LECTURE # 1

Course Objective:

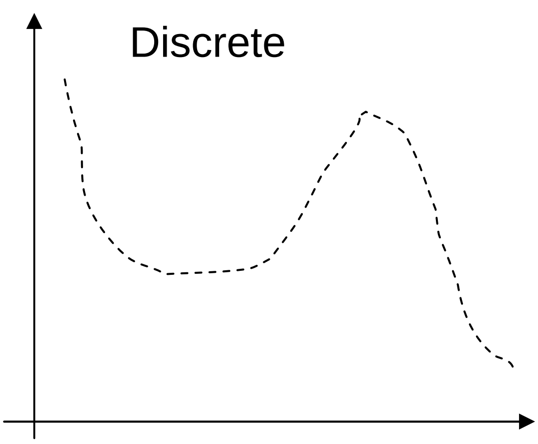
1.Express statements with the precision of formal logic
2.Analyze arguments to test their validity
3.Apply the basic properties and operations related to sets
4.Apply to sets the basic properties and operations related to relations and functions
5.Define terms recursively
6.Prove a formula using mathematical induction
7.Prove statements using direct and indirect methods
8.Compute probability of simple and conditional events
9.Identify and use the formulas of combinatorics in different problems
10.Illustrate the basic definitions of graph theory and properties of graphs
11.Relate each major topic in Discrete Mathematics to an application area in computing

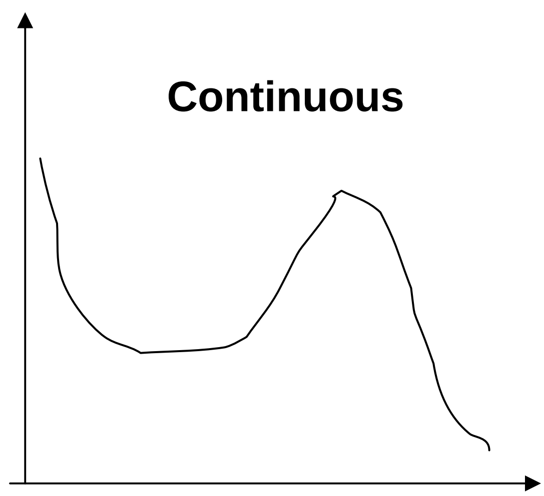
1.Recommended Books:

1.Discrete Mathematics with Applications (second edition) by Susanna S. Epp
2.Discrete Mathematics and Its Applications (fourth edition) by Kenneth H. Rosen
1.Discrete Mathematics by Ross and Wright

MAIN TOPICS:

1. Logic
2. Sets & Operations on sets
3. Relations & Their Properties
4. Functions
5. Sequences & Series
6. Recurrence Relations
7. Mathematical Induction
8. Loop Invariants
9. Loop Invariants
10. Combinatorics
11. Probability
12. Graphs and Trees





Set of Integers:

• • • • • •
3 -2 -1 0 1 2

Set of Real Numbers:

• • • • • • •
-3 -2 -1 0 1 2

What is Discrete Mathematics?:

Discrete Mathematics concerns processes that consist of a sequence of individual steps.

LOGIC:

Logic is the study of the principles and methods that distinguishes between a valid and an invalid argument.

SIMPLE STATEMENT:

A statement is a declarative sentence that is either true or false but not both.
A statement is also referred to as a **proposition**
Example: 2+2 = 4, It is Sunday today
If a proposition is true, we say that it has a **truth value** of "**true**".
If a proposition is false, its **truth value** is "**false**".
The truth values **"true"** and **"false"** are, respectively, denoted by the letters **T** and **F**.

EXAMPLES:

1.Grass is green.
2.4 + 2 = 6
2.4 + 2 = 7
3.There are four fingers in a hand.
are propositions

Not Propisitions

- Close the door.
- *x* is greater than 2.
- He is very rich

are not propositions.

Rule:

If the sentence is preceded by other sentences that make the pronoun or variable reference clear, then the sentence is a statement.

Example:

$x = 1$
$x > 2$
$x > 2$ is a statement with truth-value FALSE.

Example

Bill Gates is an American
He is very rich
He is very rich is a statement with truth-value TRUE.

UNDERSTANDING STATEMENTS:

1.$x + 2$ is positive.	Not a statement
2.May I come in?	Not a statement
3.Logic is interesting.	A statement
4.It is hot today.	A statement
5.-1 > 0	A statement
6.$x + y = 12$	Not a statement

COMPOUND STATEMENT:

Simple statements could be used to build a compound statement.

EXAMPLES: **LOGICAL CONNECTIVES**

1. "3 + 2 = 5" **and** "Lahore is a city in Pakistan"
2. "The grass is green" or " It is hot today"
3. "Discrete Mathematics is **not** difficult to me"

AND, OR, NOT are called LOGICAL CONNECTIVES.

SYMBOLIC REPRESENTATION:

Statements are symbolically represented by letters such as ***p, q, r,...***

EXAMPLES:

***p* =** "Islamabad is the capital of Pakistan"
***q* =** "17 is divisible by 3"

CONNECTIVE	MEANINGS	SYMBOL	CALLED
Negation	not	$\sim$	Tilde
Conjunction	and	$\wedge$	Hat
Disjunction	or	$\vee$	Vel
Conditional	if…then…	$\rightarrow$	Arrow
Biconditional	if and only if	$\leftrightarrow$	Double arrow

EXAMPLES:

***p* =** "Islamabad is the capital of Pakistan"
***q* =** "17 is divisible by 3"
***p* ∧ *q* =** "Islamabad is the capital of Pakistan and 17 is divisible by 3"
***p* ∨ *q* =** "Islamabad is the capital of Pakistan or 17 is divisible by 3"
~***p* =** "It is not the case that Islamabad is the capital of Pakistan" or simply "Islamabad is not the capital of Pakistan"

TRANSLATING FROM ENGLISH TO SYMBOLS:

Let p = "It is hot", and q = " It is sunny"

SENTENCE	SYMBOLIC FORM
1.It is **not** hot.	$\sim p$
2.It is hot **and** sunny.	$p \wedge q$
3.It is hot **or** sunny.	$p \vee q$
4.It is **not** hot **but** sunny.	$\sim p \wedge q$
5.It is **neither** hot **nor** sunny.	$\sim p \wedge \sim q$

EXAMPLE:

Let ***h*** = "Zia is healthy"
w = "Zia is wealthy"
s = "Zia is wise"

Translate the compound statements to symbolic form:

1.Zia is healthy and wealthy but not wise. $(h \wedge w) \wedge (\sim s)$
2.Zia is not wealthy but he is healthy and wise. $\sim w \wedge (h \wedge s)$
3.Zia is neither healthy, wealthy nor wise. $\sim h \wedge \sim w \wedge \sim s$

TRANSLATING FROM SYMBOLS TO ENGLISH:

Let m = "Ali is good in Mathematics"
c = "Ali is a Computer Science student"

Translate the following statement forms into plain English:

1.$\sim c$ Ali is not a Computer Science student
2.$c \vee m$ Ali is a Computer Science student or good in Maths.

