Principles of Physical Similarity - An Introduction

Laboratory tests are usually carried out under altered conditions of the operating variables from the actual ones in practice. These variables, in case of experiments relating to fluid flow, are pressure, velocity, geometry of the working systems and the physical properties of the working fluid.

The pertinent questions arising out of this situation are:

1. How to apply the test results from laboratory experiments to the actual problems?

2. Is it possible, to reduce the large number of experiments to a lesser one in achieving the same objective?

Answer of the above two questions lies in the principle of physical similarity. This principle is useful for the following cases:

1. To apply the results taken from tests under one set of conditions to another set of conditions

and

2. To predict the influences of a large number of independent operating variables on the performance of a system from an experiment with a limited number of operating variables.

Concept and Types of Physical Similarity

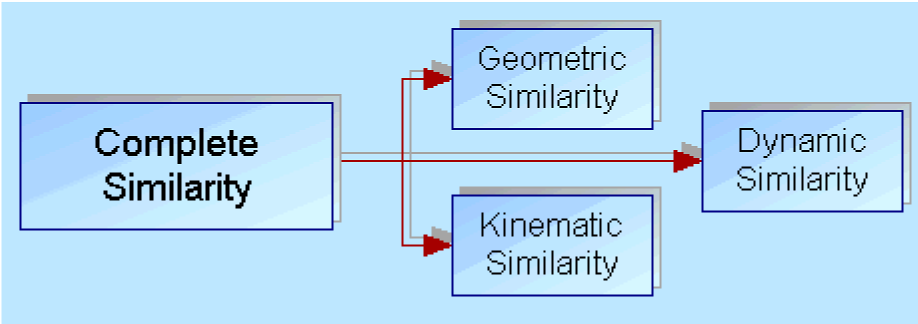
The primary and fundamental requirement for the **physical similarity** between two problems is that the **physics of the problems must be the same**.

For an example, two flows: one governed by viscous and pressure forces while the other by gravity force cannot be made physically similar. Therefore, the laws of similarity have to be sought between problems described by the same physics.

Definition of physical similarity as a general proposition.

Two systems, described by the same physics, operating under different sets of conditions are said to be physically similar in respect of certain specified physical quantities; when the ratio of corresponding magnitudes of these quantities between the two systems is the same everywhere.

In the field of mechanics, there are three types of similarities which constitute the complete similarity between problems of same kind.



Geometric Similarity : If the specified physical quantities are geometrical dimensions, the similarity is called Geometric Similarty,

Kinematic Similarity : If the quantities are related to motions, the similarity is called Kinematic Similarity

Dynamic Similarity : If the quantities refer to forces, then the similarity is termed as Dynamic Similarity.

Geometric Similarity

- Geometric Similarity implies the similarity of shape such that, the **ratio of any length in one system to the corresponding length in other system is the same everywhere**.

- This ratio is usually known as **scale factor**.

Therefore, geometrically similar objects are similar in their shapes, i.e., proportionate in their physical dimensions, but differ in size.

In investigations of physical similarity,

· the full size or **actual scale systems** are known as **prototypes**

· the **laboratory scale systems** are referred to as **models**

· use of the same fluid with both the prototype and the model is not necessary

· model need not be necessarily smaller than the prototype. The flow of fluid through an injection nozzle or a carburettor , for example, would be more easily studied by using a model much larger than the prototype.

· the model and prototype may be of identical size, although the two may then differ in regard to other factors such as velocity, and properties of the fluid.

If l_1 and l_2 are the two characteristic physical dimensions of any object, then the requirement of geometrical similarity is

$$\frac{l_{1m}}{l_{1p}} = \frac{l_{2m}}{l_{2p}} = l_r$$

(**model ratio**)

(The second suffices *m* and *p* refer to model and prototype respectively) where l_r is the scale factor or sometimes known as the model ratio. Figure 5.1 shows three pairs of geometrically similar objects, namely, a right circular cylinder, a parallelopiped, and a triangular prism.

