the densities of working fluid, gauge liquid and manometric liquid respectively.

From continuity of gauge liquid,

$$A \Delta z = a \frac{y}{2} \tag{4.6}$$

$$p_1 - p_2 = g y \left[ \rho_m - \rho_G \left(1 - \frac{a}{A}\right) - \rho_w \frac{a}{A} \right] \tag{4.7}$$

If  $a$  is very small compared to  $A$

$$p_1 - p_2 \approx (\rho_m - \rho_G) g y \tag{4.8}$$

With a suitable choice for the manometric and gauge liquids so that their densities are close ( $\rho_m \approx \rho_G$ ) a reasonable value of  $\gamma$  may be achieved for a small pressure difference.

### Hydrostatic Thrusts on Submerged Plane Surface

Due to the existence of hydrostatic pressure in a fluid mass, a normal force is exerted on any part of a solid surface which is in contact with a fluid. The individual forces distributed over an area give rise to a resultant force.

#### Plane Surfaces

Consider a plane surface of arbitrary shape wholly submerged in a liquid so that the plane of the surface makes an angle  $\theta$  with the free surface of the liquid. We will assume the case where the surface shown in the figure below is subjected to hydrostatic pressure on one side and atmospheric pressure on the other side.

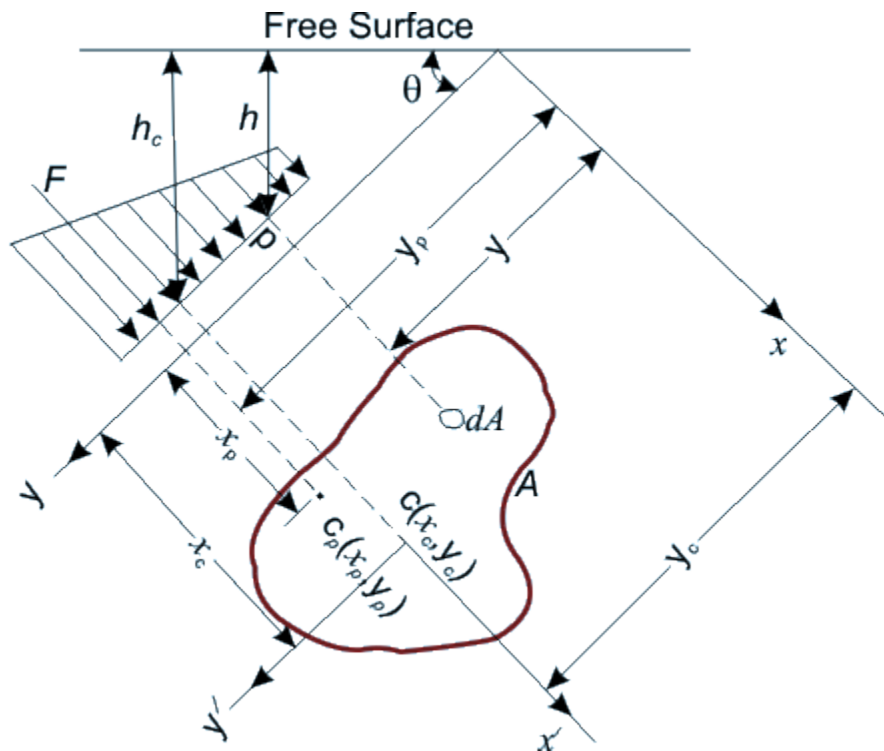


Fig 5.1 Hydrostatic Thrust on Submerged Inclined Plane Surface

Let  $p$  denotes the gauge pressure on an elemental area  $dA$ . The resultant force  $F$  on the area  $A$  is therefore



$$F = \iint_A p \, dA \quad (5.1)$$

According to Eq (3.16a) Eq (5.1) reduces to

$$F = \rho g \iint h \, dA = \rho g \sin \theta \iint y \, dA \quad (5.2)$$

Where **h is the vertical depth of the elemental area dA from the free surface** and the distance y is measured from the x-axis, the line of intersection between the extension of the inclined plane and the free surface (Fig. 5.1). The ordinate of the centre of area of the plane surface A is defined as

$$y_c = \frac{1}{A} \iint y \, dA \quad (5.3)$$

Hence from Eqs (5.2) and (5.3), we get

$$F = \rho g y_c \sin \theta A = \rho g h_c A \quad (5.4)$$

where  $h_c (= y_c \sin \theta)$  is the vertical depth (from free surface) of centre c of area .

Equation (5.4) implies that the hydrostatic thrust on an inclined plane is equal to the pressure at its centroid times the total area of the surface, i.e., the force that would have been experienced by the surface if placed horizontally at a depth  $h_c$  from the free surface (Fig. 5.2).