3. $m \wedge \sim c$ Ali is good in Maths but not a Computer Science student

A convenient method for analyzing a compound statement is to make a truth table for it.

A **truth table** specifies the truth value of a compound proposition for all possible truth values of its constituent propositions.

NEGATION (~):

If ***p*** is a statement variable, then negation of ***p, "not p"***, is denoted as **"~p"**

It has opposite truth value from p i.e.,

if p is true, ~p is false; if p is false, ~p is true.

TRUTH TABLE FOR ~ p

p	***~ p***
T	F
F	T

CONJUNCTION (∧):

If ***p*** and ***q*** are statements, then the conjunction of ***p*** and ***q*** is ***"p and q",*** denoted as ***"p ∧ q".***

It is true when, and only when, both p and q are true. If either p or q is false, or if both are false, p∧q is false.

TRUTH TABLE FOR p ∧ q

p	***q***	***p ∧ q***
T	T	T
T	F	F
F	T	F
F	F	F

DISJUNCTION (∨) or INCLUSIVE OR

If ***p*** & ***q*** are statements, then the disjunction of ***p*** and ***q*** is ***"p or q",*** denoted as ***"p ∨ q".*** It is true when at least one of p or q is true and is false only when both p and q are false.

TRUTH TABLE FOR
$p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

LECTURE #2

Truth Tables for:

1. **~ p ∧ q**
2. **~ p ∧ (q ∨ ~ r)**
3. **(p∨q) ∧ ~ (p∧q)**

Truth table for the statement form **~ p ∧ q**

p	q	~p	~ p ∧ q
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

Truth table for ~ p ∧ (q ∨ ~ r)

p	q	r	~ r	q ∨ ~ r	~ p	~ p ∧ (q ∨ ~ r)
T	T	T	F	T	F	F
T	T	F	T	T	F	F
T	F	T	F	F	F	F
T	F	F	T	T	F	F
F	T	T	F	T	T	T
F	T	F	T	T	T	T
F	F	T	F	F	T	F
F	F	F	T	T	T	T

Truth table for **(p∨q) ∧ ~ (p∧q)**

p	q	p∨q	p∧q	~ (p∧q)	(p∨q) ∧ ~ (p∧q)
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Double Negative Property ~(~p) º p

p	~p	~(~p)
T	F	T
F	T	F

Example

"It is not true that I am not happy"

Solution:

Let **p** = "I am happy"

then ~ **p** = "I am not happy"

and ~**(~ p)** = "It is not true that I am not happy"

Since ~(~**p**) ≡ **p**

Hence the given statement is equivalent to:

"**I am happy**"

$\sim(p \wedge q)$ and $\sim p \wedge \sim q$ are not logically equivalent

p	q	~p	~q	$p\wedge q$	$\sim(p\wedge q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	F	T	T

Different truth values in row 2 and row 3

DE MORGAN'S LAWS:

1)The negation of an **and** statement is logically equivalent to the **or** statement in which each component is negated.

Symbolically $\sim(p \wedge q) \equiv \sim p \vee \sim q$.

2)The negation of an **or** statement is logically equivalent to the **and** statement in which each component is negated.

Symbolically: $\sim(p \vee q) \equiv \sim p \wedge \sim q$.

$\sim(p \vee q) \equiv \sim p \wedge \sim q$

p	q	~p	~q	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Same truth values

Application:

Give negations for each of the following statements:

a.The fan is slow **or** it is very hot.

b.Akram is unfit **and** Saleem is injured.

Solution

a.The fan is **not** slow **and** it is **not** very hot.

b.Akram is **not** unfit **or** Saleem is **not** injured.

INEQUALITIES AND DEMORGAN'S LAWS:

Use DeMorgan's Laws to write the negation of

$$-1 < x \leq 4$$

for some particular real no. x

$-1 < x \leq 4$ means $x > -1$ **and** $x \leq 4$

By DeMorgan's Law, the negation is:

$x > -1$ or $x \leq 4$Which is equivalent to: $x \leq -1$ **or** $x > 4$

EXERCISE:

1. $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
2. Are the statements $(p\wedge q)\vee r$ and $p \wedge (q \vee r)$ logically equivalent?

TAUTOLOGY:

A tautology is a statement form that is always true regardless of the truth values of the statement variables.

A tautology is represented by the symbol "**t**"..

EXAMPLE: The statement form $p \vee \sim p$ is tautology

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

$p \vee \sim p \equiv t$

LECTURE #3
APPLYING LAWS OF LOGIC

Using law of logic, simplify the statement form

$p \vee [\sim(\sim p \wedge q)]$

Solution:

$p \vee [\sim(\sim p \wedge q)]$	$\equiv p \vee [\sim(\sim p) \vee (\sim q)]$	DeMorgan's Law
	$\equiv p \vee [p \vee (\sim q)]$	Double Negative Law
	$\equiv [p \vee p] \vee (\sim q)$	Associative Law for $\vee$
	$\equiv p \vee (\sim q)$	Indempotent Law

Which is the simplified statement form.

EXAMPLE Using Laws of Logic, verify the logical equivalence

$\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p$

$\sim(\sim p \wedge q) \wedge (p \vee q)$	$\equiv (\sim(\sim p) \vee \sim q) \wedge (p \vee q)$	DeMorgan's Law
	$\equiv (p \vee \sim q) \wedge (p \vee q)$	Double Negative Law
	$\equiv p \vee (\sim q \wedge q)$	Distributive Law
	$\equiv p \vee c$	Negation Law
	$\equiv p$	Identity Law

SIMPLIFYING A STATEMENT:

"You will get an A if you are hardworking and the sun shines, or you are hardworking and it rains."

Rephrase the condition more simply.

Solution:

Let p = "You are hardworking'

q = "The sun shines"

r = "It rains" .The condition is then $(p \wedge q) \vee (p \wedge r)$

And using distributive law in reverse,

$\mathbf{(p \wedge q) \vee (p \wedge r) \equiv p \wedge (q \vee r)}$

Putting $p \wedge (q \vee r)$ back into English, we can rephrase the given sentence as

"You will get an A if you are hardworking and the sun shines or it rains.

EXERCISE:

Use Logical Equivalence to rewrite each of the following sentences more simply.

1.It is not true that I am tired and you are smart.

{I am not tired or you are not smart.}

2.It is not true that I am tired or you are smart.

{I am not tired and you are not smart.}

3.I forgot my pen or my bag and I forgot my pen or my glasses.

{I forgot my pen or I forgot my bag and glasses.

4.It is raining and I have forgotten my umbrella, or it is raining and I have forgotten my hat.

{It is raining and I have forgotten my umbrella or my hat.}

CONDITIONAL STATEMENTS:

Introduction

Consider the statement:

"**If you earn an A in Math, then I'll buy you a computer**."

This statement is made up of two simpler statements:

p: "You earn an A in Math," and

q: "I will buy you a computer."

The original statement is then saying :

if p is true, then q is true, or, more simply, **if** p, **then** q.