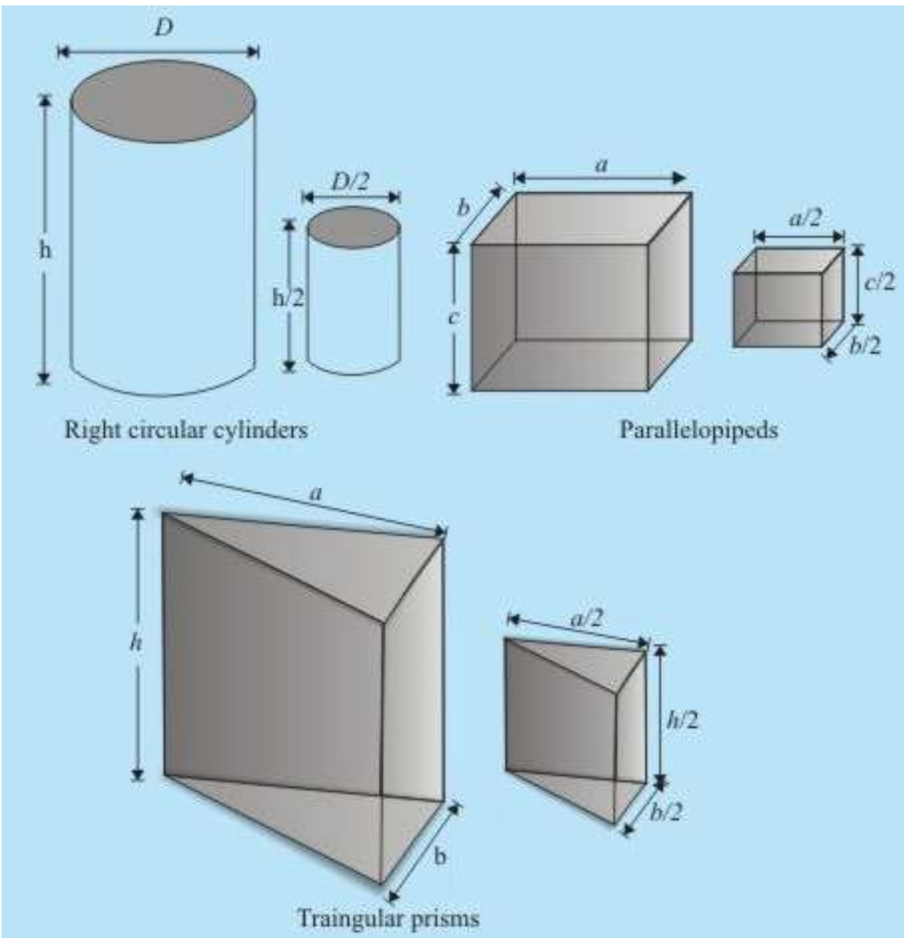


Fig 17.1 Geometrically Similar Objects

In all the above cases model ratio is 1/2

Geometric similarity is perhaps the most obvious requirement in a model system designed to correspond to a given prototype system.

A perfect geometric similarity is not always easy to attain. **Problems in achieving perfect geometric similarity** are:

· For a small model, the surface roughness might not be reduced according to the scale factor (unless the model surfaces can be made very much smoother than those of the prototype). If for any reason the scale factor is not the same throughout, a distorted model results.

· Sometimes it may so happen that to have a perfect geometric similarity within the available laboratory space, physics of the problem changes. For example, in case of large prototypes, such as rivers, the size of the model is limited by the available floor space of the laboratory; but if a very low scale factor is used in reducing both the horizontal and vertical lengths, this may result in a stream so shallow that surface tension has a considerable effect and, moreover, the flow may be laminar instead of turbulent. In this situation, a distorted model may be unavoidable (a lower scale factor ”for horizontal lengths while a relatively higher scale factor for vertical lengths. The extent to which perfect geometric similarity should be sought therefore depends on the problem being investigated, and the accuracy required from the solution.

Kinematic Similarity

Kinematic similarity refers to **similarity of motion.**

Since motions are described by distance and time, it implies **similarity of lengths (i.e., geometrical similarity**) and, in addition**, similarity of time intervals**.

If the corresponding lengths in the two systems are in a fixed ratio, the velocities of corresponding particles must be in a fixed ratio of magnitude of corresponding time intervals.

If the ratio of corresponding lengths, known as the **scale factor, is l_r** and **the ratio of corresponding time intervals is t_r**, then the magnitudes of corresponding **velocities are in the ratio l_r/t_r** and the magnitudes of corresponding **accelerations are in the ratio l_r/t^2_r**.

A well-known **example** of kinematic similarity is found in a planetarium. Here the galaxies of stars and planets in space are reproduced in accordance with a certain length scale and in simulating the motions of the planets, a fixed ratio of time intervals (and hence velocities and accelerations) is used.

When fluid motions are kinematically similar, the **patterns formed by streamlines are geometrically similar** at corresponding times.

Since the impermeable boundaries also represent streamlines, **kinematically similar flows are possible only past geometrically similar boundaries.**

Therefore, **geometric similarity is a necessary condition for the kinematic similarity** to be achieved, but not the sufficient one.

For example, geometrically similar boundaries may ensure geometrically similar streamlines in the near vicinity of the boundary but not at a distance from the boundary.

Dynamic Similarity

Dynamic similarity is the **similarity of forces** .