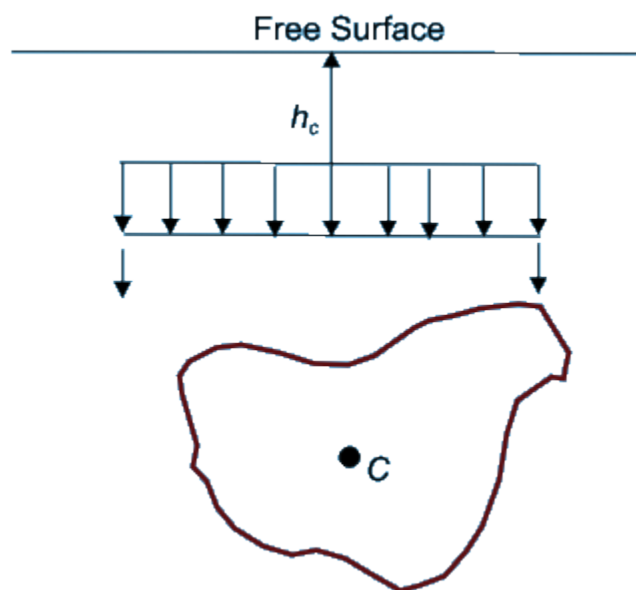


Fig 5.2 Hydrostatic Thrust on Submerged Horizontal Plane Surface

The point of action of the resultant force on the plane surface is called the centre of pressure  $C_p$ . Let  $x_p$  and  $y_p$  be the distances of the centre of pressure from the y and x axes respectively. Equating the moment of the resultant force about the x axis to the summation of the moments of the component forces, we have

$$y_p F = \int y dF = \rho g \sin \theta \int \int y^2 dA \quad (5.5)$$

Solving for  $y_p$  from Eq. (5.5) and replacing F from Eq. (5.2), we can write

$$y_p = \frac{\int \int y^2 dA}{\int \int y dA} \quad (5.6)$$

In the same manner, the x coordinate of the centre of pressure can be obtained by taking moment about the y-axis. Therefore,

$$x_p F = \int x dF = \rho g \sin \theta \int \int xy dA$$

From which,

$$x_p = \frac{\int \int xy dA}{\int \int y dA} \quad (5.7)$$

The two double integrals in the numerators of Eqs (5.6) and (5.7) are the moment of inertia about the x-axis  $I_{xx}$  and the product of inertia  $I_{xy}$  about x and y axis of the plane area respectively. By applying the theorem of parallel axis

$$I_{xx} = \int \int y^2 dA = I_{x'x'} + Ay_c^2 \quad (5.8)$$

$$I_{xy} = \int \int xy dA = I_{x'y'} + Ax_c y_c \quad (5.9)$$

where,  $I_{x'x'}$  and  $I_{x'y'}$  are the moment of inertia and the product of inertia of the surface about the centroidal axes  $(x' - y')$ ,  $x_c$ , and  $y_c$  are the coordinates of the center c of the area with respect to x-y axes.

With the help of Eqs (5.8), (5.9) and (5.3), Eqs (5.6) and (5.7) can be written as

$$y_p = \frac{I_{x'y'}}{Ay_c} + y_c \quad (5.10a)$$

$$x_p = \frac{I_{x'y'}}{Ay_c} + x_c \quad (5.10b)$$

The first term on the right hand side of the Eq. (5.10a) is always positive. Hence, the centre of pressure is always at a higher depth from the free surface than that at which the centre of area lies. This is obvious because of the typical variation of hydrostatic pressure with the depth from the free surface. When the plane area is symmetrical about the  $y'$  axis,  $I_{x'y'} = 0$ , and  $x_p = x_c$ .

### Hydrostatic Thrusts on Submerged Curved Surfaces

On a curved surface, the direction of the normal changes from point to point, and hence the pressure forces on individual elemental surfaces differ in their directions. Therefore, a scalar summation of them cannot be made. Instead, the resultant thrusts in certain directions are to be determined and these forces may then be combined vectorially. An arbitrary submerged curved surface is shown in Fig. 5.3. A rectangular Cartesian coordinate system is introduced whose  $xy$  plane coincides with the free surface of the liquid and  $z$ -axis is directed downward below the  $x - y$  plane.

