UNIT - 3.

Integral Calculus.

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Some important formulae:-

(1) $\int e^x dx = e^x$

(2) $\int \sin x\, dx = -\cos x$

(3) $\int \tan x\, dx = -\log \cos x$

(4) $\int \sec x\, dx = \log(\sec x + \tan x)$

(5) $\int \sec^2 x\, dx = \tan x$

(6) $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1}\frac{x}{a}$.

(7) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log \frac{a+x}{a-x}$

(8) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \frac{x-a}{x+a}$

(9) $\int \sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a}$

(10) $\int \sqrt{a^2+x^2}\, dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2} \sinh^{-1}\frac{x}{a}$

(11) $\int \sqrt{x^2-a^2}\, dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2} \cosh^{-1}\frac{x}{a}$

(12) $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx)$

(13) $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$

(14) $\int a^x dx = \frac{a^x}{\log_e a}$

(15) $\int \cos x\, dx = \sin x$

(16) $\int \cot x\, dx = \log \sin x$

(17) $\int \operatorname{cosec} x\, dx = \log(\operatorname{cosec} x - \cot x)$

(18) $\int \operatorname{cosec}^2 x\, dx = -\cot x$.

(19) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$

(20) $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a}$.

(21) $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a}$.

UNIT – III :-

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Topics to be covered in this unit :-

Reduction formulae
Application of integration to rectification
Qudrature
Volume of revolution
Centre of gravity

Reduction formulae:-

The integrals which cannot be evaluated by standard methods of integrating function of a single variable are evaluated by using reduction formulae.

Reduction formula for:-

(1) $\int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx$

(2) $\int \cos^n x\,dx = \frac{\sin x\cos^{n-1}x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx$

(3) $\int_0^{\pi/2} \sin^m x\cos^n x\,dx = \frac{(m-1)(m-3)\cdots(n-1)(n-3)}{(m+n)(m+n-2)(m+n-4)} \cdots \left(\frac{\pi}{2}\right)$ provided m, n are even.

(4) $\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - I_{n-2}$.

(5) $\int \sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$.

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(6) $\int x^n e^{ax}\, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$

(7) $\int x^m (\log x)^n\, dx = \frac{x^{m+1}}{m+1} (\log x)^n - \frac{n}{m+1} I_{m,n-1}$.

(8) $\int x^n \sin mx\, dx = -\frac{x^n \cos mx}{m} + \frac{n}{m^2} x^{n-1} \sin mx$

$- \frac{n(n-1)}{m^2} I_{n-2}$

(9) $\int x^n \cos mx\, dx = \frac{x^n \sin mx}{m} + \frac{n}{m^2} x^{n-1} \cos mx$

$- \frac{n(n-1)}{m^2} I_{n-2}$

(10) $\int \cos^m x \sin nx\, dx = -\frac{\cos^m x \cos nx}{m+n}$

$+ \frac{m}{m+n} I_{m-1,\, n-1}$

Proof of above formulae:-

(1) $\int \sin^n x\, dx$.

$= \int \sin^{n-1} x \sin x\, dx$. {Integrate by parts}.

$= \sin^{n-1} x(\cos x) - \int (n-1) \sin^{n-2} x \cos x(-\cos x) dx$

$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x(1 - \sin^2 x) dx$

$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$$

$$n\int \sin^n x\,dx = -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx$$

$$\therefore \int \sin^n x\,dx = \frac{-\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx.$$

similarly it can be proved for $\int \cos^n x\,dx$.

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(2) $\int_0^{\pi/2} \sin^m x\cos^n x\,dx.$

Soln Now for $\int \sin^m x\cos^n x\,dx.$

$$I_{m,n} = \int \sin^{m-1}x\cos^n x\sin x\,dx \quad \{\text{By parts}\}.$$

$$= \sin^{m-1}x\left(\frac{-\cos^{n+1}x}{n+1}\right) - \int (m-1)\sin^{m-2}x\cos x\left(\frac{-\cos^{n+1}x}{n+1}\right)dx$$

$$= -\frac{\sin^{m-1}x\cos^{n+1}x}{n+1} + \frac{m-1}{m+1}\int \sin^{m-2}x(1-\sin^2 x)\cos^n x\,dx$$

$$= -\frac{\sin^{m-1}x\cos^{n+1}x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2}x\cos^n x\,dx$$

$$-\frac{m-1}{n+1}\int \sin^m x\cos^n x\,dx.$$

$$I_{m,n} + \frac{m-1}{n+1}\int \sin^m x\cos^n x\,dx =$$

$$-\frac{\sin^{m-1}x\cos^{n+1}x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2}x\cos^n x\,dx.$$

$$\left(1+\frac{m-1}{n+1}\right)\int \sin^m x\cos^n x\,dx = -\frac{\sin^{m-1}x\cos^{n+1}x}{n+1}$$

$$+\frac{m-1}{n+1}\int \sin^{m-2}x\cos^n x\,dx.$$

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$$\therefore \int \sin^m x \cos^n x \, dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x \, dx.$$

$$\therefore \int_0^{\pi/2} \sin^m x \cos^n x \, dx$$

$$= \left\{ -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} \right\}_0^{\pi/2} + \frac{m-1}{m+n} \int_0^{\pi/2} \sin^{m-2} x \cos^n x \, dx.$$

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,\, n}.$$

where $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$.

③ $\int \tan^n x\,dx$

$$\int \tan^n x\,dx = \int \tan^{n-2} x \tan^2 x\,dx.$$

$$I_n = \int \tan^{n-2} x\,[\sec^2 x - 1]\,dx.$$

$$= \int \tan^{n-2} x \sec^2 x - \int \tan^{n-2} x\,dx.$$

$$\therefore \quad = \tan^{n-2} x \tan x - \int (n-2)\tan^{n-3} x \sec^2 x \tan x\,dx$$

$$= \tan^{n-2} x \tan x - (n-2)\int \tan^{n-2} x \sec^2 x\,dx$$

$$= \tan^{n-2} x \tan x - (n-2)\int \tan^{n-2} x\,(\tan^2 x + 1)\,dx.$$

$$= \tan^{n-2} x \tan x - (n-2)\int (\tan^n x + \tan^{n-2} x)\,dx$$

$$= \tan^{n-2} x \tan x - (n-2) I_n - (n-2) I_{n-2}$$

$$1 + (n-2) I_n = \tan^{n-1} x - (n-2) I_{n-2}$$

$$I_n = \frac{\tan^{n-1} x}{n-1} - \frac{n-2}{n-1} I_{n-2}.$$

Similarly it can be proved for $\int \sec^n x\,dx$ also.

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④ $\int x^n e^{ax}\,dx$.