In dynamically similar systems, the **magnitudes of forces** at correspondingly similar points in each system are **in a fixed ratio.**

In a system involving flow of fluid, different forces due to different causes may act on a fluid element. These forces are as follows:

Viscous Force (due to viscosity)	$\vec{F_v}$
Pressure Force (due to different in pressure)	$\vec{F_p}$
Gravity Force (due to gravitational attraction)	$\vec{F_g}$
Capillary Force (due to surface tension)	$\vec{F_c}$
Compressibility Force (due to elasticity)	$\vec{F_e}$

According to Newton 's law, the resultant F_R of all these forces, will cause the acceleration of a fluid element. Hence

$$\vec{F_R} = \vec{F_v} + \vec{F_p} + \vec{F_g} + \vec{F_c} + \vec{F_e} \quad (17.1)$$

Moreover, the **inertia force** $\vec{F_i}$ **is defined as equal and opposite to the resultant accelerating force** $\vec{F_R}$

$\vec{F_i} = -\vec{F_R}$

Therefore Eq. 17.1 can be expressed as

$$\vec{F_v} + \vec{F_p} + \vec{F_g} + \vec{F_c} + \vec{F_e} + \vec{F_i} = 0$$

For dynamic similarity, the magnitude ratios of these forces have to be same for both the prototype and the model. **The inertia force** $\vec{F_i}$ **is usually taken as the common one to describe the ratios** as (or putting in other form we equate the the non dimensionalised forces in the two systems)

$$\frac{|\vec{F_v}|}{|\vec{F_i}|}, \frac{|\vec{F_p}|}{|\vec{F_i}|}, \frac{|\vec{F_g}|}{|\vec{F_i}|}, \frac{|\vec{F_c}|}{|\vec{F_i}|}, \frac{|\vec{F_e}|}{|\vec{F_i}|}$$

Magnitudes of Different Forces

A fluid motion, under all such forces is characterised by

1. Hydrodynamic parameters like pressure, velocity and acceleration due to gravity,
2. Rheological and other physical properties of the fluid involved, and
3. Geometrical dimensions of the system.

It is important to express the magnitudes of different forces in terms of these parameters, to know the extent of their influences on the different forces acting on a flluid element in the course of its flow.

Inertia Force $\vec{F}_i$

- The inertia force acting on a fluid element is equal in magnitude to the mass of the element multiplied by its acceleration.
- The mass of a fluid element is proportional to ρl^3 where, ρ is the density of fluid and l is the characteristic geometrical dimension of the system.
- The acceleration of a fluid element in any direction is the rate at which its velocity in that direction changes with time and is therefore proportional in magnitude to some characteristic velocity V divided by some specified interval of time t. The time interval t is proportional to the characteristic length l divided by the characteristic velocity V, so that the acceleration becomes proportional to V^2/l.

The magnitude of inertia force is thus proportional to

$$\boxed{\rho l^3 V^2 / l = \rho l^2 V^2}$$

This can be written as,

$$\left|\vec{F}_i\right| \propto \rho l^2 V^2 \qquad (18.1a)$$

Viscous Force $\vec{F}_v$

The viscous force arises from shear stress in a flow of fluid.

Therefore, we can write

Magnitude of viscous force $\vec{F}_v$ = shear stress X surface area over which the shear stress acts

Again, shear stress = μ (viscosity) X rate of shear strain

where, rate of shear strain $\propto$ velocity gradient $\propto \frac{V}{l}$ and surface area $\propto l^2$

Hence

$$\left|\vec{F}_v\right| \propto \mu \frac{V}{l} l^2$$

$$\propto \mu V l \qquad (18.1b)$$

Pressure Force $\vec{F}_p$

The pressure force arises due to the difference of pressure in a flow field.

Hence it can be written as

$$\left|\vec{F}_p\right| \propto \Delta p\, l^2 \qquad (18.1c)$$

(where, Dp is some characteristic pressure difference in the flow.)

Gravity Force $\vec{F}_g$

The gravity force on a fluid element is its weight. Hence,

$$\left|\vec{F}_g\right| \propto p l^3 g \qquad (18.1d)$$

(where g is the acceleration due to gravity or weight per unit mass)

Capillary or Surface Tension Force $\vec{F}_c$

The capillary force arises due to the existence of an interface between two fluids.

- The surface tension force acts tangential to a surface .
- It is equal to the coefficient of surface tension σ multiplied by the length of a linear element on the surface perpendicular to which the force acts.

Therefore,

$$\left|\vec{F}_c\right| \propto \sigma l \qquad (18.1e)$$

Compressibility or Elastic Force $\vec{F}_e$

Elastic force arises due to the compressibility of the fluid in course of its flow.