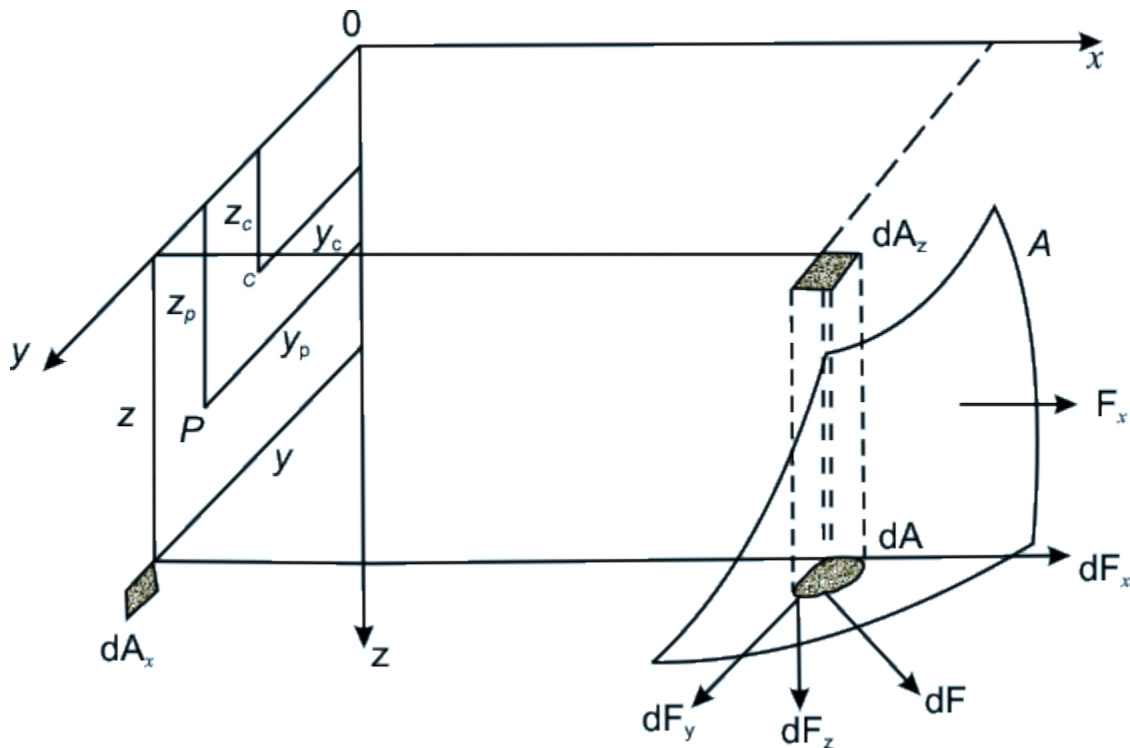


Fig 5.3 Hydrostatic thrust on a Submerged Curved Surface

Consider an elemental area  $dA$  at a depth  $z$  from the surface of the liquid. The hydrostatic force on the elemental area  $dA$  is

$$dF = \rho g z dA \quad (5.11)$$

and the force acts in a direction normal to the area  $dA$ . The components of the force  $dF$  in  $x$ ,  $y$  and  $z$  directions are

$$dF_x = l dF = l \rho g z dA \quad (5.12a)$$

$$dF_y = m dF = m \rho g z dA \quad (5.12b)$$

$$dF_z = n dF = n \rho g z dA \quad (5.13c)$$

Where  $l$ ,  $m$  and  $n$  are the direction cosines of the normal to  $dA$ . The components of the surface element  $dA$  projected on  $yz$ ,  $xz$  and  $xy$  planes are, respectively

$$dA_x = l dA \quad (5.13a)$$

$$dA_y = m dA \quad (5.13b)$$

$$dA_z = n dA \quad (5.13c)$$

Substituting Eqs (5.13a-5.13c) into (5.12) we can write

$$dF_x = \rho g z dA_x \quad (5.14a)$$

$$dF_y = \rho g z dA_y \quad (5.14b)$$

$$dF_z = \rho g z dA_z \quad (5.14c)$$

Therefore, the components of the total hydrostatic force along the coordinate axes are

$$F_x = \iint \rho g z dA_x = \rho g z_c A_x \quad (5.15a)$$

$$F_y = \iint \rho g z dA_y = \rho g z_c A_y \quad (5.15b)$$

$$F_z = \iint \rho g z dA_z \quad (5.15c)$$

where  $z_c$  is the  $z$  coordinate of the centroid of area  $A_x$  and  $A_y$  (the projected areas of curved surface on  $yz$  and  $xz$  plane respectively). If  $z_p$  and  $y_p$  are taken to be the coordinates of the point of action of  $F_x$  on the projected area  $A_x$  on  $yz$  plane, , we can write

$$z_p = \frac{1}{A_x z_c} \iint z^2 dA_x = \frac{I_{yy}}{A_x z_c} \quad (5.16a)$$

$$y_p = \frac{1}{A_x z_c} \iint yz dA_x = \frac{I_{yz}}{A_x z_c} \quad (5.16b)$$

where  $I_{yy}$  is the moment of inertia of area  $A_x$  about y-axis and  $I_{yz}$  is the product of inertia of  $A_x$  with respect to axes y and z. In the similar fashion,  $z'_p$  and  $x'_p$  the coordinates of the point of action of the force  $F_y$  on area  $A_y$ , can be written as

$$z'_p = \frac{1}{A_y z_c} \iint z^2 dA_y = \frac{I_{xx}}{A_y z_c} \quad (5.17a)$$

$$x'_p = \frac{1}{A_y z_c} \iint xz dA_y = \frac{I_{xz}}{A_y z_c} \quad (5.17b)$$

where  $I_{xx}$  is the moment of inertia of area  $A_y$  about x axis and  $I_{xz}$  is the product of inertia of  $A_y$  about the axes x and z.

We can conclude from Eqs (5.15), (5.16) and (5.17) that for a curved surface, the component of hydrostatic force in a horizontal direction is equal to the hydrostatic force on the projected plane surface perpendicular to that direction and acts through the centre of pressure of the projected area. From Eq. (5.15c), the vertical component of the hydrostatic force on the curved surface can be written as

$$F_z = \rho g \iint z dA_z = \rho g \forall \quad (5.18)$$

where  $\forall$  is the volume of the body of liquid within the region extending vertically above the submerged surface to the free surface of the liquid. Therefore, the vertical component of hydrostatic force on a submerged curved surface is equal to the weight of the liquid volume vertically above the solid surface of the liquid and acts through the center of gravity of the liquid in that volume.