Solⁿ $\int x^n e^{ax}\,dx$ {by parts}.

$$I_n = x^n \frac{e^{ax}}{a} - \int n x^{n-1} \frac{e^{ax}}{a}\,dx.$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1},$$

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(5) $\int x^m (\log x)^n \, dx$.

Sol^n Let $I_{m,n} = \int x^m (\log x)^n \, dx$.

By parts

$$= (\log x)^n \cdot \frac{x^{m+1}}{m+1} - \int n (\log x)^{n-1} \frac{1}{x} \frac{x^{m+1}}{m+1} \, dx$$

$$= \frac{x^{m+1}}{m+1} (\log x)^n - \frac{n}{m+1} \int x^m (\log x)^{n-1} \, dx$$

$$\therefore I_{m,n} = \frac{x^{m+1}}{m+1} (\log x)^n - \frac{n}{m+1} I_{m,n-1}.$$

(6) $\int x^n \sin mx \, dx$.

Sol^n Let $I_{m,n} = \int x^n \sin mx \, dx$

By parts

$$= x^n \left(\frac{-\cos mx}{m} \right) - \int n x^{n-1} \left(\frac{-\cos mx}{m} \right) dx.$$

$$= -\frac{x^n \cos mx}{m} + \frac{n}{m} \int x^{n-1} \cos mx \, dx.$$

Again by parts

$$= -\frac{x^n \cos mx}{m} + \frac{n}{m} \left\{ x^{n-1} \frac{\sin mx}{m} - \int (n-1) x^{n-2} \frac{\sin mx}{m} \, dx \right.$$

$$= \frac{-x^n \cos mx}{m} + \frac{n}{m^2} x^{n-1} \sin mx - \frac{n(n-1)}{m^2} I_{n-2}$$

Similarly it can be solved for $\int x^n \cos mx \, dx$.

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(7) $\int \cos^m x \sin nx \, dx$

Solⁿ Let $I_{m,n} = \int \cos^m x \sin^n x \, dx$.

By parts :-

$$= -\cos^m x \frac{\cos nx}{n} - \int m \cos^{m-1} x (-\sin x) \left(-\frac{\cos nx}{n}\right) dx$$

$$= \frac{-1}{n} \cos^m x \cos nx - \frac{m}{n} \int \cos^{m-1} x \cos nx \sin x \, dx$$

since $\sin(n-1)x = \sin nx \cos x - \cos nx \sin x$

$\therefore \cos nx \sin x = \sin nx \cos x - \sin(n-1)x$

Substituting in above equation

$$= \frac{-1}{n} \cos^m x \cos nx - \frac{m}{n} \int \cos^{m-1} x \left\{ \sin nx \cos x - \sin(n-1)x \right\} dx$$

$$= \frac{-1}{n} \cos^m x \cos nx - \frac{m}{n} \left\{ I_{m,n} - I_{m-1,n-1} \right\}$$

$$\left(1 + \frac{m}{n}\right) I_{m,n} = -\frac{1}{n} \cos^m x \cos nx + I_{m-1,n-1}$$

$$I_{m,n} = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

PROBLEMS :-

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Q 1. Evaluate the integral.

$$\int_0^{2a} x^3 \sqrt{2ax - x^2}\, dx .$$

Sol^n Put $x = 2a\sin^2\theta$

$dx = 4a \sin\theta \cos\theta\, d\theta$

$$\therefore \int_0^{2a} x^3 \sqrt{2ax - x^2}\, dx$$

$$= \int_0^{\pi/2} (2a\sin^2\theta)^{7/2} \sqrt{2a} \cos\theta \; 4a\sin\theta\cos\theta\, d\theta$$

$$= 2^6 a^5 \int_0^{\pi/2} \sin^8\theta \cos^2\theta\, d\theta = 2^6 a^5 \frac{7\cdot 5\cdot 3\cdot 1\cdot 1}{10\cdot 8\cdot 6\cdot 4\cdot 2}\,\frac{\pi}{2}$$

$$\left\{ \text{Since } \int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{(m-1)(m-3)\cdots(n-1)(n-3)\cdots}{(m+n)(m+n-2)(m+n-4)\cdots}\,\frac{\pi}{2} \right.$$

where m, n are both even. $\}$.

$$= \frac{7\pi a^5}{8}$$

Q. Evaluate $\displaystyle\int_0^2 x^{5/2} \sqrt{2-x}\, dx .$

Sol^n Put $x = 2\sin^2\theta$.

$dx = 4\sin\theta\cos\theta\, d\theta$.

when $x = 0$, $\theta = 0$

$x = 2$ $\quad \theta = \pi/2$.

$$\therefore \int_0 x^{5/2}\sqrt{2-x}\,dx = \int_0 2^{5/2}\sin^5\theta\,\sqrt{2}\cos\theta \cdot 4\sin\theta\cos\theta\,d\theta$$

$$= 32\int_0^{\pi/2}\sin^6\theta\cos^2\theta\,d\theta$$

$$= \cancel{32}\cdot\frac{5\cdot 3\cdot 1\cdot 1}{\cancel{8}\cdot \cancel{6}_2\cdot \cancel{4}\cdot 2}\,\frac{\pi}{2}$$

$$= \frac{5}{8}\pi$$

Q. find the reduction formula for $\int e^{ax}\sin^n x\,dx$.

Sol^n Let $I_n = \int e^{ax}\sin^n x\,dx$. $\{$By parts$\}$

$$= \sin^n x\,\frac{e^{ax}}{a} - \int n\sin^{n-1}x\cos x\,\frac{e^{ax}}{a}\,dx$$

$$= \frac{e^{ax}\sin^n x}{a} - \frac{n}{a}\Big[\sin^{n-1}x\cos x\cdot\frac{e^{ax}}{a} - \int (n-1)\sin^{n-2}x\cos x^2 + \sin^{n-1}x(-\sin x)\frac{e^{ax}}{a}\,dx\Big]$$

$$= \frac{e^{ax}\sin^{n-1}x}{a^2}(a\sin x - n\cos x) + \frac{n}{a^2}\int (n-1)\sin^{n-2}x\,(1-\sin^2 x) - \sin^n x\Big]e^{ax}\,dx.$$

$$= \frac{e^{ax}\sin^{n-1}x}{a^2}(a\sin x - n\cos x) + \frac{n(n-1)}{a^2}I_{n-2} - \frac{n^2}{a^2}I_n.$$

$$I_n + \frac{n^2}{a^2} I_n = \frac{e^{ax}\sin^{n-1}x}{a}(a\sin x - n\cos x)$$

$$+ \frac{n(n-1)}{a^2} I_{n-2}$$

$$\therefore I_n = \frac{e^{ax}\sin^{n-1}x\,(a\sin x - n\cos x)}{a^2+n^2} + \frac{n(n-1)}{a^2+n^2} I_{n-2}$$