- For a given compression (a decrease in volume), the increase in pressure is proportional to the bulk modulus of elasticity E
- This gives rise to a force known as the elastic force.

Hence, for a given compression $\Delta \mathbf{p} \ \alpha \ \mathbf{E}$

$$\left|\vec{F}_e\right| \propto El^2 \qquad (18.1f)$$

The flow of a fluid in practice does not involve all the forces simultaneously.

Therefore, the pertinent dimensionless parameters for dynamic similarity are derived from the ratios of significant forces causing the flow.

Dynamic Similarity of Flows governed by Viscous, Pressure and Inertia Forces

The criterion of dynamic similarity for the **flows controlled by viscous, pressure and inertia forces** are derived from the ratios of the representative magnitudes of these forces with the help of Eq. (18.1a) to (18.1c) as follows:

$$\frac{\text{Viscous force}}{\text{Inertia Force}} = \frac{\left|\vec{F}_v\right|}{\left|\vec{F}_i\right|} \propto \frac{\mu V l}{\rho V^2 l^2} = \frac{\mu}{\rho V l} \qquad (18.2a)$$

|

$$\frac{\text{Pressure force}}{\text{Inertia Force}} = \frac{\left|\vec{F}_p\right|}{\left|\vec{F}_i\right|} \propto \frac{\Delta p l^2}{\rho V^2 l^2} = \frac{\Delta p}{\rho V^2} \qquad (18.2b)$$

The term $\rho l V / \mu$ **is known as Reynolds number, Re** after the name of the scientist who first developed it and is thus proportional to the magnitude ratio of inertia force to viscous force .(Reynolds number plays a vital role in the analysis of fluid flow)

The term $\Delta p / \rho V^2$ **is known as Euler number, Eu** after the name of the scientist who first derived it. The dimensionless terms Re and Eu represent the critieria of dynamic similarity for the flows which are affected only by viscous, pressure and inertia forces. Such instances, for example, are

1. the full flow of fluid in a completely closed conduit,
2. flow of air past a low-speed aircraft and

3. the flow of water past a submarine deeply submerged to produce no waves on the surface.

Hence, for a complete dynamic similarity to exist between the prototype and the model for this class of flows, the Reynolds number, Re and Euler number, Eu have to be same for the two (prototype and model). Thus

$$\frac{\rho_p\, l_p V_p}{\mu_p} = \frac{\rho_m\, l_m V_m}{\mu_m} \tag{18.2c}$$

$$\frac{\Delta P_p}{\rho_p V_p^2} = \frac{\Delta P_m}{\rho_m V_m^2} \tag{18.2d}$$

where, the suffix p and suffix m refer to the parameters for prototype and model respectively.

In practice, the pressure drop is the dependent variable, and hence it is compared for the two systems with the help of Eq. (18.2d), while the equality of Reynolds number (Eq. (18.2c)) along with the equalities of other parameters in relation to kinematic and geometric similarities are maintained.

- The characteristic geometrical dimension l **and** the reference velocity **V** in the expression of the Reynolds number **may be any geometrical dimension and any velocity which are significant in determining the pattern of flow.**
- For internal flows through a closed duct, the hydraulic diameter of the duct D_h and the average flow velocity at a section are invariably used for l and V respectively.
- The hydraulic diameter D_h is defined as D_h= 4A/P where A and P are the cross-sectional area and wetted perimeter respectively.

Dynamic Similarity of Flows with Gravity, Pressure and Inertia Forces

A flow of the type in which **significant forces** are **gravity force, pressure force and inertia force,** is found **when a free surface is present.**

Examples can be

1. the flow of a liquid in an open channel.
2. the wave motion caused by the passage of a ship through water.
3. the flows over weirs and spillways.

The condition for dynamic similarity of such flows requires

- the equality of the Euler number Eu (the magnitude ratio of pressure to inertia force), and

- the equality of the magnitude ratio of gravity to inertia force at corresponding points in the systems being compared.
  Thus ,

$$\frac{\text{Gravity force}}{\text{Inertia Force}} = \frac{|\vec{F}_g|}{|\vec{F}_i|} \propto \frac{\rho l^3 g}{\rho V^2 l^2} = \frac{lg}{V^2} \qquad (18.2e)$$

- In practice, it is often convenient to use the square root of this ratio so to deal with the first power of the velocity.

- From a physical point of view, equality of $(lg)^{1/2}/V$ implies equality of lg/V^2 as regard to the concept of dynamic similarity.

The **reciprocal of the term** $(lg)^{1/2}/V$ **is known as Froude number (** after William Froude who first suggested the use of this number in the study of naval architecture.)