### Buoyancy

- When a body is either wholly or partially immersed in a fluid, a lift is generated due to the net vertical component of hydrostatic pressure forces experienced by the body.
- This lift is called the buoyant force and the phenomenon is called buoyancy
- Consider a solid body of arbitrary shape completely submerged in a homogeneous liquid as shown in Fig. 5.4. **Hydrostatic pressure forces act on the entire surface of the body.**

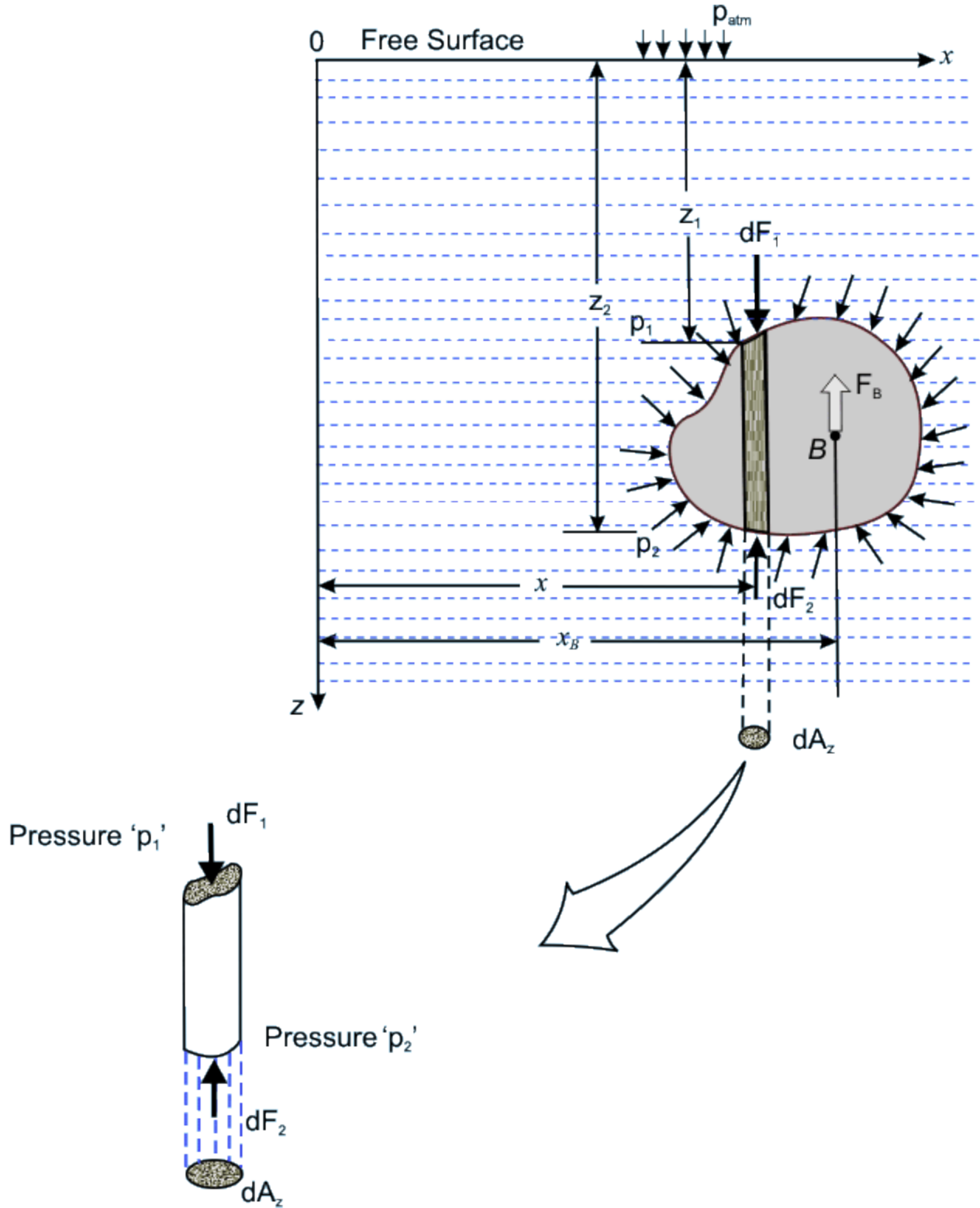


Fig 5.4 Buoyant Force on a Submerged Body

To calculate the vertical component of the resultant hydrostatic force, the body is considered to be divided into a number of elementary vertical prisms. The vertical forces acting on the two ends of such a prism of cross-section  $dA_z$  (Fig. 5.4) are respectively

$$dF_1 = (p_{atm} + p_1) dA_z = (p_{atm} + \rho g z_1) dA_z \quad (5.19a)$$

$$dF_2 = (p_{atm} + p_2) dA_z = (p_{atm} + \rho g z_2) dA_z \quad (5.19b)$$

Therefore, the buoyant force (the net vertically upward force) acting on the elemental prism of volume  $dV$  is -

$$dF_B = dF_2 - dF_1 = \rho g (z_2 - z_1) dA_z = \rho g dV \quad (5.19c)$$

Hence the buoyant force  $F_B$  on the entire submerged body is obtained as

$$F_B = \iiint_V \rho g dV = \rho g V \quad (5.20)$$

Where  $V$  is the **total volume of the submerged body**. The line of action of the force  $F_B$  can be found by taking moment of the force with respect to z-axis. Thus

$$x_B F_B = \int x dF_B \quad (5.21)$$

Substituting for  $dF_B$  and  $F_B$  from Eqs (5.19c) and (5.20) respectively into Eq. (5.21), the **x coordinate of the center of the buoyancy** is obtained as

$$x_B = \frac{1}{V} \iiint_V x dV \quad (5.22)$$

which is **the centroid of the displaced volume**. It is found from Eq. (5.20) that the buoyant force  $F_B$  equals to the weight of liquid displaced by the submerged body of volume  $V$ . This phenomenon was discovered by Archimedes and is known as the Archimedes principle.

### ARCHIMEDES PRINCIPLE

The buoyant force on a submerged body

- The Archimedes principle states that the buoyant force on a submerged body is equal to the weight of liquid displaced by the body, and acts vertically upward through the centroid of the displaced volume.
- Thus the net weight of the submerged body, (the net vertical downward force experienced by it) is reduced from its actual weight by an amount that equals the buoyant force.