$\therefore$ Put $a=1$, $n=3$,

$$I_3 = \frac{e^x \sin^2 x\,(\sin x - 3\cos x)}{1^2+9} + \frac{3\cdot 2}{1+9} I_1.$$

Now Evaluating I_1.

$$I_n = \int e^{ax} \sin^n x\,dx.$$

$$I_1 = \int e^{ax} \sin x\,dx.$$

$$= \int e^{ax} \sin x\,dx.$$

$$= \sin x \frac{e^{ax}}{a} - \int \cos x \frac{e^{ax}}{a}\,dx.$$

$$= \frac{e^{ax}}{a} \sin x - \frac{1}{a}\left\{\cos x \frac{e^{ax}}{a} + \int \sin x \frac{e^{ax}}{a}\,dx\right\}$$

$$= \frac{e^{ax}}{a} \sin x - \frac{1}{a^2} e^{ax} \cos x + \frac{1}{a^2}\int \sin x\, e^{ax}\,dx$$

$$\left(1+\frac{1}{a^2}\right) I_1 = \frac{e^{ax}}{a} \sin x - \frac{1}{a^2} e^{ax} \cos x.$$

$$\therefore \frac{a}{1+a^2} e^{ax} \sin x - \frac{1}{1+a^2} e^{ax} \cos x.$$

$$= \frac{e^{ax}}{1+a^2}\left[a\sin x - \cos x\right].$$

$$\therefore \quad I_3 = \frac{e^x \sin^2 x(\sin x - 3\cos x)}{10} + \frac{3\cdot 2}{10}\left[\frac{e^x}{2}(\sin x - \cos x)\right].$$

Q. If $u_n = \int_0^{\pi/2} x^n \sin x \, dx$, prove that

$u_n + n(n-1)u_{n-2} = n(\pi/2)^{n-1}$. Hence evaluate U_2

Soln $\quad U_n = \int_0^{\pi/2} x^n \sin x \, dx$.

Integrating by parts

$$U_n = \left[x^n(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} nx^{n-1}(-\cos x)\,dx$$

$$= n\int_0^{\pi/2} x^{n-1}\cos x \, dx.$$

Again integrating by parts

$$U_n = n\left[\left(x^{n-1}\sin x\right)_0^{\pi/2} - (n-1)\int_0^{\pi/2} x^{n-2}\sin x \, dx\right]$$

$$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)U_{n-2}$$

$$\Rightarrow U_n + n(n-2)U_{n-2} = \left(\frac{\pi}{2}\right)^{n-1}\cdot n \longrightarrow (1)$$

Put $n=5$

$$U_5 = 5\left(\frac{1}{2}\pi\right)^4 - 5\cdot 4\, U_3$$

$\therefore$ Put $n = 3$ in Eqn (1)

$$U_3 = 3\left(\frac{\pi}{2}\right)^2 - 3\cdot 2\, U_1$$

Now since $U_n = \int_0^{\pi/2} x^n \sin x\, dx$

Put $n = 1$

$$U_1 = \int_0^{\pi/2} x \sin x\, dx$$

$$= \left\{x(-\cos x)\right\}_0^{\pi/2} - \int_0^{\pi/2} (-\cos x)\, dx$$

$$= \left[\sin x\right]_0^{\pi/2} = 1$$

$$\therefore\ U_3 = 3\left(\frac{\pi}{2}\right)^2 - 6 = \frac{3}{4}\pi^2 - 6$$

$$\therefore\ U_5 = \frac{5\pi^4}{16} - 15\pi^2 + 120$$

Q. If $I_n = \int_0^{\pi/2} x \cos^n x\, dx$,

prove that $I_n = \frac{n-1}{n} I_{n-2} - \frac{1}{n^2}$. Evaluate I_4.

Soln
$$I_n = \int_0^{\pi/2} x \cos^n x\, dx$$

By parts

$$I_n = \int_0^{\pi/2} \underset{\text{I}}{\underline{x \cos^{n-1} x}} \cdot \underset{\text{II}}{\underline{\cos x}}\, dx$$

$$= \left\{ (x \cos^{n-1} x) \cdot \sin x \right\}_0^{\pi/2} - \int_0^{\pi/2} \left\{ \cos^{n-1} x + x(n-1)\cos^{n-2} x\, (-\sin x) \right\} \sin x\, dx$$

$$= -\int_0^{\pi/2} \cancel{\cos^{n-1} x + x(n-1)\cos^{n-2} x - \sin^2 x\, dx}$$

$$= \left[(x\cos^{n-1} x) \sin x \right] + \int \cos^{n-1} x + x(n-1)\cos^{n-2} x \sin^2 x\, dx$$

$$\cancel{= x \sin x \cos^{n-1} x}$$

$$= \left[(x\cos^{n-1} x) \sin x \right]_0^{\pi/2} + \int_0^{\pi/2} \left[\cos^{n-1} x \sin x + x(n-1)\cos^{n-2} x \sin^2 x \right] dx$$

$$= -\int \cos^{n-1} x \sin x\, dx + \int x(n-1)\cos^{n-2} x \sin^2 x\, dx$$

$$I_n = -\int \cos^{n-1} x \sin x\, dx + \int x(n-1)\cos^{n-2} x\,(1-\cos^2 x)\, dx$$

$$= -\int \cos^{n-1} x \sin x\, dx + \int x(n-1)\cos^{n-2} x\, dx$$

$$- \int x(n-1)\cos^n x\, dx$$

$$I_n = -\int \cos^{n-1} x \sin x\, dx + (n-1) I_{n-2} - (n-1) I_n$$

$$[\cancel{1} + (n-\cancel{1})] I_n = (n-1) I_{n-2} - \int \cos^{n-1} x \sin x\, dx$$

$$I_n = \frac{(n-1)}{n} I_{n-2} - \frac{1}{n} \int \cos^{n-1} x \sin x\, dx$$

$$= \frac{(n-1)}{n} I_{n-2} - \frac{1}{n^2}$$

Q. If $I_{m,n} = \int_0 \cos^m x \cos nx\, dx$, prove that

$$I_{m,n} = \frac{m(m-1)}{m^2-n^2} I_{m-2,n}.$$

Solⁿ/ $$I_{m,n} = \int_0^{\pi/2} \cos^m x \cos nx\, dx.$$

$$= \left\{ \cos^m x \left(\frac{\sin nx}{n}\right)\right\}_0^{\pi/2} - \int_0^{\pi/2} -m\cos^{m-1} x \sin x \frac{\sin nx}{n}\, dx.$$

$$= \frac{m}{n} \int_0^{\pi/2} \cos^{m-1} x \cdot \sin x \cdot \sin nx\, dx.$$

By parts

$$= \frac{m}{n} \left[\left\{ -\cos^{m-1} x \sin x \cdot \frac{\cos nx}{n} \right\}_0^{\pi/2} - \int_0^{\pi/2} \left[\left\{ \cos^{m-1} x \cos x + (m-1)\cos^{m-2} x (-\sin x) \sin x \right\} \cdot \frac{\cos nx}{n} \right] \right]$$