Hence **Froude number,** $Fr = V/(lg)^{1/2}$ **.**

Therefore, **the primary requirement for dynamic similarity between the prototype and the model involving flow of fluid with gravity as the significant force, is the equality of Froude number, Fr,** i.e.,

$$\frac{(l_p g_p)^{1/2}}{V_p} = \frac{(l_m g_m)^{1/2}}{V_m} \qquad (18.2f)$$

Dynamic Similarity of Flows with Surface Tension as the Dominant Force

Surface tension forces are important in certain classes of practical problems such as ,

1. flows in which capillary waves appear
2. flows of small jets and thin sheets of liquid injected by a nozzle in air
3. flow of a thin sheet of liquid over a solid surface.

Here the significant parameter for dynamic similarity is the magnitude ratio of the surface tension force to the inertia force.

This can be written as $$\frac{|\vec{F}_e|}{|\vec{F}_i|} \propto \frac{\sigma l}{\rho V^2 l^2} = \frac{\sigma}{\rho V^2 l} \qquad (18.2g)$$

The term $\sigma/\rho V^2 l$ **is usually known as Weber number, Wb** (after the German naval architect Moritz Weber who first suggested the use of this term as a relevant parameter.)

Thus for dynamically similar flows $(Wb)_m = (Wb)_p$

i.e., $$\frac{\sigma_m}{\rho_m V_m^2 L_m} = \frac{\sigma_p}{\rho_p V_p^2 L_p^2}$$

Dynamic Similarity of Flows with Elastic Force

When the compressibility of fluid in the course of its flow becomes significant, the elastic force along with the pressure and inertia forces has to be considered.

Therefore, the magnitude ratio of inertia to elastic force becomes a relevant parameter for dynamic similarity under this situation.

Thus we can write,

$$\boxed{\frac{\text{Inertia force}}{\text{Elastic Force}} = \frac{|\vec{F}_i|}{|\vec{F}_e|} \propto \frac{\rho V^2 l^2}{E l^2} = \frac{\rho V^2}{E}} \qquad (18.2h)$$

The parameter $\rho V^2/E$ **is known as Cauchy number** ,(after the French mathematician A.L. Cauchy)

If we consider the **flow to be isentropic** , then it can be written

$$\frac{|\vec{F}_i|}{|\vec{F}_e|} \propto \frac{\rho V^2}{E_s} \qquad (18.2i)$$

(where E_s is the isentropic bulk modulus of elasticity)

Thus for dynamically similar flows $(cauchy)_m = (cauchy)_p$

ie., $\frac{\rho_m V_m^2}{(E_s)_m} = \frac{\rho_p V_p^2}{(E_s)_p}$

- The velocity with which a sound wave propagates through a fluid medium equals to $\sqrt{E_s/\rho}$.
- Hence, the term $\rho V^2/E_s$ can be written as V^2/a^2 where **a is the acoustic velocity** in the fluid medium.

The ratio V/a is known as Mach number, Ma (after an Austrian physicist Earnst Mach)

It has been shown in Chapter 1 that the effects of compressibility become important when the Mach number exceeds 0.33.

The situation arises in the flow of air past high-speed aircraft, missiles, propellers and rotory compressors. In these cases equality of Mach number is a condition for dynamic similarity. Therefore,

$(Ma)_p=(Ma)_m$

i.e.

$$\boxed{V_p/a_p = V_m/a_m} \qquad (18.2j)$$

Ratios of Forces for Different Situations of Flow

Pertinent Dimensionless term as the croterion of dynamic similarity in different situations of fluid flow	Representative magnitude ration of the forces	Name	Recommended symbol
$\rho l V/\mu$	$\frac{\text{Inertia force}}{\text{Viscous force}}$	Reynolds number	Re
$\Delta p/\rho V^2$	$\frac{\text{Pressure force}}{\text{Inertia force}}$	Euler number	Eu

$V/(lg)^{1/2}$	$\frac{\text{Inertia force}}{\text{Gravity force}}$	Froude number	Fr
$\sigma/\rho V^2 l$	$\frac{\text{Surface Tesion force}}{\text{Inertia force}}$	Weber number	Wb
$V/\sqrt{E_s/\rho}$	$\frac{\text{Inertia force}}{\text{Elastic force}}$	Mach number	Ma

The Application of Dynamic Similarity - The Dimensional Analysis

The concept:

A physical problem may be characterised by a group of dimensionless similarity parameters or variables rather than by the original dimensional variables.

This gives a clue to the reduction in the number of parameters requiring separate consideration in an experimental investigation.