We can also phrase this as p **implies** q, and we write $\mathbf{p \rightarrow q}$.

CONDITIONAL STATEMENTS OR IMPLICATIONS:

If p and q are statement variables, the conditional of q by p is "If p then q" or "p implies q" and is denoted $p \to q$.
It is false when p is true and q is false; otherwise it is true. The arrow "$\to$ " is the **conditional** operator, and in $p \to q$ the statement **p is called the hypothesis (or antecedent)** and q is called the conclusion (or consequent).

TRUTH TABLE:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

PRACTICE WITH CONDITIONAL STATEMENTS:

Determine the truth value of each of the following conditional statements:

1. "If 1 = 1, then 3 = 3." **TRUE**
2. "If 1 = 1, then 2 = 3." **FALSE**
3. "If 1 = 0, then 3 = 3." **TRUE**
4. "If 1 = 2, then 2 = 3." **TRUE**
5. "If 1 = 1,then 1 = 2 and 2 = 3." **FALSE**
6. "If 1 = 3 or 1 = 2 then 3 = 3." **TRUE**

ALTERNATIVE WAYS OF EXPRESSING IMPLICATIONS:

The implication $\mathbf{p \to q}$ could be expressed in many alternative ways as:

•"if p then q"
•"p implies q"
•"if p, q"
•"p only if q"
•"p is sufficient for q"
•"not p unless q"
•"q follows from p"
•"q if p"
•"q whenever p"
•"q is necessary for p"

EXERCISE:

Write the following statements in the form "if p, then q" in English.

a)Your guarantee is good only if you bought your CD less than 90 days ago.
If your guarantee is good, then you must have bought your CD player less than 90 days ago.

b)To get tenure as a professor, it is sufficient to be world-famous.
If you are world-famous, then you will get tenure as a professor.

c)That you get the job implies that you have the best credentials.
If you get the job, then you have the best credentials.

d)It is necessary to walk 8 miles to get to the top of the Peak.
If you get to the top of the peak, then you must have walked 8 miles.

TRANSLATING ENGLISH SENTENCES TO SYMBOLS:

Let p and q be propositions:

p = "you get an A on the final exam"
q = "you do every exercise in this book"
r = "you get an A in this class"

Write the following propositions using p, q,and r and logical connectives.

1.To get an A in this class it is necessary for you to get an A on the final.

SOLUTION $\mathbf{p \to r}$

2.You do every exercise in this book; You get an A on the final, implies, you get an A in the class.

SOLUTION $p \wedge q \rightarrow r$

3. Getting an A on the final and doing every exercise in this book is sufficient For getting an A in this class.

SOLUTION $p \wedge q \rightarrow r$

TRANSLATING SYMBOLIC PROPOSITIONS TO ENGLISH:

Let **p**, **q**, and **r** be the propositions:

p = "you have the flu"
q = "you miss the final exam"
r = "you pass the course"

Express the following propositions as an English sentence.

1. $p \rightarrow q$

If you have flu, then you will miss the final exam.**2.** $\sim q \rightarrow r$

If you don't miss the final exam, you will pass the course.**3.** $\sim p \wedge \sim q \rightarrow r$

If you neither have flu nor miss the final exam, then you will pass the course.

HIERARCHY OF OPERATIONS FOR LOGICAL CONNECTIVES

- $\sim$ (negation)
- $\wedge$ (conjunction), $\vee$ (disjunction)
- $\rightarrow$ (conditional)

Construct a truth table for the statement form $p \vee \sim q \rightarrow \sim p$

p	q	~ q	~p	p ∨ ~q	**p ∨ ~ q → ~ p**
T	T	F	F	T	F
T	F	T	F	T	F
F	T	F	T	F	T
F	F	T	T	T	T

Construct a truth table for the statement form $(p \rightarrow q) \wedge (\sim p \rightarrow r)$

p	q	r	p→q	~p	~p→r	(p→q)∧(~ p →r)
T	T	T	T	F	T	T
T	T	F	T	F	T	T
T	F	T	F	F	T	F
T	F	F	F	F	T	F
F	T	T	T	T	T	T
F	T	F	T	T	F	F
F	F	T	T	T	T	T
F	F	F	T	T	F	F

LOGICAL EQUIVALENCE INVOLVING IMPLICATION

Use truth table to show $p \rightarrow q \equiv \sim q \rightarrow \sim p$

p	q	~q	~p	$p\to q$	$\sim q \to \sim p$
T	T	F	F	T	T
T	F	T	F	F	F
F	T	F	T	T	T
F	F	T	T	T	T

same truth values

Hence the given two expressions are equivalent.

IMPLICATION LAW

$p\to q \equiv \sim p\vee q$

p	q	$p\to q$	~p	$\sim p\vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

same truth values

NEGATION OF A CONDITIONAL STATEMENT:

Since $p\to q \equiv \sim p\vee q$ therefore

$\sim (p \to q) \equiv \sim (\sim p \vee q)$

$\equiv \sim (\sim p) \wedge (\sim q)$ by De Morgan's law

$\equiv p \wedge \sim q$ by the Double Negative law

Thus the negation of "**if p then q**" is logically equivalent to "**p and not q**".

Accordingly, the negation of an if-then statement does not start with the word if.

EXAMPLES

Write negations of each of the following statements:

1.If Ali lives in Pakistan then he lives in Lahore.
2.If my car is in the repair shop, then I cannot get to class.
3.If x is prime then x is odd **or** x is 2.
4.If n is divisible by 6, then n is divisible by 2 **and** n is divisible by 3.

SOLUTIONS:

1. Ali lives in Pakistan and he does not live in Lahore.
2. My car is in the repair shop and I can get to class.
3.x is prime but x is not odd **and** x is not 2.
4.n is divisible by 6 but n is not divisible by 2 **or** by 3.

INVERSE OF A CONDITIONAL STATEMENT:

The inverse of the conditional statement **$p \to q$** is **$\sim p \to \sim q$**
A conditional and its inverse are not equivalent as could be seen from the truth table.

p	q	p→q	~p	~q	~p →~q
T	T	T	F	F	T
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	T	T	T

different truth values in rows 2 and 3

WRITING INVERSE:

1. If today is Friday, then 2 + 3 = 5.
If today is not Friday, then $2 + 3 \neq 5$.
2. If it snows today, I will ski tomorrow.
If it does not snow today I will not ski tomorrow.
3. If P is a square, then P is a rectangle.
If P is not a square then P is not a rectangle.
4. If my car is in the repair shop, then I cannot get to class.
If my car is not in the repair shop, then I shall get to the class.

CONVERSE OF A CONDITIONAL STATEMENT:

The converse of the conditional statement **$p \to q$** is **$q \to p$**
A conditional and its converse are not equivalent.
i.e., $\to$ is not a commutative operator.

p	q	p→q	q→p
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

not the same

WRITING CONVERSE:

1.If today is Friday, then 2 + 3 = 5.
If 2 + 3 = 5, then today is Friday.
2.If it snows today, I will ski tomorrow.
I will ski tomorrow only if it snows today.
3. If P is a square, then P is a rectangle.