The buoyant force on a partially immersed body

- According to Archimedes principle, the buoyant force of a partially immersed body is equal to the weight of the displaced liquid.
- Therefore the buoyant force depends upon the density of the fluid and the submerged volume of the body.
- For a floating body in static equilibrium and in the absence of any other external force, the buoyant force must balance the weight of the body.

### Stability of Unconstrained Submerged Bodies in Fluid

- The equilibrium of a body submerged in a liquid requires that the weight of the body acting through its centre of gravity should be colinear with an equal hydrostatic lift acting through the centre of buoyancy.
- In general, if the body is not homogeneous in its distribution of mass over the entire volume, the location of **centre of gravity G does not coincide with the centre of volume, i.e., the centre of buoyancy B.**
- Depending upon the relative locations of G and B, a floating or submerged body attains three different states of equilibrium-

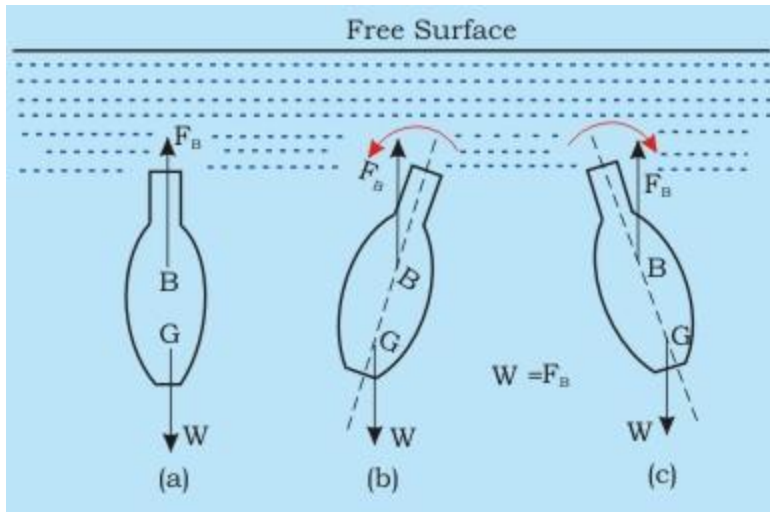
Let us suppose that a body is given a small angular displacement and then released. Then it will be said to be in

- **Stable Equilibrium:** If the body **returns to its original position** by retaining the originally vertical axis as vertical.
- **Unstable Equilibrium:** If the body **does not return to its original position but moves further** from it.
- **Neutral Equilibrium:** If the body **neither returns to its original position nor increases its displacement further**, it will simply adopt its new position.

### Stable Equilibrium

Consider a submerged body in equilibrium whose centre of gravity is located below the centre of buoyancy (Fig. 5.5a). If the body is tilted slightly in any direction, the buoyant force and the weight always produce a restoring couple trying to return the body to its original position (Fig. 5.5b, 5.5c).



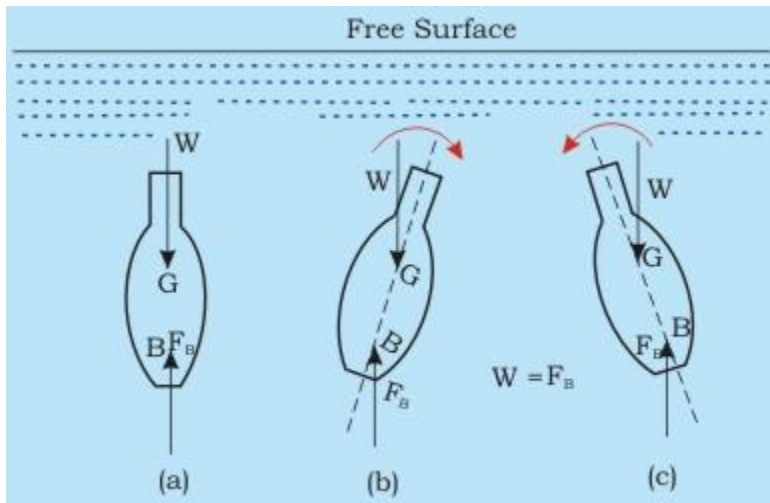


[Click to play the Demonstration](#)

Fig 5.5 A Submerged body in Stable Equilibrium

### Unstable Equilibrium

On the other hand, if point G is above point B (Fig. 5.6a), any disturbance from the equilibrium position will create a destroying couple which will turn the body away from its original position (5.6b, 5.6c).

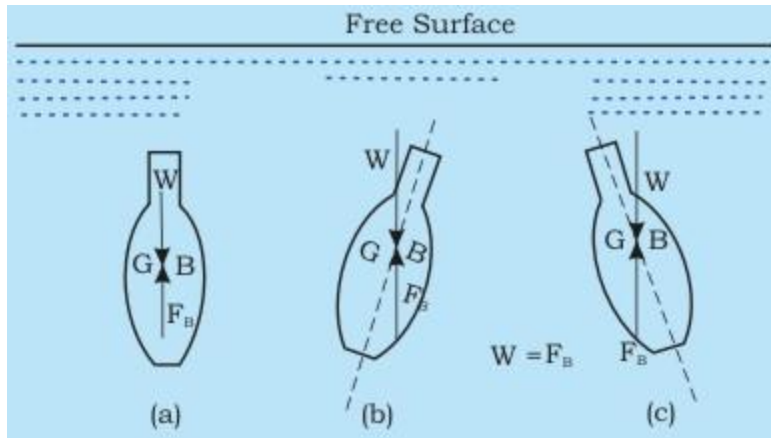


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Fig 5.6 A Submerged body in Unstable Equilibrium

### Neutral Equilibrium

When the centre of gravity G and centre of buoyancy B coincides, the body will always assume the same position in which it is placed (Fig 5.7) and hence it is in neutral equilibrium.