$$= \frac{m}{n^2} \int_0^{\pi/2} \left[\cos^m x - (m-1)\cos^{m-2} x \sin^2 x \right] \cos nx\, dx.$$

$$= \frac{m}{n^2} \int_0^{\pi/2} \cos^m x \cos nx\, dx - \frac{m(m-1)}{n^2} \int_0^{\pi/2} \cos^{m-2} x \sin^2 x \cos nx\, dx$$

$$= \frac{m}{n^2} I_{m,n} - \frac{m(m-1)}{n^2} \int_0^{\pi/2} \cos^{m-2} x (1-\cos^2 x) \cos nx\, dx$$

$$= \frac{m}{n^2} I_{m,n} - \frac{m(m-1)}{n^2} \int_0^{\pi/2} \cos^{m-2} x \cos nx\, dx$$

$$+ \frac{m(m-1)}{n^2} \int_0^{\pi/2} \cos^m x \cos nx\, dx.$$

$$= \frac{m}{n^2} I_{m,n} - \frac{m(m-1)}{n^2} I_{m-2,n} + \frac{m(m-1)}{n^2} I_{m,n}$$

$$I_{m,n}\left\{1 - \frac{m}{n^2} - \frac{(m^2-m)}{n^2}\right\} = -\frac{m(m-1)}{n^2} I_{m-2,n}$$

$$I_{m,n}\left\{\frac{n^2 - \cancel{m} - m^2 + \cancel{m}}{n^2}\right\} = -\frac{m(m-1)}{n^2} I_{m-2,n}.$$

$$I_{m,n}\left\{\frac{n^2 - m^2}{n^2}\right\} = -\frac{m(m-1)}{n^2} I_{m-2,n}.$$

$$I_{m,n} = \frac{m(m-1)}{m^2 - n^2} I_{m-2,n}.$$

LENGTH OF CURVES (RECTIFICATION).

Rectification :- Process of finding length of an arc of a curve between two given points is called rectification.

x)

Cos x dx

I :- Length of the arc of the curve $y = f(x)$ between two points $x = a$ and $x = b$ is

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

II :- Length of arc of the curve $x = f(x)$ between the points where $y = a$ and $y = b$ is

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$$\int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

III Length of arc of the curve $x = f(t)$, $y = \phi(t)$ between points $t = a$ and $t = b$ is

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

IV Length of arc of curve $r = f(\theta)$ between points $\theta = \alpha$ and $\theta = \beta$ is

$$\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

V Length of the arc of the curve $\theta = f(r)$ between the points $r = a$ and $r = b$ is

$$\int_a^b \sqrt{1+\left(\frac{r\,d\theta}{dr}\right)^2}\, dr$$

PROBLEMS :-

Q: Find the perimeter of the loop of the curve $3ay^2 = x(x-a)^2$

Sol^n $y = \dfrac{\sqrt{x}\,(x-a)}{\sqrt{3a}}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{3a}}\left[\frac{3}{2}x^{1/2} - \frac{a}{2}x^{-1/2}\right]$$

$$= \frac{1}{2\sqrt{3a}}\,\frac{3x-a}{\sqrt{x}}$$

$$\text{Perimeter of loop} = 2\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

$$= 2\int_0^a \sqrt{1+\frac{(3x-a)^2}{12ax}}\,dx$$

$$= 2\int_0^a \frac{\sqrt{9x^2+6ax+a^2}}{\sqrt{12ax}}\,dx$$

$$= \frac{1}{\sqrt{3a}}\int_0^a \frac{3x+a}{\sqrt{x}}\,dx$$

$$\cancel{= \frac{1}{\sqrt{3a}}\int_0^a \frac{\sqrt{9x^2+6ax+a^2}}{\sqrt{}}}$$

$$= \frac{1}{\sqrt{3a}}\int_0^a \left(3x^{1/2} + ax^{-1/2}\right)dx$$

$$= \frac{1}{\sqrt{3a}}\left[\frac{3x^{3/2}}{3/2} + \frac{ax^{1/2}}{1/2}\right]_0^a$$

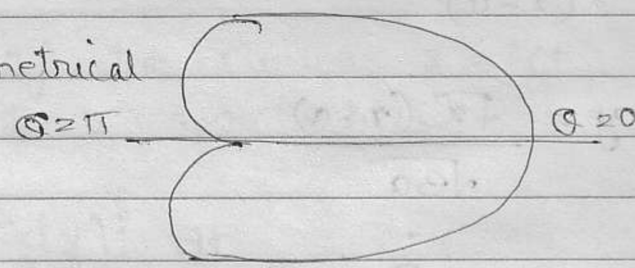
$$= \frac{4a}{\sqrt{3}}.$$

Q: find the entire length of the cardiod

$r = a(1+\cos\theta)$

Sol^n

Cardiod is symmetrical about intial line



So θ varies from 0 to π

$r = a(1+\cos\theta)$

$\dfrac{dr}{d\theta} = -a\sin\theta$

Length of curve $= 2\int_0^{\pi} \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}\, d\theta$

$= 2\int_0^{\pi} \sqrt{a(1+\cos\theta)^2 + (-a\sin\theta)^2}\, d\theta$

$= 2a \int_0^{\pi} \sqrt{2(1+\cos\theta)}\, d\theta$

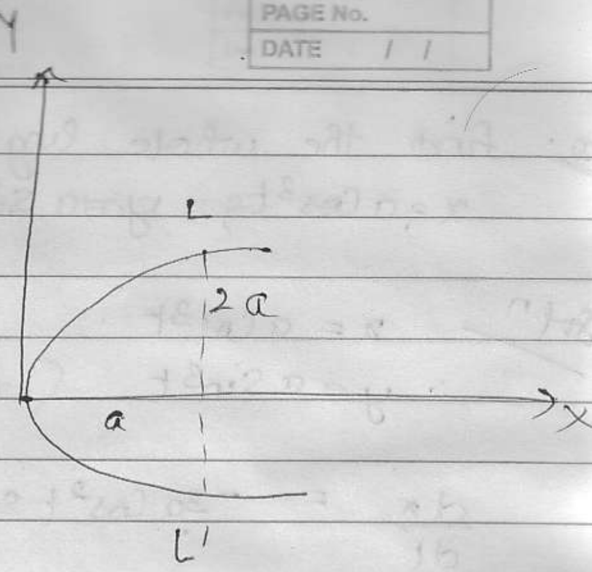
$= 4a \int_0^{\pi} \cos\frac{\theta}{2}\, d\theta$

$= 4a \left[\dfrac{\sin\frac{\theta}{2}}{\frac{1}{2}}\right]_0^{\pi}$

$= 8a.$

Q: Find the length of arc of parabola $y^2 = 4ax$ from the vertex to an end of the latus rectum.