If P is a rectangle then P is a square.

4. If my car is in the repair shop, then I cannot get to class.

If I cannot get to the class, then my car is in the repair shop.

CONTRAPOSITIVE OF A CONDITIONAL STATEMENT:

The contrapositive of the conditional statement $\mathbf{p \rightarrow q}$ is $\mathbf{\sim q \rightarrow \sim p}$

A conditional and its contrapositive are equivalent. Symbolically, $\mathbf{p \rightarrow q \equiv \sim q \rightarrow \sim p}$

1.If today is Friday, then 2 + 3 = 5.

If $2 + 3 \neq 5$, then today is not Friday.

2.If it snows today, I will ski tomorrow.

I will not ski tomorrow only if it does not snow today.

3. If P is a square, then P is a rectangle.

If P is not a rectangle then P is not a square.

4. If my car is in the repair shop, then I cannot get to class.

If I get to the class, then my car is not in the repair shop.

LECTURE # 4
BICONDITIONAL

If p and q are statement variables, the biconditional of p and q is
"**p if, and only if, q**" and is denoted **p↔q**. *if and only if* abbreviated **iff.**
The double headed arrow " ↔" is the **biconditional operator.**
TRUTH TABLE FOR
p↔q.

p	q	p ↔q
T	T	T
T	F	F
F	T	F
F	F	T

EXAMPLES:
True or false?
1."**1+1 = 3** if and only if **earth is flat**"
TRUE
2. "**Sky is blue** iff **1 = 0**"
FALSE3. "**Milk is white** iff **birds lay eggs**"
TRUE
4. "**33 is divisible by 4** if and only if **horse has four legs**"
FALSE
5. "**$x > 5$ iff $x^2 > 25$**"
FALSE

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

p	q	p↔q	p→q	q→p	(p→q)∧(q→p)
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

same truth values

REPHRASING BICONDITIONAL:
p↔q is also expressed as:
"p is necessary and sufficient for q"
"if p then q, and conversely"
"p is equivalent to q"

EXERCISE:
Rephrase the following propositions in the form "p if and only if q" in English.
1.If it is hot outside you buy an ice cream cone, and if you buy an ice cream

cone it is hot outside.

Sol You buy an ice cream cone if and only if it is hot outside.

2.For you to win the contest it is necessary and sufficient that you have the only winning ticket.

Sol You win the contest if and only if you hold the only winning ticket.

3.If you read the news paper every day, you will be informed and conversely.

Sol You will be informed if and only if you read the news paper every day.**4.It rains if it is a weekend day, and it is a weekend day if it rains.**

Sol It rains if and only if it is a weekend day.

5.The train runs late on exactly those days when I take it.

Sol The train runs late if and only if it is a day I take the train.

6.This number is divisible by 6 precisely when it is divisible by both 2 and 3.

Sol This number is divisible by 6 if and only if it is divisible by both 2 and 3.

TRUTH TABLE FOR

$(p\to q) \leftrightarrow (\sim q\to \sim p)$

p	q	$p\to q$	$\sim q$	$\sim p$	$\sim q\to \sim p$	$(p\to q) \leftrightarrow (\sim q\to \sim p)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

TRUTH TABLE FOR

$(p\leftrightarrow q)\leftrightarrow(r\leftrightarrow q)$

p	q	r	$p\leftrightarrow q$	$r\leftrightarrow q$	$(p\leftrightarrow q)\leftrightarrow(r\leftrightarrow q)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	F	T
T	F	F	F	T	F
F	T	T	F	T	F
F	T	F	F	F	T
F	F	T	T	F	F
F	F	F	T	T	T

TRUTH TABLE FOR

$p \wedge \sim r \leftrightarrow q\vee r$

Here $p \wedge \sim r \leftrightarrow q \vee r$ means $(p \wedge (\sim r)) \leftrightarrow (q \vee r)$

p	q	r	~r	p∧~r	q∨r	p ∧ ~r ↔q∨r
T	T	T	F	F	T	F
T	T	F	T	T	T	T
T	F	T	F	F	T	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	T	F	T	F	T	F
F	F	T	F	F	T	F
F	F	F	T	F	F	T

LOGICAL EQUIVALENCE INVOLVING BICONDITIONAL

Show that **~p↔q** and **p↔~q** are logically equivalent

p	q	~p	~q	~p↔q	p↔~q
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	F	F

same truth values

EXERCISE:

Show that **~(p⊕q)** and **p↔q** are logically equivalent

p	q	p⊕q	~(p⊕q)	p↔q
T	T	F	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

same truth values

LAWS OF LOGIC:

1.Commutative Law: $p \leftrightarrow q \equiv q \leftrightarrow p$

2.Implication Laws: $p \to q \equiv \sim p \vee q$
$\equiv \sim(p \wedge \sim q)$
3.Exportation Law: $(p \wedge q) \to r \equiv p \to (q \to r)$
4.Equivalence: $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
5.Reductio ad absurdum $p \to q \equiv (p \wedge \sim q) \to c$

APPLICATION:

Rewrite the statement forms without using the symbols $\to$ or $\leftrightarrow$

1.$p \wedge \sim q \to r$ **2. $(p \to r) \leftrightarrow (q \to r)$**

SOLUTION

1.$p \wedge \sim q \to r \equiv (p \wedge \sim q) \to r$ order of operations
$\equiv \sim(p \wedge \sim q) \vee r$ implication law
2.$(p \to r) \leftrightarrow (q \to r) \equiv (\sim p \vee r) \leftrightarrow (\sim q \vee r)$ implication law
$\equiv [(\sim p \vee r) \to (\sim q \vee r)] \wedge [(\sim q \vee r) \to (\sim p \vee r)]$
equivalence of biconditional
$\equiv [\sim(\sim p \vee r) \vee (\sim q \vee r)] \wedge [\sim(\sim q \vee r) \vee (\sim p \vee r)]$
implication law

Rewrite the statement form **$\sim p \vee q \to r \vee \sim q$** to a logically equivalent form that uses only **~** and $\wedge$

SOLUTION

STATEMENT	**REASON**
$\sim p \vee q \to r \vee \sim q$	Given statement form
$\equiv (\sim p \vee q) \to (r \vee \sim q)$	Order of operations
$\equiv \sim[(\sim p \vee q) \wedge \sim (r \vee \sim q)]$	Implication law $p \to q \equiv \sim(p \wedge \sim q)$
$\equiv \sim[\sim(p \wedge \sim q) \wedge (\sim r \wedge q)]$	De Morgan's law

Show that **$\sim(p \to q) \to p$** is a tautology without using truth tables.