[Click to play the Demonstration](#)

Fig 5.7 A Submerged body in Neutral Equilibrium

Therefore, it can be concluded that a **submerged body will be in stable, unstable or neutral equilibrium if its centre of gravity is below, above or coincident with the center of buoyancy respectively (Fig. 5.8).**

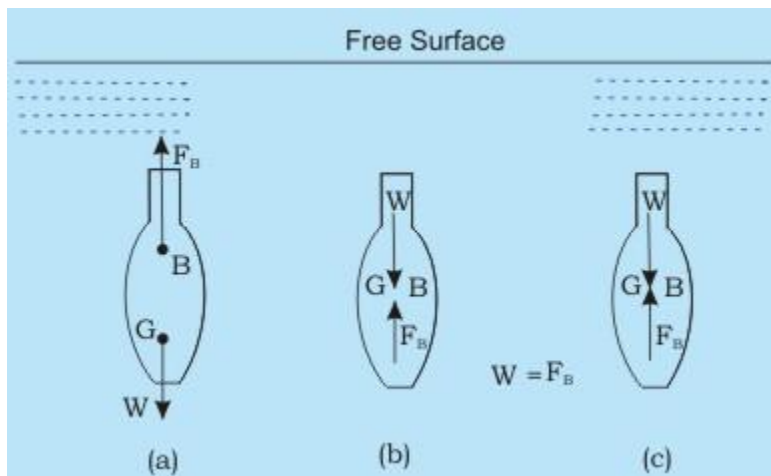


Fig 5.8 States of Equilibrium of a Submerged Body

(a) STABLE EQUILIBRIUM (B) UNSTABLE EQUILIBRIUM (C) NEUTRAL EQUILIBRIUM

### Stability of Floating Bodies in Fluid

- When the body undergoes an angular displacement about a horizontal axis, the shape of the immersed volume changes and so the centre of buoyancy moves relative to the body.
- As a result of above observation stable equilibrium can be achieved, under certain condition, even when G is above B.

Figure 5.9a illustrates a floating body -a boat, for example, in its equilibrium position.

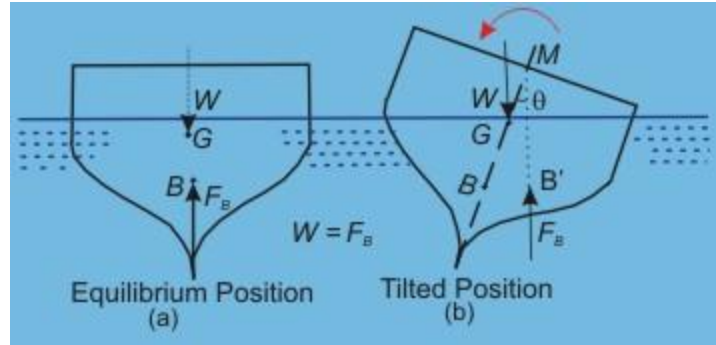


Fig 5.9 A Floating body in Stable equilibrium

Important points to note here are

- The force of buoyancy  $F_b$  is equal to the weight of the body  $W$
- Centre of gravity  $G$  is above the centre of buoyancy in the same vertical line.
- Figure 5.9b shows the situation after the body has undergone a small angular displacement  $q$  with respect to the vertical axis.
- The centre of gravity  $G$  remains unchanged relative to the body (This is not always true for ships where some of the cargo may shift during an angular displacement).
- During the movement, the volume immersed on the right hand side increases while that on the left hand side decreases. Therefore the centre of buoyancy moves towards the right to its new position  $B'$ .

Let the new **line of action of the buoyant force** (which is **always vertical**) through  $B'$  intersect the axis  $BG$  (the old vertical line containing the centre of gravity  $G$  and the old centre of buoyancy  $B$ ) at  $M$ . For small values of  $q$  the **point  $M$**  is practically constant in position and is **known as metacentre**. For the body shown in Fig. 5.9,  $M$  is above  $G$ , and the couple acting on the body in its displaced position is a restoring couple which tends to turn the body to its original position. If  $M$  were below  $G$ , the couple would be an overturning couple and the original equilibrium would have been unstable. When  $M$  coincides with  $G$ , the body will assume its new position without any further movement and thus will be in neutral equilibrium. **Therefore, for a floating body, the stability is determined not simply by the relative position of  $B$  and  $G$ , rather by the relative position of  $M$  and  $G$ .** The distance of metacentre above  $G$  along the line  $BG$  is known as metacentric height  $GM$  which can be written as

$$GM = BM - BG$$

Hence the **condition of stable equilibrium for a floating body** can be expressed in terms of **metacentric height** as follows:

**$GM > 0$  ( $M$  is above  $G$ )**

**Stable equilibrium**

**GM = 0 (M coinciding with G)**

**Neutral equilibrium**

**GM < 0 (M is below G)**

**Unstable equilibrium**

The angular displacement of a boat or ship about its longitudinal axis is known as 'rolling' while that about its transverse axis is known as "pitching".