Solⁿ. $y^2 = 4ax$



Vertex of parabola (0,0)
one extreme of latus rectum (a, 2a)

$y = 0$ to $y = 2a$

$\therefore\quad y^2 = 4ax$

$$2y = 4a\frac{dx}{dy}$$

$$\frac{dx}{dy} = \frac{y}{2a}$$

$$\therefore\quad S = \int_0^{2a} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_0^{2a} \sqrt{\left(1+\frac{y^2}{4a^2}\right)}\, dy$$

$$= \frac{1}{2a}\int_0^{2a} \sqrt{y^2+4a^2}\, dy$$

$$= \frac{1}{4a}\left[y\sqrt{y^2+4a^2} + 4a^2\log\left\{y+\sqrt{y^2+4a^2}\right\}\right]_0^{2a}$$

$$= a\left[\sqrt{2} + \log\left(1+\sqrt{2}\right)\right]$$

Q. find the whole length of the curve
$x = a\cos^3 t$, $y = a\sin^3 t$

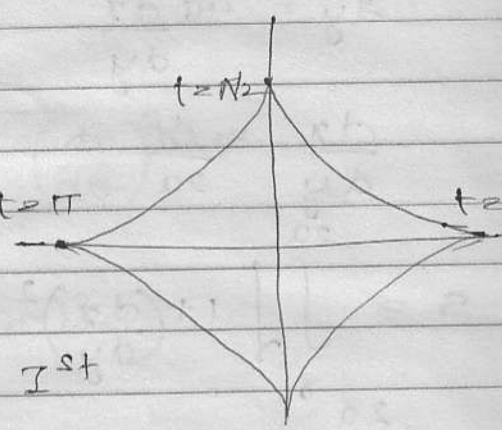
Sol^n $x = a\cos^3 t$

$y = a\sin^3 t$

$$\frac{dx}{dt} = -3a\cos^2 t\sin t$$

$$\frac{dy}{dt} = 3a\sin^2 t\cos t$$

This is an astroid



So whole length of curve is

= 4 × Length of curve in Ist quadrant

$$= 4 \times \int_0^{\pi/2} \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}\, dt$$

$$= 4\int_0^{\pi/2} \sqrt{\left[9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t\right]}\, dt$$

$$= 12a^2 \int_0^{\pi/2} \sqrt{\cos^2 t\sin^2 t}\ (1)\ dt$$

$$= 12a^2 \int_0^{\pi/2} \cos t\sin t\, dt$$

$$= 6a^2 \int_0^{\pi/2} \sin 2t\, dt = 6a^2$$

Q. find the length of the curve

$$y = \log \frac{e^x - 1}{e^x + 1} \quad \text{from } x = 1 \text{ to } x = 2.$$

Sol^n $y = \log(e^x - 1) - \log(e^x + 1)$

$$\frac{dy}{dx} = \frac{e^x}{e^x - 1} - \frac{e^x}{e^x + 1}$$

$$= \frac{2e^x}{e^{2x} - 1}.$$

Length of curve $x = 1$ to $x = 2$

$$s = \int_1^2 \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}\, dx$$

$$= \int_1^2 \sqrt{\left[1 + \frac{4e^{2x}}{(e^{2x} - 1)^2}\right]^{1/2}} dx$$

$$= \int_1^2 \left[\frac{(e^{2x} + 1)^2}{(e^{2x} - 1)^2}\right]^{1/2} dx$$

$$= \int_1^2 \left(\frac{e^{2x} + 1}{e^{2x} - 1}\right) dx$$

$$= \int_1^2 \left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right) dx$$

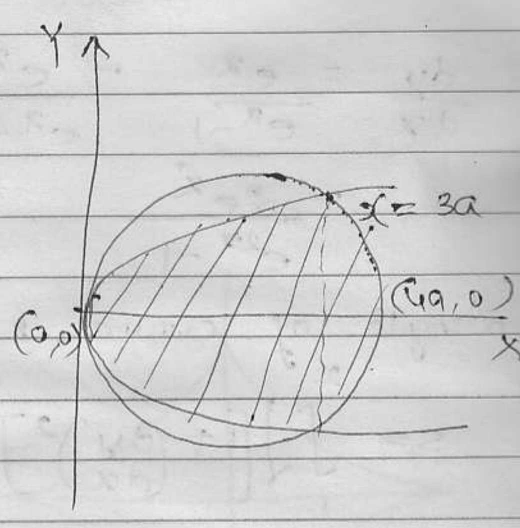
$$= \left[\log(e^x - e^{-x})\right]_1^2$$

$$= \log(e^2 - e^{-2}) - \log(e - e^{-1})$$

$$= \log\left(e + \frac{1}{e}\right)$$

Q. find the area common to the parabola $y^2 = ax$ and the circle $x^2+y^2 = 4ax$.

Soln
$$y^2 = ax$$
$$x^2+y^2 = 4ax$$



$$\therefore \quad x^2 + ax = 4ax$$
$$x^2 = 3ax$$
$$x(x-3a) = 0$$
$$x = 0, 3a.$$

Area common

$$= 2\left[\int_0^{3a} y\,dx + \int_{3a}^{4a} y\,dx\right]$$

$$= 2\left[\int_0^{3a} \sqrt{ax}\,dx + \int_{3a}^{4a} \sqrt{4ax - x^2}\,dx\right]$$

$$= 2\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^{3a} + 2\int_{3a}^{4a} \sqrt{4a^2 - (x-2a)^2}\,dx$$

$$= \frac{4\sqrt{a}}{3}(3a)^{3/2} + 2\left[\frac{1}{2}(x-2a)\sqrt{4a^2-(x-2a)^2} + \frac{4a^2}{2}\sin^{-1}\frac{x-2a}{2a}\right]_{3a}^{4a}$$

$$= 4\sqrt{3}a^2 + 2\left[0 - \frac{1}{2}a\sqrt{3}a\right] + 2a^2\left[\frac{\pi}{2} - \frac{\pi}{6}\right]$$

$$= 4\sqrt{3}a^2 - \sqrt{3}a^2 + \frac{4}{3}\pi a^2$$

$$= \left(3\sqrt{3} + \frac{4}{3}\pi\right)a^2$$

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Q: find the area of the ellipse

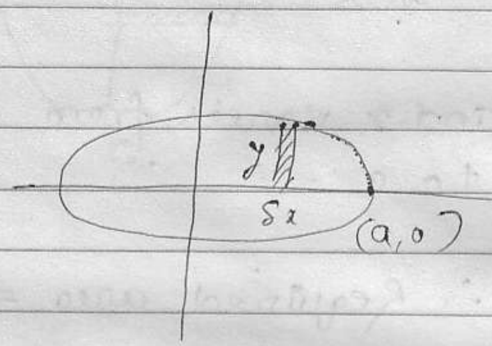
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Soln