SOLUTION

STATEMENT	**REASON**
$\sim(p \to q) \to p$	Given statement form
$\equiv \sim[\sim(p \wedge \sim q)] \to p$	Implication law $p \to q \equiv \sim(p \wedge \sim q)$
$\equiv (p \wedge \sim q) \to p$	Double negation law
$\equiv \sim(p \wedge \sim q) \vee p$	Implication law $p \to q \equiv \sim p \vee q$
$\equiv (\sim p \vee q) \vee p$	De Morgan's law
$\equiv (q \vee \sim p) \vee p$	Commutative law of
$\vee \equiv q \vee (\sim p \vee p)$	Associative law of
$\vee \equiv q \vee t$	Negation law
$\equiv t$	Universal bound law

EXERCISE:

Suppose that p and q are statements so that $p \to q$ is false. Find the truth values of each of the following:

1.$\sim p \to q$
2.$p \vee q$
3.$q \leftrightarrow p$

SOLUTION

1.TRUE
2.TRUE
3.FALSE

LECTURE # 5
EXAMPLE

An interesting teacher keeps me awake. I stay awake in Discrete Mathematics class.
Therefore, my Discrete Mathematics teacher is interesting.
Is the above argument valid?

ARGUMENT:

An **argument** is a list of statements called **premises** (or **assumptions** or **hypotheses**) followed by a statement called the **conclusion.**

P_1 Premise
P_2 Premise
P_3 Premise
. . . .
P_n Premise
____
$\therefore C$ Conclusion

NOTE The symbol \ read "therefore," is normally placed just before the conclusion.

VALID AND INVALID ARGUMENT:

An argument is **valid** if the conclusion is true when all the premises are true.
Alternatively, an argument is valid if conjunction of its premises imply conclusion.That is
$(P_1 \wedge P_2 \wedge P_3 \wedge . . . \wedge P_n) \to C$ is a tautology.
An argument is **invalid** if the conclusion is false when all the premises are true.
Alternatively, an argument is invalid if conjunction of its premises does not imply conclusion.

EXAMPLE:

Show that the following argument form is valid:

$$p \to q$$
$$p$$
$$\therefore \; q$$

SOLUTION

premises conclusion

p	q	p→q	p	q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

critical row

EXAMPLE Show that the following argument form is invalid:

$$p \to q$$
$$q$$
$$\therefore \; p$$

SOLUTION

premises conclusion

p	q	p→q	q	p
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	F

critical row

EXERCISE:

Use truth table to determine the argument form

$$\begin{array}{l} \mathbf{p \vee q} \\ \mathbf{p \rightarrow \sim q} \\ \mathbf{p \rightarrow r} \\ \therefore\ \mathbf{r} \end{array}$$

is valid or invalid.

premises conclusion

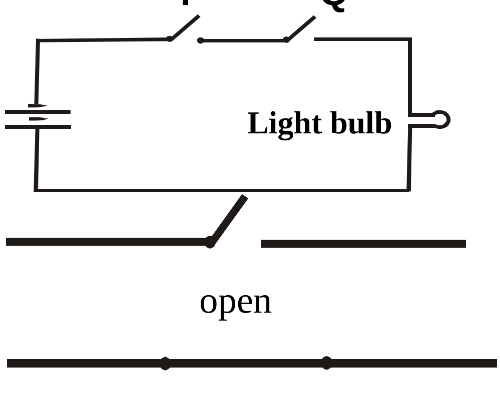
p	q	r	p∨q	p→~q	p→r	r
T	T	T	T	F	T	T
T	T	F	T	F	F	F
T	F	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	T	T	T	T
F	T	F	T	T	T	F
F	F	T	F	T	T	T
F	F	F	F	T	T	F

critical rows

The argument form is invalid

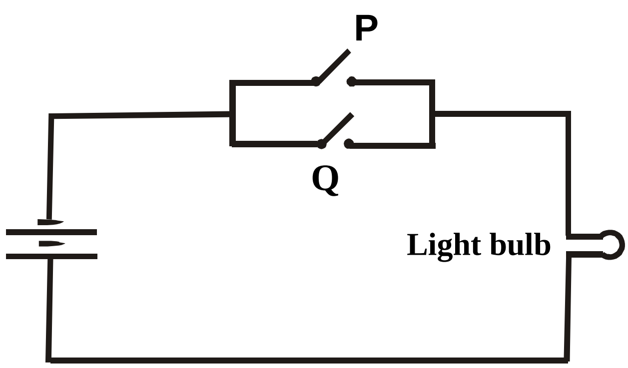
LECTURE #6

SWITCHES IN SERIES



Switches		Light Bulb
P	**Q**	**State**
Closed	Closed	On
Closed	Open	Off
Open	Closed	Off
Open	Open	Off

SWITCHES IN PARALLEL:



Switches		Light Bulb
P	**Q**	**State**
Closed		On
Closed	Open	On
Open	Closed	On
Open	Open	Off

SWITCHES IN SERIES:

Switches		Light Bulb
P	**Q**	**State**
Closed	Closed	On
Closed	Open	Off
Open	Closed	Off
Open	Open	Off

$\Leftrightarrow$

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

SWITCHES IN PARALLEL:

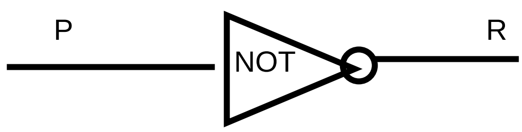
Switches		**Light Bulb**
P	**Q**	**State**
Closed	Closed	On
Closed	Open	On
Open	Closed	On
Open	Open	Off

⇔

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

1.NOT-gate

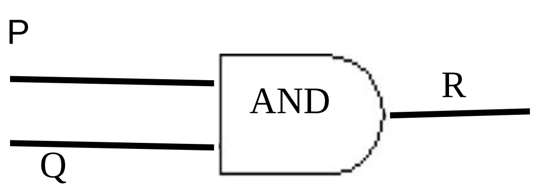
A NOT-gate (or inverter) is a circuit with one input and one output signal. If the input signal is 1, the output signal is 0. Conversely, if the input signal is 0, then the output signal is 1.



Input	Output
P	R
1	0
0	1

2.AND-gate

An AND-gate is a circuit with two input signals and one output signal.
If both input signals are 1, the output signal is 1. Otherwise the output signal is 0.
Symbolic representation & Input/Output Table

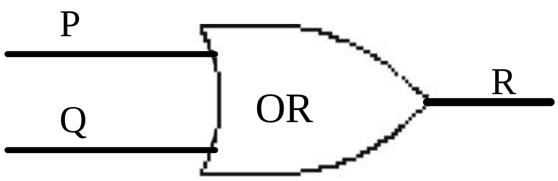


Input		**Output**
P	**Q**	**R**
1	**1**	**1**
1	**0**	**0**
0	**1**	**0**
0	**0**	**0**

3. OR-gate

An OR-gate is a circuit with two input signals and one output signal.
If both input signals are 0, then the output signal is 0. Otherwise, the output signal is 1.