### Floating Bodies Containing Liquid

If a floating body carrying liquid with a free surface undergoes an angular displacement, the liquid will also move to keep its free surface horizontal. Thus not only does the centre of buoyancy B move, but also the centre of gravity G of the floating body and its contents move in the same direction as the movement of B. Hence the stability of the body is reduced. For this reason, liquid which has to be carried in a ship is put into a number of separate compartments so as to minimize its movement within the ship.

### **Period of Oscillation**

The restoring couple caused by the buoyant force and gravity force acting on a floating body displaced from its equilibrium position is  $W \cdot GM \sin \theta$  (Fig. 5.9). Since the torque equals to mass moment of inertia (i.e., second moment of mass) multiplied by angular acceleration, it can be written

$$W(GM) \sin \theta = -I_M \frac{d^2 \theta}{dt^2} \quad (5.23)$$

Where  $I_M$  represents the mass moment of inertia of the body about its axis of rotation. The minus sign in the RHS of Eq. (5.23) arises since the torque is a retarding one and decreases the angular acceleration. If  $\theta$  is small,  $\sin \theta = \theta$  and hence Eq. (5.23) can be written as

$$\frac{d^2 \theta}{dt^2} + \frac{W \cdot GM}{I_M} \theta = 0 \quad (5.24)$$

Equation (5.24) represents a simple harmonic motion. The time period (i.e., the time of a complete oscillation from one side to the other and back again) equals to  $2\pi (I_M / W \cdot GM)^{1/2}$ . The oscillation of the body results in a flow of the liquid around it and this flow has been disregarded here. In practice, of course, viscosity in the liquid introduces a damping action which quickly suppresses the oscillation unless further disturbances such as waves cause new angular displacements.

Exercise Problems - Chapter 2

1. For the system shown in Fig 5.10, determine the air pressure  $p_A$  which will make the pressure at N one

[3.33 kPa]

fourth of that at M.

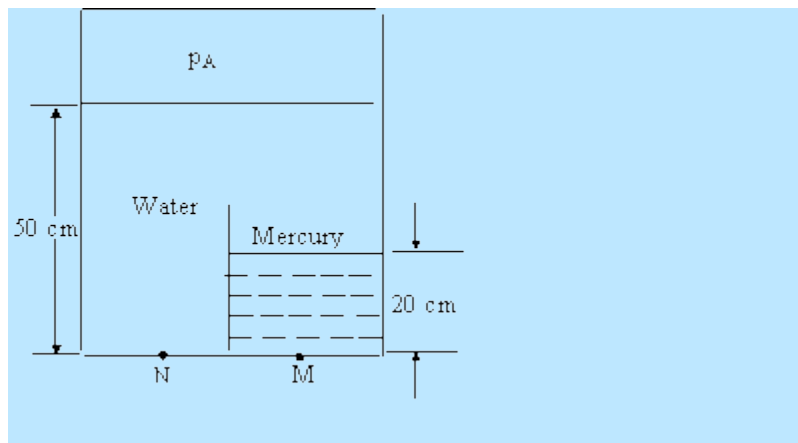


Fig 5.10

2. Consider the pipe and manometer system as shown in Fig 5.11. The pipe contains water. Find the value of manometer reading  $h$ , and the difference in pressure between A and B if there is no flow. If there is a flow from A towards B and the manometer reading is  $h = 60$  mm, then determine the static pressure difference  $p_A - p_B$

[0, 2.94 kPa; 3.53 kPa]