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$



Limits of a are from
$x = 0$ to $x = a$

$\therefore$ Required area of 1 portion is

$$\int_0^a y\,dx$$

$$\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{b}{a}\left[\frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

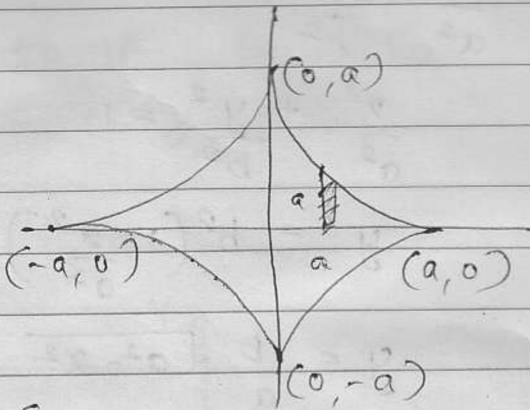
$$= \frac{b}{2a}\left[a^2\sin^{-1}(1)\right] = \frac{\pi ab}{4}$$

Area of whole ellipse $= 4 \times \frac{\pi ab}{4}$

$$= \pi ab$$

Q. Find the area of $x = a\cos^3 t$, $y = a\sin^3 t$

Soln Since the curve is symmetrical about both the axis



and x varies from 0 to a.

$$\therefore \text{Required area} = 4\int_0^a y\,dx$$

$$= 4\int_0^a y\,dx$$

Given curve is $x^{2/3} + y^{2/3} = a^{2/3}$

$$= 4\int_0^a \left(a^{2/3} - x^{2/3}\right)^{3/2} dx \qquad \text{Put } x = a\sin^3\theta$$

$$= 4\int_0^{\pi/2} \left(a^{2/3} - a^{2/3}\sin^2\theta\right)^{3/2} 3a\sin^2\theta\cos\theta\,d\theta$$

$$= 4\int_0^{\pi/2} a\left(1 - \sin^2\theta\right)^{3/2} 3a\sin^2\theta\cos\theta\,d\theta$$

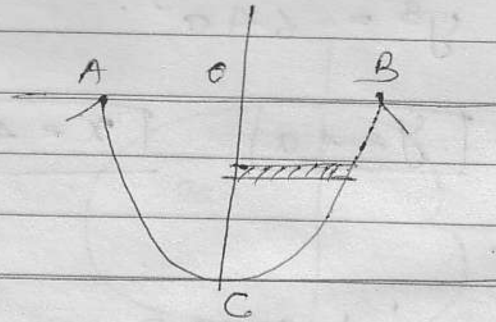
$$= 12a^2\int_0^{\pi/2} \sin^2\theta\cos^4\theta\,d\theta$$

$$= 12a^2\left[\frac{1\cdot 3\cdot 1}{6\cdot 4\cdot 2}\,\frac{\pi}{2}\right] = \frac{3\pi a^2}{8}$$

Q: Find the area included between the cycloid

$x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$

Sol^n



Required area

$= 2 \times$ Area of BOC.

$$= 2\int_{y=0}^{y=2a} x\,dy$$

$$= 2\int_0^{\pi} x \frac{dy}{d\theta}\,d\theta$$

$$= 2\int_0^{\pi} a(\theta + \sin\theta)\, a\sin\theta\, d\theta$$

$$= 2a^2\int_0^{\pi} \theta\sin\theta\, d\theta + 2a^2\int_0^{\pi} \sin^2\theta\, d\theta$$

$$= 2a^2\left[-\theta\cos\theta + \sin\theta\right]_0^{\pi} + 4a^2\int_0^{\pi/2} \sin^2\theta\, d\theta$$

$$= 2a^2\pi + 4a^2 \frac{1}{2}\cdot\frac{1}{2}\pi$$

$$= 3\pi a^2$$

Q: Find the area common to $y^2 = 4ax$ and $x^2 = 4ay$.

Sol^n $y^2 = 4ax \rightarrow (1)$

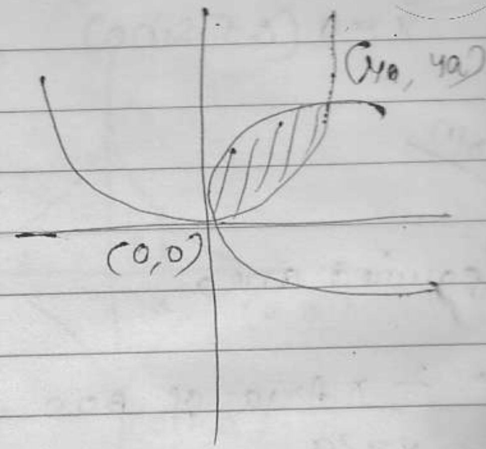
$x^2 = 4ay \rightarrow (2)$

$x = \frac{y^2}{4a}$ Put in (2)

$$\frac{y^4}{16a^2} = 4ay$$

$$y^3 = 64a^3.$$

$\boxed{y = 4a}$ $\boxed{x = 4a}$



$\therefore$ Required area =

$$\int_0^{4a} y\,dx.$$

$$\int_0^{4a} 2\sqrt{ax}\,dx$$

$$= 2\sqrt{a}\int_0^{4a}\sqrt{x}\,dx.$$

$$= 2\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^{4a}$$

$$= \frac{2\sqrt{a}}{3/2}\left[4a^{3/2}\right]$$

$$= \frac{4\sqrt{a}}{3} \times 4 \left[a^{3/2}\right]$$

$$= \frac{16a^2}{3}$$

Area of polar curves:-

Area bounded by the curve $r = f(\theta)$ is

$$\frac{1}{2}\int^{\beta} r^2\,d\theta.$$

In terms of double integral $\int_{\alpha}^{\beta} \int_{f_1(\theta)}^{f_2(\theta)} r\,d\theta\,dr$.

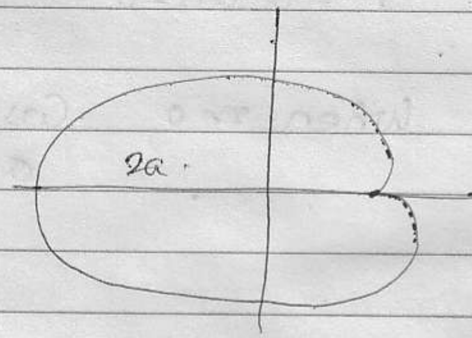
where $r = f_1(\theta)$, $r = f_2(\theta)$. and $\theta = \alpha$ and $\theta = \beta$.