For an **example,** if the Reynolds number $Re = \rho V D_h / \mu$ is considered as the independent variable, in case of a flow of fluid through a closed duct of hydraulic diameter D_h, then a change in Re may be caused through a change in flow velocity V only. Thus a range of Re can be covered simply by the variation in V without varying other independent dimensional variables ρ, D_h and μ.

In fact, the variation in the Reynolds number physically implies the variation in any of the dimensional parameters defining it, though the change in Re, may be obtained through the variation in anyone parameter, say the velocity V.

A number of such **dimensionless parameters** in relation to dynamic similarity are shown in Table 5.1. Sometimes it becomes diffcult to derive these parameters straight forward from an estimation of the representative order of magnitudes of the forces involved. An alternative **method of determining these dimensionless parameters by a mathematical technique is known as dimensional analysis** .

The Technique:

The requirement of dimensional homogeneity imposes conditions on the quantities involved in a physical problem, and these restrictions, placed in the form of an algebraic function by the requirement of dimensional homogeneity, play the central role in dimensional analysis.

There are two existing approaches;

- one due to Buckingham known as **Buckingham's pi theorem**
- other due to Rayleigh known as **Rayleigh's Indicial method**

In our next slides we'll see few examples of the dimensions of physical quantities.

Dimensions of Physical Quantities

All physical quantities are expressed by magnitudes and units.

*For **example*** , the velocity and acceleration of a fluid particle are 8m/s and $10m/s^2$ respectively. Here the dimensions of velocity and acceleration are ms^{-1} and ms^{-2} respectively.

In SI (System International) units, the primary physical quantities which are assigned base dimensions are the mass, length, time, temperature, current and luminous intensity. Of these, the first four are used in fluid mechanics and they are symbolized as M (mass), L (length), T (time), and θ (temperature).

- Any physical quantity can be expressed in terms of these primary quantities by using the basic mathematical definition of the quantity.
- The resulting expression is known as the dimension of the quantity.

Let us take some **examples:**

1. **Dimension of Stress**

Shear stress τ is defined as force/area. Again, force = mass × acceleration

Dimensions of acceleration = Dimensions of velocity/Dimension of time.

$$= \frac{\text{Dimension of Distance}}{(\text{Dimension of Time})^2}$$

$$= \frac{L}{T^2}$$

Dimension of area = (Length)^2 =L^2

Hence, dimension of shear stress

$$\tau = \left(ML/T^2\right)/L^2 = ML^{-1}T^{-2} \quad (19.1)$$

2. Dimension of Viscosity

Consider Newton's law for the definition of viscosity as

$$\tau = \mu du/dy$$

or,

$$\mu = \frac{\tau}{(du/dy)}$$

The dimension of velocity gradient du/dy can be written as

dimension of du/dy= dimension of u/dimension of y = (L / T)/L = T $^{-1}$

The dimension of shear stress τ is given in Eq. (19.1).

Hence dimension of

$$\mu = \frac{\text{Dimension of } \tau}{\text{Dimension of du/dy}} = \frac{ML^{-1}T^{-2}}{T^{-1}}$$

$$= ML^{-1}T^{-1}$$

Dimensions of Various Physical Quantities in Tabular Format

Physical Quantity	Dimension
Mass	M
Length	L
Time	T

Temperature	θ
Velocity	LT^{-1}
Angular velocity	T^{-1}
Acceleration	LT^{-2}
Angular Acceleration	T^{-2}
Force, Thrust, Weight	MLT^{-2}
Stress, Pressure	$ML^{-1}T^{-2}$
Momentum	MLT^{-1}
Angular Momentum	$ML^{2}T^{-1}$
Moment, Torque	$ML^{2}T^{-2}$
Work, Energy	$ML^{2}T^{-2}$
Power	$ML^{2}T^{-3}$
Stream Function	$L^{2}T^{-1}$
Vorticity, Shear Rate	T^{-1}
Velocity Potential	$L^{2}T^{-1}$
Density	ML^{-3}
Coefficient of Dynamic Viscosity	$ML^{-1}T^{-1}$

Coefficient of Kinematic Viscosity	L^2T^{-1}
Surface Tension	MT^{-2}
Bulk Modulus of Elasticity	$ML^{-1}T^{-2}$

Buckingham's Pi Theorem

Assume, a physical phenomenon is described by **m number of independent variables like x_1 , x_2 , x_3 , ..., x_m**

The phenomenon may be expressed analytically by an implicit functional relationship of the controlling variables as

$$f(x_1, x_2, x_3, \ldots\ldots, x_m) = 0 \quad (19.2)$$

Now if **n be the number of fundamental dimensions like mass, length, time, temperature etc** ., involved in these m variables, then according to Buckingham's p theorem -

The phenomenon can be described in terms of (m - n) independent dimensionless groups like π_1 ,π_2 , ..., π_{m-n} , where p terms, represent the dimensionless parameters and consist of different combinations of a number of dimensional variables out of the m independent variables defining the problem.