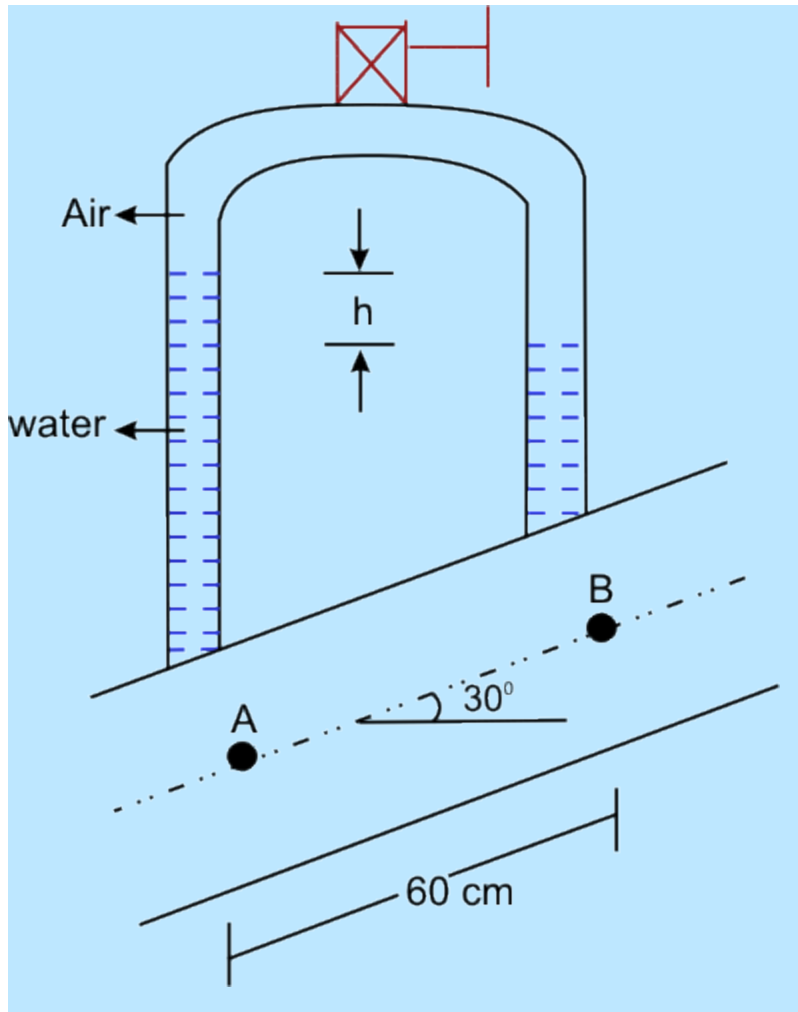


Fig 5.11

3. Determine the air pressure above the water surface in the tank if a force of  $8\text{ kN}$  is required to hold the hinged door in position as shown in Fig 5.12.

[10.76 kPa]

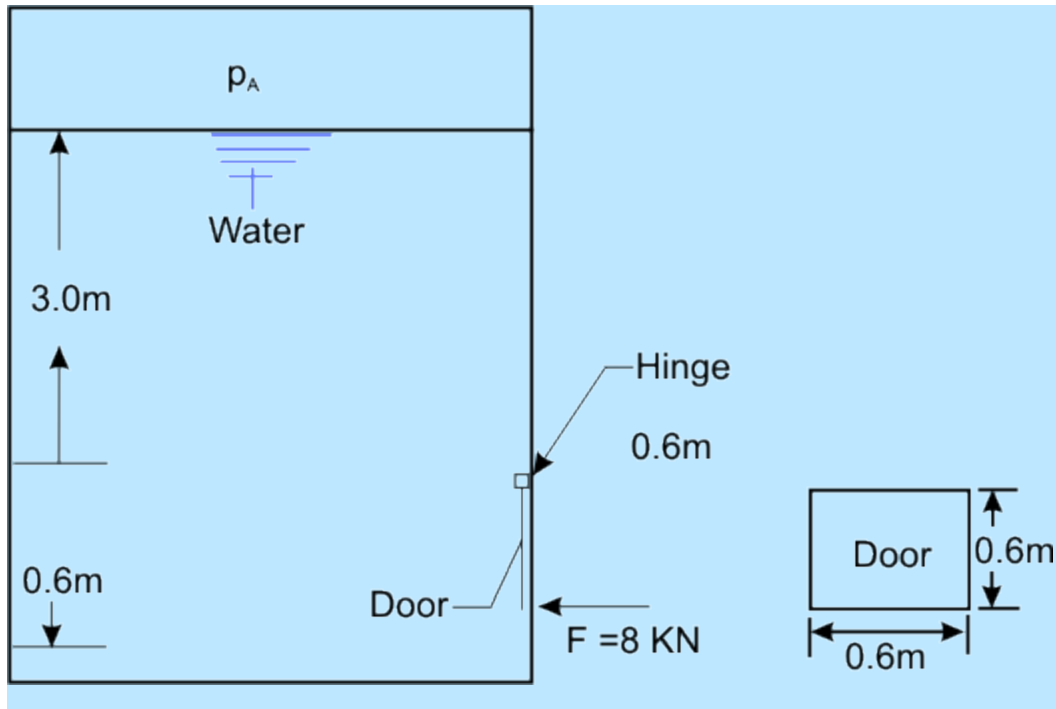


Fig 5.12

4. The profile of the inner face of a dam takes the form of a parabola with the equation  $18y = x^2$ , where  $y$  is the height above the base and  $x$  is the horizontal distance of the face from the vertical reference line. The water level is 27m above the base. Determine the thrust on the dam (per meter with) due to the water pressure, its inclination to the vertical and the point where the line of action of this force

[ 5.28 MN/m,  $42^\circ 33'$ , 30.29 m from face ]

intersects the free water surface

5. A solid uniform cylinder of length 150 mm and diameter 75 mm is to float upright in water. Determine

[ 0.641 kg and 0.663 kg ]

the limits within which its mass should lie.

6. A long prism, the cross-section of which is an equilateral triangle of side **a**, floats in water with one side horizontal and submerged to a depth **h**. Find

(a) **h/a** as a function of the specific gravity, **S** of the prism.

(b) The metacentric height in terms of side **a**, for small angle of rotation if specific gravity, **S=0.8**.

7. A metal sphere of volume  $V_m = 0.1 \text{ m}^3$ , specific gravity  $s_m = 2$  and fully immersed in water is attached by a flexible wire to a buoy of volume  $V_B = 1 \text{ m}^3$  and specific gravity  $s_B = 0.1$ . Calculate the tension **T** in the wire and volume of the buoy that is submerged. Refer to Fig 5.13.

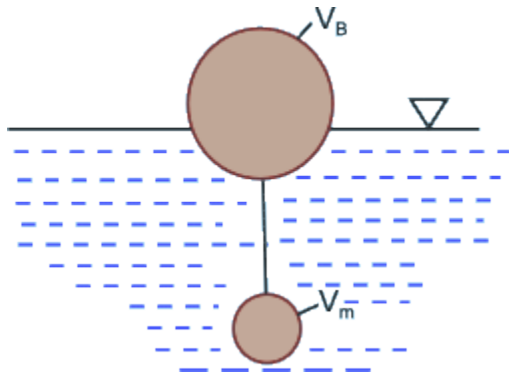


Fig 5.13