Q: Find the area of the cardioid $r = a(1-\cos\theta)$

4a)

Sol^n

θ varies from 0 to π



Area of curve

$$= 2 \times \frac{1}{2} \int_0^{\pi} r^2 d\theta$$

$$= \int_0^{\pi} a^2 (1-\cos\theta)^2 d\theta$$

$$= a^2 \int_0^{\pi} \left(2\sin^2\frac{\theta}{2}\right)^2 d\theta$$

$$= 4a^2 \int_0^{\pi} \sin^4\frac{\theta}{2}\, d\theta$$

Let $\frac{\theta}{2} = \phi$.

$$= 4a^2 \int_0^{\pi/2} \sin^4\phi \; 2\,d\phi.$$

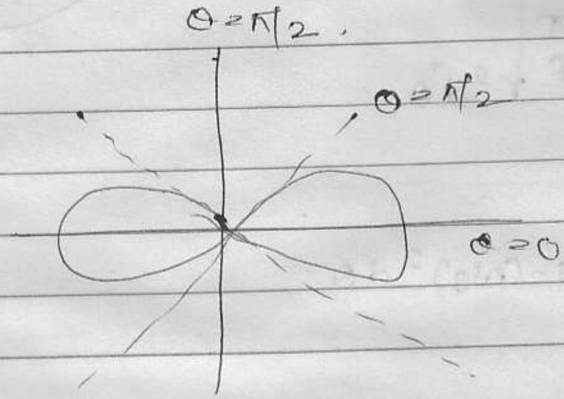
$$= \frac{3\pi a^2}{2}$$

Q: find area of $r^2 = a^2 \cos 2\theta$.

Soln $r^2 = a^2 \cos 2\theta$

When $r = 0$, $\cos 2\theta = 0$

$\theta = \pm\frac{\pi}{4}$.



Required area $= 2\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sqrt{\cos 2\theta}} r\,d\theta\,dr$

$= \frac{2}{2}\int_0^{\pi/4} \left[r^2\right]_0^{a\sqrt{\cos 2\theta}} d\theta$.

$= a^2 \int_0^{\pi/4} \cos 2\theta \, d\theta$

Let $2\theta = \phi$.

$= 2a^2 \int_0^{\pi/2} \cos\phi \, d\phi$.

$= \frac{a^2}{2}$.

Required area $= 2 \times$ Area of 1 loop.

$= a^2$

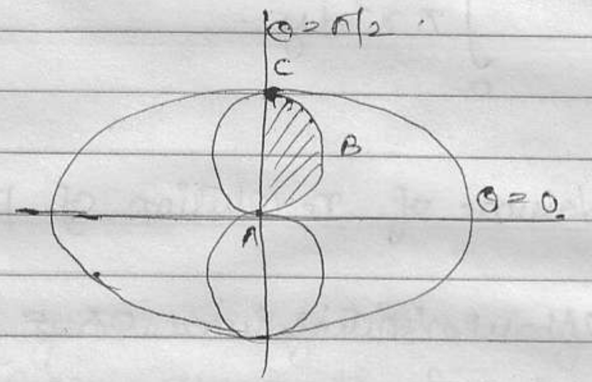
Q. Show that the area included between the cardioids $r = a(1+\cos\theta)$ and $r = a(1-\cos\theta)$ is $a^2(3\pi - 8)/2$.

Solⁿ

$$r = a(1+\cos\theta)$$
$$r = a(1-\cos\theta)$$

$$\cancel{a}(1+\cos\theta) = \cancel{a}(1-\cos\theta)$$

$$2\cos\theta = 0$$
$$\cos\theta = 0$$
$$\theta = \pm\frac{\pi}{2}.$$



Required area.

$$= 4 \times \text{area of } ABC.$$

$$= 4\int_0^{\pi/2}\int_0^{a(1-\cos\theta)} r\,d\theta\,dr.$$

$$= 4\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{a(1-\cos\theta)} d\theta$$

$$= 2a^2\int_0^{\pi/2}(1-\cos\theta)^2\,d\theta$$

$$= 2a^2\int_0^{\pi/2}(1+\cos^2\theta - 2\cos\theta)\,d\theta$$

$$= 2a^2\left[\theta - 2\sin\theta\right]_0^{\pi/2} + 2a^2\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$= 2a^2\left[\frac{\pi}{2} - 2\right] + 2a^2\,\frac{1}{2}\,\frac{\pi}{2} = \frac{a^2}{2}(3\pi - 8)$$

VOLUME OF REVOLUTION

Revolution about x axis

$$\int_a^b \pi y^2 dx$$

Revolution about y axis

$$\int_a^b \pi x^2 dy$$

Volume of revolution of polar curves

(i) About initial line $OX = \int_\alpha^\beta \frac{2\pi}{3} r^3 \sin\theta\, d\theta \qquad (\theta = 0)$

(ii) About initial line $OY \quad (\theta = \frac{\pi}{2}) = \int_\alpha^\beta \frac{2\pi}{3} r^3 \cos\theta\, d\theta$

Q: Find the volume formed by revolution of loop of the curve

$$y^2(a+x) = x^2(3a-x)$$

Sol^n Equating x to zero :-

$$(3a - x) = 0$$
$$x = 3a$$

x varies from 0 to $3a$.

$\therefore$ Volume of the loop $= \int_0^{3a} \pi y^2 dx$

$$= \pi \int_0^{\pi} \frac{x^2(3a-x)}{a+x} dx$$

$$= \pi \int_0^{\pi} \frac{-x^3 + 3ax^2}{x+a} dx$$

solving

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$$= \pi \int_0^{3a} \left[-x^2 + 4ax - 4a^2 + \frac{4a^3}{x+a} \right] dx$$

$$= \pi \left[-\frac{x^3}{3} + 4a\frac{x^2}{2} - 4a^2 x + 4a^3 \log(x+a) \right]_0^{3a}$$

$$= \pi \left[-\frac{27a^3}{3} + 2a \cdot 9a^2 - 4a^2 \cdot 3a + 4a^3 \log 4a - (4a^3 \log a) \right]$$

$$= \pi a^3 (-3 + 4 \log 4)$$
$$= \pi a^3 (8 \log 2 - 3)$$

Q. Find volume of solid generated by revolving the laminscate $r^2 = a^2 \cos 2\theta$ about the line $\theta = \frac{\pi}{2}$

Sol^n for a laminscate $\theta = 0$ to $\theta = \frac{\pi}{4}$.