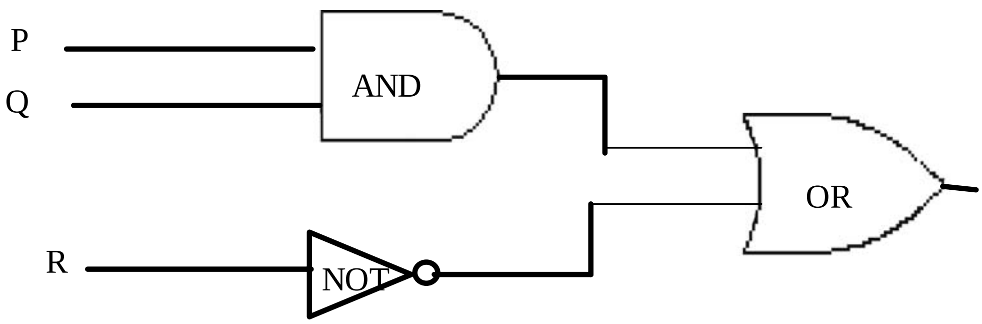
Symbolic representation & Input/Output Table



Input		Output
P	**Q**	**R**
1	1	1
1	0	1
0	1	1
0	0	0

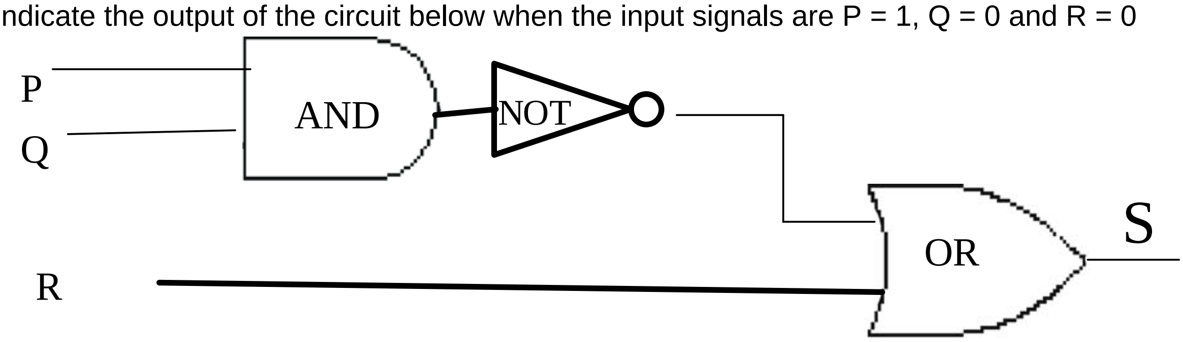
COMBINATIONAL CIRCUIT:

A Combinational Circuit is a compound circuit consisting of the basic logic gates such as NOT, AND, OR.

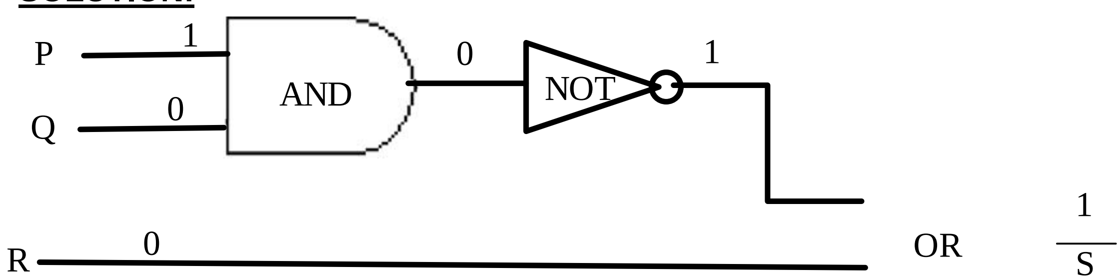


DETERMINING OUTPUT FOR A GIVEN INPUT:

Indicate the output of the circuit below when the input signals are P = 1, Q = 0 and R = 0



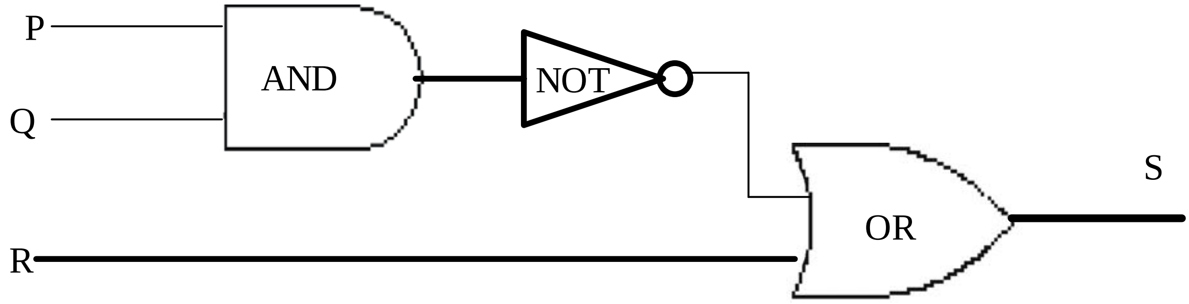
SOLUTION:



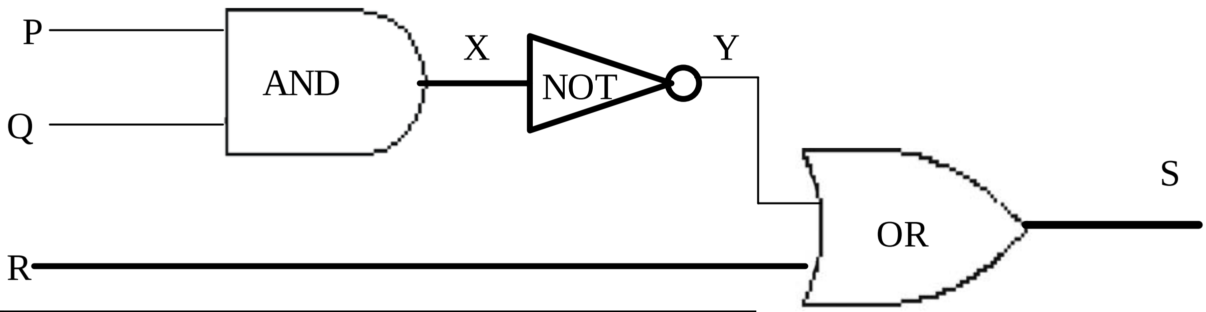
Output S = 1:

CONSTRUCTING THE INPUT/OUTPUT TABLE FOR A CIRCUIT

Construct the input/output table for the following circuit.