Therefore. the analytical version of the phenomenon given by Eq. (19.2) can be reduced to

$$\boxed{F(\pi_1, \pi_2, \ldots\ldots\ldots, \pi_{m-n}) = 0} \quad (19.3)$$

according to Buckingham's pi theorem

- This physically implies that the **phenomenon** which is basically described by m independent dimensional variables, **is ultimately controlled by (m-n) independent dimensionless parameters known as π terms.**

Alternative Mathematical Description of (π) Pi Theorem

A physical problem described by m number of variables involving n number of fundamental dimensions (n < m) leads to a system of n linear algebraic equations with m variables of the form

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots a_{1m}x_m = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots a_{2m}x_m = b_2$$

....

$$a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots a_{nm}x_m = b_n \qquad (19.4)$$

or in a matrix form,

$$\boxed{Ax = b} \qquad (19.5)$$

where, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_m \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ .. \\ b_m \end{bmatrix}$$

Determination of π terms

- A group of n (n = number of fundamental dimensions) variables out of m (m = total number of independent variables defining the problem) variables is first chosen to form a basis so that all n dimensions are represented . These n variables are referred to as repeating variables.
- Then the p terms are formed by the product of these repeating variables raised to arbitrary unknown integer exponents and anyone of the excluded (m -n) variables.

*For **example*** , if x_1 x_2 ...x_n are taken as the repeating variables. Then

$$\pi_2 = x_1^{a_1} x_2^{a_2} \ldots\ldots x_n^{a_n} x_{n+2}$$

$$\pi_1 = x_1^{a_1} x_2^{a_2} \ldots\ldots x_n^{a_n} x_{n+1}$$

$$\pi_{m-n} = x_1^{a_1} x_2^{a_2} \ldots\ldots x_n^{a_n} x_m$$

- The sets of integer exponents **a_1, a_2 . . . a_n are different for each p term.**
- Since p terms are dimensionless, it requires that when all the variables in any p term are expressed in terms of their fundamental dimensions, the exponent of all the fundamental dimensions must be zero.
- This leads to a system of n linear equations in a, a_2 . . . a_n which gives a unique solution for the exponents. This gives the values of a_1 a_2 . . . a_n for each p term and hence the p terms are uniquely defined.

In selecting the repeating variables, the following points have to be considered:

1. The repeating variables must include among them all the n fundamental dimensions, not necessarily in each one but collectively.
2. The dependent variable or the output parameter of the physical phenomenon should not be included in the repeating variables.

No physical phenomena is represented when -

- **m < n** because there is no solution **and**
- **m = n** because there is a unique solution of the variables involved and hence all the parameters have fixed values.

. **Therefore all feasible phenomena are defined with m > n .**

- **When m = n + 1,** then, according to the Pi theorem, the number of pi term is one and the phenomenon can be expressed as

$$f(\pi_1) = 0$$

where, the non-dimensional term π_1 is some specific combination of n + 1 variables involved in the problem.

When m > n+ 1 ,

1. the number of π terms are more than one.
2. A number of choices regarding the repeating variables arise in this case.

Again, it is true that if one of the repeating variables is changed, it results in a different set of π terms. Therefore the interesting question is **which set of repeating variables is to be chosen** , to arrive at the correct set of π terms to describe the problem. The **answer to this question lies in the fact that different sets of π terms resulting from the use of different sets of repeating variables are not independent. Thus, anyone of such interdependent sets is meaningful in describing the same physical phenomenon.**

From any set of such π terms, one can obtain the other meaningful sets from some combination of the π terms of the existing set without altering their total numbers (m-n) as fixed by the Pi theorem.

Rayleigh's Indicial Method

This alternative method is also b**ased on the fundamental principle of dimensional homogeneity** of physical variables involved in a problem.

Procedure-

1. The dependent variable is identified and expressed as a product of all the independent variables raised to an unknown integer exponent.
2. Equating the indices of n fundamental dimensions of the variables involved, n independent equations are obtained .
3. These n equations are solved to obtain the dimensionless groups.

Example

Let us illustrate this method by solving the pipe flow problem

. **Step 1 -** ----- Here, the dependent variable Δp/l can be written as

$$\frac{\Delta p}{l} = AV^a D_h^b \rho^c \mu^d$$

(where, A is a dimensionless constant.)