Required volume = 2 (volume of half loop in first quadrant)

$$= 2 \int_0^{\pi/4} \frac{2}{3} \pi r^3 \cos\theta \, d\theta$$

$r = a(\cos 2\theta)^{1/2}$.

$$= \frac{4\pi}{3} \int_0^{\pi/4} a^3 (\cos 2\theta)^{3/2} \cos\theta \, d\theta$$

$$= \frac{4\pi a^3}{3} \int_0^{\pi/4} (1 - 2\sin^2\theta)^{3/2} \cos\theta \, d\theta$$

Let $\sqrt{2} \sin\theta = \sin\phi$.

$$= \frac{4\pi a^3}{3} \int_0^{\pi/2} (1 - \sin^2\phi)^{3/2} \frac{1}{\sqrt{2}} \cos\phi \, d\phi$$

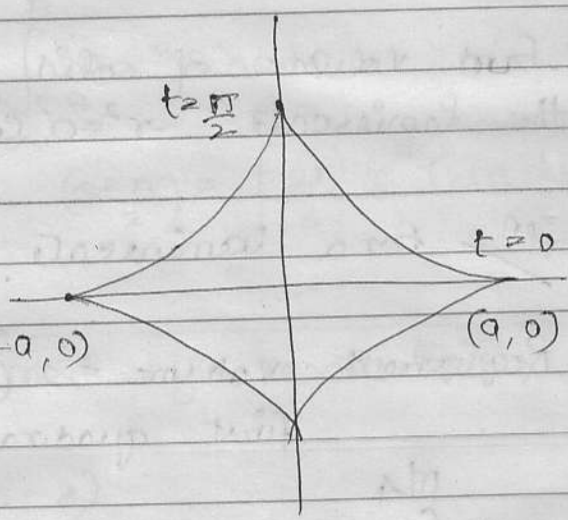
$$= \frac{4\pi a^3}{3\sqrt{2}} \int_0^{\pi/2} \cos^4\phi \, d\phi .$$

$$= \frac{4\pi a^3}{3\sqrt{2}} \frac{3\cdot 1}{4\cdot 2} \frac{\pi}{2}$$

$$= \frac{\pi a^3}{4\sqrt{2}}$$

Q: find the volume of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ (or $x = a\cos^3 t$, $y = a\sin^3 t$) about the x axis

Soln t varies from 0 to $\frac{\pi}{2}$



Required Volume

$$= 2\int \pi y^2 \, dx .$$

$$= 2\int_0^a \pi y^2 \, dx .$$

$$x^{2/3} + y^{2/3} = a^{2/3} .$$

$$\therefore y^{2/3} = (a^{2/3} - x^{2/3})$$

$$\therefore = 2\pi \int_0^{\pi/2} a^2 \sin^2\theta \cos^6\theta \, 3a\cos\theta \, d\theta .$$

Put $x = a\sin^3\theta$.

$$= 6\pi a^3 \int_0^{\pi/2} \sin^2\theta \cos^7\theta \, d\theta$$

$$= \frac{3\pi a^3}{105}$$

Fid the volume formed by revolution of the curve $r = a + b\cos\theta$ about the initial line.

Soln $r = a + b\cos\theta$

θ varies from 0 to π

$\therefore$ Required volume $\displaystyle\int_0^{\pi} \frac{2}{3}\pi r^3 \sin\theta \, d\theta$

$$= \frac{2\pi}{3} \int_0^{\pi} (a + b\cos\theta)^3 \sin\theta \, d\theta$$

2/3 $$= \frac{2\pi}{3} \left[\frac{(a + b\cos\theta)^4}{4\,(-b)} \right]_0^{b}$$

$$= -\frac{\pi}{6b} \left[(a-b)^4 - (a+b)^4 \right]$$

$$= \frac{\pi}{6b} \left[(a+b)^4 - (a-b)^4 \right]$$

$$= \frac{\pi}{6b} \left[\left\{ (a+b)^2 - (a-b)^2 \right\} \left\{ (a+b)^2 + (a-b)^2 \right\} \right]$$

$$= \frac{4\pi a}{3} (a^2 + b^2)$$

Q. Find the volume obtained by revolving $x = a(t - \sin t)$ $y = a(1 - \cos t)$ about its base

Soln $x = a(t - \sin t)$
$y = a(1 - \cos t)$

Base of a cycloid is actualy the x axis.
$\therefore$ $\theta = 0$ to $\theta = 2\pi$

Required volume $= \int_{0}^{2\pi} \pi y^2 dx$

$$= \int_{0}^{2\pi} \pi y^2 \frac{dx}{dt} \cdot dt$$

$$= 2\pi \int_{0}^{\pi} a^2 (1-\cos t)^2 \, a (1-\cos t)\, dt$$

$$= 2\pi a^3 \int_{0}^{\pi} \left(2 \sin^2 \frac{t}{2}\right)^3 dt$$

$$= 16\pi a^3 \int_{0}^{\pi} \sin^6 \frac{t}{2}\, dt$$

Let $\frac{t}{2} = \phi$

$$= 32\pi a^3 \int_{0}^{\pi/2} \sin^6 \phi \, d\phi$$

$$= 5\pi^2 a^3 .$$

~~MOMENT OF INERTIA :-~~

~~If particles~~

Let (x_1, y_1) (x_2, y_2) .... (x_n, y_n) be the centre of gravity of masses $m_1, m_2 \ldots m_n$ then (ξ, η) =

$$\xi = \frac{\Sigma mx}{\Sigma m}, \quad \eta = \frac{\Sigma my}{\Sigma m}$$

where $\Sigma mx = m_1 x_1 + m_2 x_2 + \ldots$

Also $$\xi = \frac{\int_{x=a}^{x=b} x \, ds}{L}$$

and $$\eta = \frac{\int_{x=a}^{x=b} y \, ds}{L}$$

Centre of gravity of an arc : -

$$\eta = \frac{\frac{1}{2}\int_a^b y^2 \, dx}{\alpha} \quad \{\alpha \text{ is area}\}$$

$$\xi = \frac{\int_a^b xy \, dx}{\alpha}$$

Q: Find the centroid of the area enclosed by the parabola $y^2 = 4ax$ and the double ordinate $x = h$.

Solⁿ Let (ξ, η) be centre of gravity. $\eta = 0$ and there is symmetry about x axis

$$\therefore \bar{x} = \frac{\int_0^h xy\,dx}{\int_0^h y\,dx}$$

$$= \frac{\int_0^h x^{3/2}\,dx}{\int_0^h x^{1/2}\,dx} = \frac{3}{5}h.$$

Moment of inertia:—

If particles of masses $m_1, m_2 \ldots m_n$ be situated at points whose perpendicular distances from a straight line are $r_1, r_2 \ldots r_n$ then Σmr^2

where $m_1 r_1^2 + m_2 r_2^2 + \ldots + m_n r_n^2$ are called moment of inertia of the system about the given line.

The moment of inertia is useful in dynamics of rigid bodies. Thus Kinetic Energy of a body rotating with angular velocity ω about an axis AB is equal to

$$\frac{1}{2}(\text{moment of inertia of body about AB}) \times \omega^2$$

If moment of inertia of a body of mass M about any axis AB be MK^2, then K is called radius of gyration of body about AB.