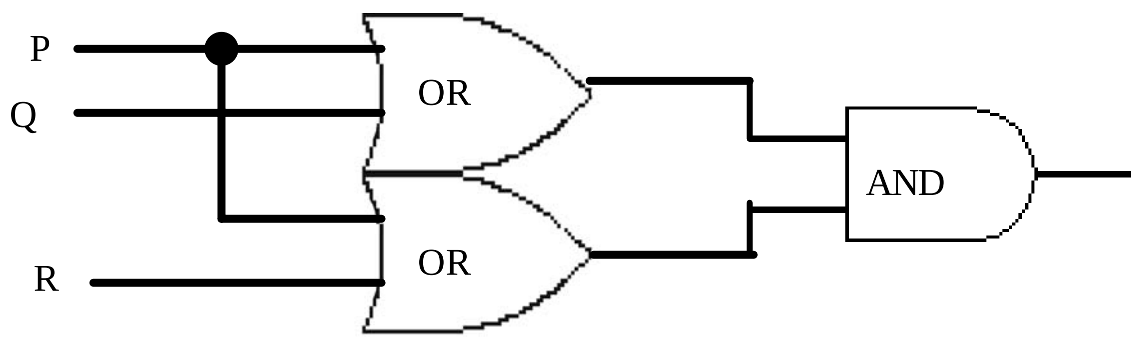


LABELING INTERMEDIATE OUTPUTS:

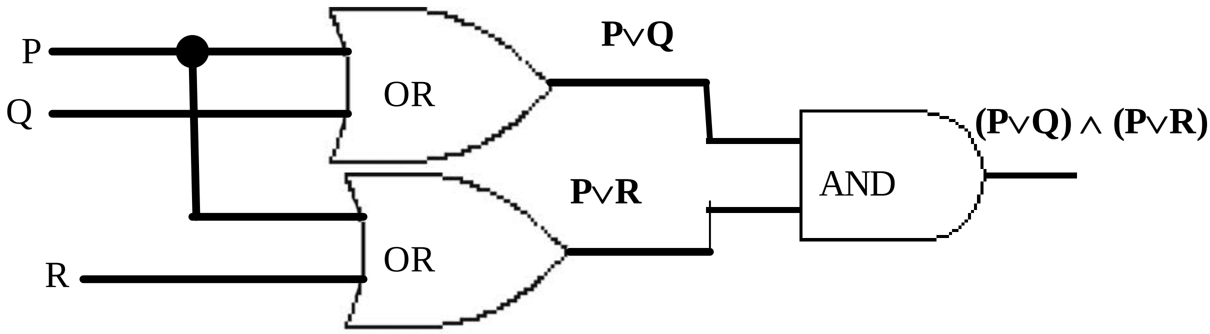


P	Q	R	X	Y	S
1	1	1	1	0	1
1	1	0	1	0	0
1	0	1	0	1	1
1	0	0	0	1	1
0	1	1	0	1	1
0	1	0	0	1	1
0	0	1	0	1	1
0	0	0	0	1	1

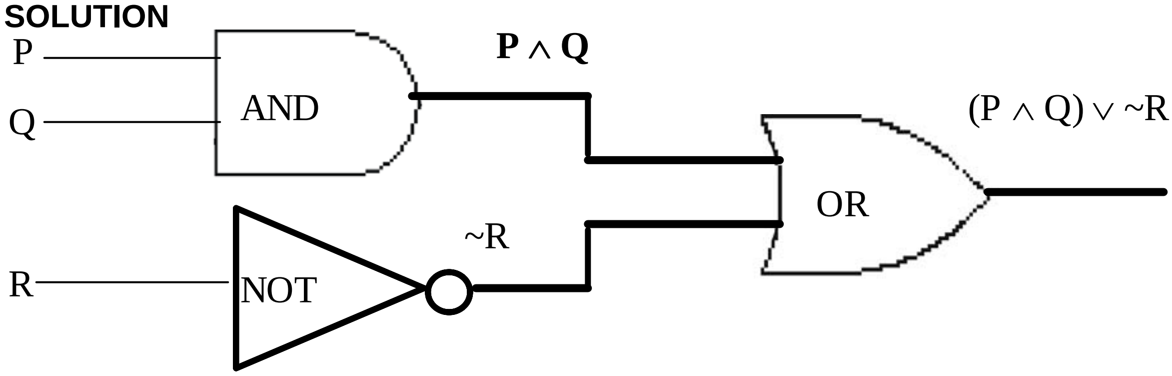
FINDING A BOOLEAN EXPRESSION FOR A CIRCUIT



SOLUTION:
Trace through the circuit from left to right, writing down the output of each logic gate.



Hence **(P∨Q) ∧(P∨R)** is the Boolean expression for this circuit.
CIRCUIT CORRESPONDING TO A BOOLEAN EXPRESSION
EXERCISE
Construct circuit for the Boolean expression (P∧Q) ∨ ~R
SOLUTION



CIRCUIT FOR INPUT/OUTPUT TABLE:

INPUTS			OUTPUT
P	Q	R	S
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	0
0	0	0	0

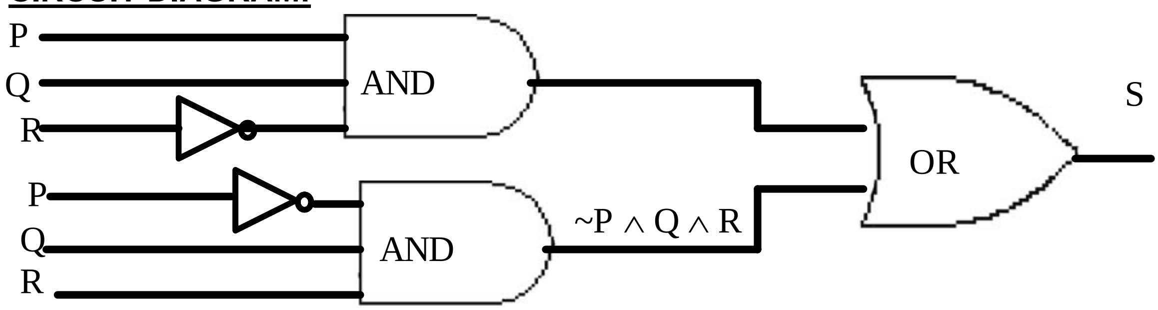
SOLUTION:

INPUTS			OUTPUT
P	Q	R	S
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	0
0	0	0	0

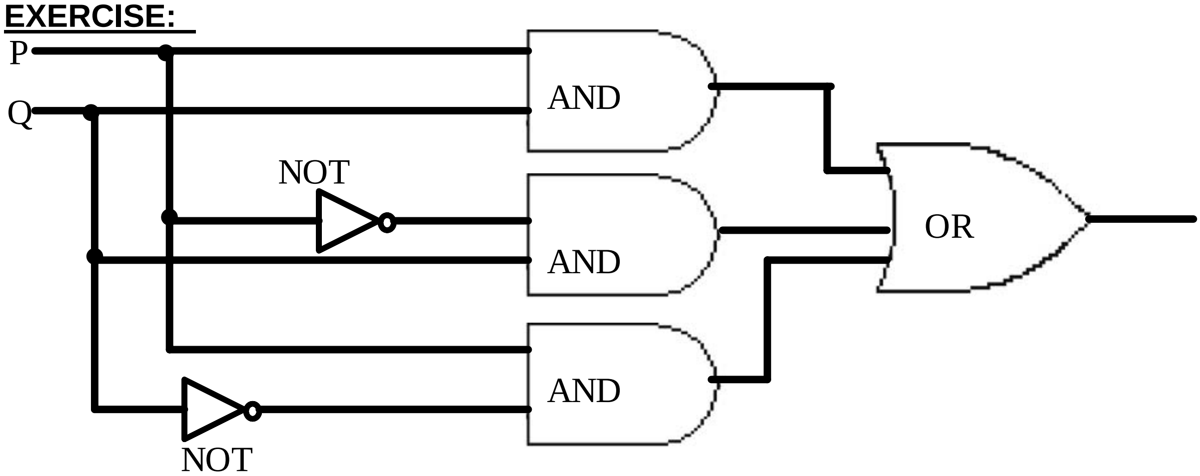
$\longrightarrow$ $P \wedge Q \wedge \sim R$