Step 2 -----Inserting the dimensions of each variable in the above equation, we obtain,

$$ML^{-2}T^{-2} = A(LT^{-1})^a (L)^b (ML^{-3})^c (ML^{-1}T^{-1})^d$$

Equating the indices of M, L, and T on both sides, we get **,**

c + d = 1
a + b - 3c - d = -2
-a - d = -2

Step 3 -----There are three equations and four unknowns. Solving these equations in terms of the unknown d, we have

a = 2- d
b = -d - 1
c = 1- d

Hence , we can be written

$$\frac{\Delta p}{l} = AV^{2-d} D_h^{-d-1} \rho^{1-d} \mu^d$$

$$\frac{\Delta p}{l} = \frac{AV^2 \rho}{D_h} \left(\frac{\mu}{VD_h \rho} \right)^d$$

or, $$\frac{\Delta p D_h}{l \rho V^2} = A \left(\frac{\mu}{VD_h \rho} \right)^d$$

Therefore we see that there are two independent dimensionless terms of the problem, namely,

- **Both Buckingham's method and Rayleigh's method of dimensional analysis determine only the relevant independent dimensionless parameters of a problem, but not the exact relationship between them.**

For example, the numerical values of A and d can never be known from dimensional analysis. They are found out from experiments.

If the system of equations is solved for the unknown c, it results,

$$\frac{\Delta p}{l} \frac{D_h^2}{V \mu} = A \left(\frac{VD_h \rho}{\mu} \right)^c$$

Therefore different interdependent sets of dimensionless terms are obtained with the change of unknown indices in terms of which the set of indicial equations are solved. This is similar to the situations arising with different possible choices of repeating variables in Buckingham's Pi theorem.

Exercise Problems

1. A 1/6 model automobile is tested in a wind tunnel with same air properties as the prototype. The prototype automobile runs on the roads at a velocity of 60 km/hr. For dynamically similar conditions,

the drag measured on the model is 500 N. Determine the drag of the prototype and the power required to overcome this drag.

(500N, 8.33 KN)

2. A model is built of a flow phenomenon which is governed by the action of gravity and surface tension force. Show that the length scale ratio which will ensure complete similarity between model and the prototype is $L_r = \sqrt{\sigma_r / \rho_r}$

3. The speed of propagation U of a capillary wave in deep water is known to be a function of density ρ , wave length λ , and surface tension α . Using Dimensional Analysis, find out a relationship of U with ρ , λ , and α . (b) For a given surface tension and wavelength, how does the propagation speed changes if the density is halved ?

(increased by a factor of $\sqrt{2}$)

4. A hydraulic jump occurring in a stilling basin is to be studied in a 1:36 scale model. The prototype jump has an initial velocity of 10 m/s, an entrance Froude number of 6.0 and a power loss of 2 kW per meter width of basin. Determine (a) the corresponding model velocity, (b) model Froude number and (c) power loss per meter width of the model

(a 1·67 m/s, (b) 6· 0, (c) 0·26 W)

5. A model of a reservoir having a free water surface within it is drained in 3 minutes by opening a sluice gate. The geometrical scale of the model is 1/100. How long would it take to empty the prototype?

(30 minutes)

6. Assuming that nothing is known about the particle motion under gravity beyond $x = G(v_0, g, t)$ where x , v_0 , *g*, and *t* are respectively the displacement, initial velocity, gravitational acceleration, and time. Perform a dimensional analysis to explain the situation.

7. The tensile force inside a pendulum is known to depend on the mass, length, period, and angular amplitude of the pendulum. Perform a dimensional analysis.

8. The pressure drop in pipe flows of liquids is found to depend on the time required to pass a volume of a given liquid through, on this volume, and on the density as well as the viscosity of the liquid. Perform a dimensional analysis in a step by step manner, with the pressure drop and the density displayed as leading quantities.

9. Using the long steps of dimensional analysis, reduce the relationship n = G (*g*, *A*, ρ , *M*) for the frequency n of the wing beat of a flying insect, where g stands for the gravitational acceleration; A, the wing area; ρ , the air density; and M, the mass of the insect. Choose *n* and *A* as the leading quantities.

10. The shape of a drop of liquid pulsates as it falls. The period of oscillation is observed to depend on the surface tension, the mean radius of the drop, and the liquid density. Perform a dimensional analysis to express the period of oscillation..

11. Liquid flows across an orifice loses useful power which is dependent on the liquid density and viscosity, the volume flow rate, and the orifice diameter. Perform a dimensional analysis (with the objective of analysing power loss).

12. During the flow through a pipe, it is observed that there exists a critical average flow velocity $\overline{V}_{cr}$, beyond which the flow becomes turbulent. It is also known that $\overline{V}_{cr}$ is influenced by the diameter of the pipe, the density and the viscosity of the fluid. Perform a dimensional analysis to explain the situation.