$\longrightarrow$ $\sim P \wedge Q \wedge R$

CIRCUIT DIAGRAM:

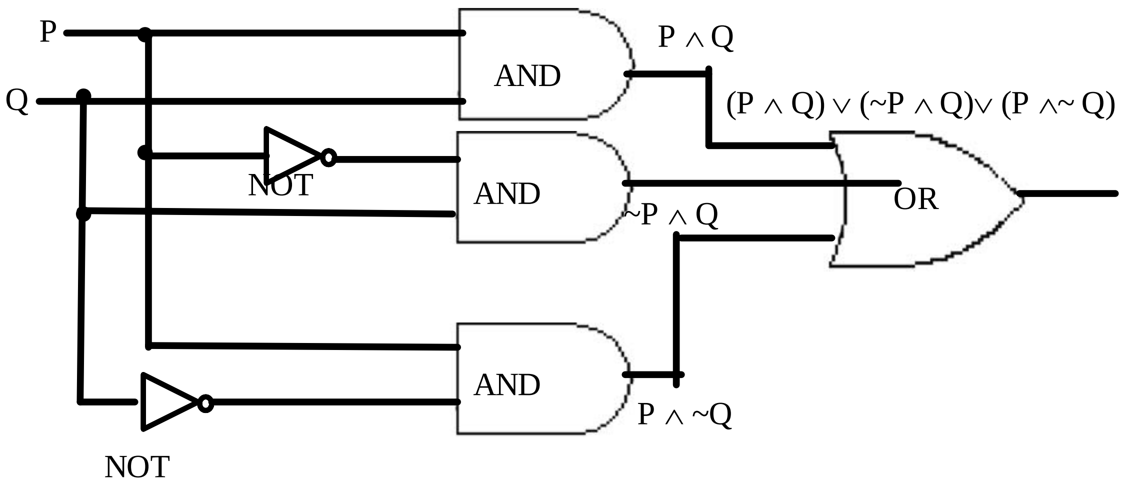


EXERCISE:



SOLUTION:

We find the Boolean expressions for the circuits and show that they are logically equivalent, when regarded as statement forms.



STATEMENT	**REASON**
$(P \wedge Q) \vee (\sim P \wedge Q) \wedge (P \wedge \sim Q)$	
$\equiv (P \wedge Q) \wedge (\sim P \wedge Q) \wedge (P \wedge \sim Q)$	
$\equiv (P \wedge \sim P) \wedge Q \wedge (P \wedge \sim Q)$	Distributive law
$\equiv t \wedge Q \wedge (P \wedge \sim Q)$	Negation law
$\equiv Q \wedge (P \wedge \sim Q)$	Identity law
$\equiv (Q \wedge P) \wedge (Q \wedge \sim Q)$	Distributive law
$\equiv (Q \wedge P) \wedge t$	Negation law
$\equiv (Q \vee P) \vee t$	
$\equiv Q \vee P$	identity law
$\equiv P \vee Q$	Commutative law

Thus $(P \wedge Q) \wedge (\sim P \wedge Q) \wedge (P \wedge \sim Q) \equiv P \wedge Q$

Accordingly, the two circuits are equivalent.

LECTURE # 7

A well defined collection of {distinct}objects is called a set.

- The objects are called the elements or members of the set.
- Sets are denoted by capital letters A, B, C …, X, Y, Z.
- The elements of a set are represented by lower case letters a, b, c, … , x, y, z.
- If an object x is a member of a set A we write x ÎA, which reads "x belongs to A" or "x is in A" or "x is an element of A", otherwise we write x ÏA, which reads "x does not belong to A" or "x is not in A" or "x is not an element of A".

TABULAR FORM

Listing all the elements of a set, separated by commas and enclosed within braces or curly brackets{}.

EXAMPLES

In the following examples we write the sets in Tabular Form.

A = {1, 2, 3, 4, 5} is the set of first five **Natural Numbers**.

B = {2, 4, 6, 8, …, 50} is the set of **Even numbers** up to 50.

C = {1, 3, 5, 7, 9 …} is the set of **positive odd numbers.**

NOTE

The symbol "…" is called an ellipsis. It is a short for "and so forth."

DESCRIPTIVE FORM:

Stating in words the elements of a set.

EXAMPLES

Now we will write the same examples which we write in Tabular Form ,in the Descriptive Form.

A = set of first five Natural Numbers.(is the Descriptive Form)

B = set of positive even integers less or equal to fifty. (is the Descriptive Form)

C = {1, 3, 5, 7, 9, …} (is the Descriptive Form)

C = set of positive odd integers. (is the Descriptive Form)

SET BUILDER FORM:

Writing in symbolic form the common characteristics shared by all the elements of the set.

EXAMPLES:

Now we will write the same examples which we write in Tabular as well as Descriptive Form ,in Set Builder Form .

A = {x ÎN / x<=5} (is the Set Builder Form)

B = {x Î E / 0 < x <=50} (is the Set Builder Form)

C = {x ÎO / 0 < x } (is the Set Builder Form)

SETS OF NUMBERS:

1. **Set of Natural Numbers**

   N = {1, 2, 3, … }

2. **Set of Whole Numbers**

   W = {0, 1, 2, 3, … }

3. **Set of Integers**

   Z = {…, -3, -2, -1, 0, +1, +2, +3, …}

   = {0, ±1, ±2, ±3, …}

{"Z" stands for the first letter of the German word for integer: Zahlen.}

4. **Set of Even Integers**

$E = \{0, \pm 2, \pm 4, \pm 6, \ldots\}$

5. **Set of Odd Integers**

   $O = \{\pm 1, \pm 3, \pm 5, \ldots\}$

6. **Set of Prime Numbers**

   $P = \{2, 3, 5, 7, 11, 13, 17, 19, \ldots\}$

7. **Set of Rational Numbers (or Quotient of Integers)**

   Q = {x | x = ; p, q ∈Z, q ≠ 0}

8. **Set of Irrational Numbers**

   $Q = Q' = \{ x \mid x \text{ is not rational}\}$

   For example, $\sqrt{2}$, $\sqrt{3}$, π, e, etc.

9. **Set of Real Numbers**

   $R = Q \cup Q'$

10. **Set of Complex Numbers**

    $C = \{z \mid z = x + iy;\ x, y \in R\}$

SUBSET:

If A & B are two sets, A is called a subset of B, written $A \subseteq B$, if, and only if, any element of A is also an element of B.

Symbolically:

$A \subseteq B \Leftrightarrow \text{if } x \in A \text{ then } x \in B$

REMARK:

1. When $A \subseteq B$, then B is called a superset of A.
2. When A is not subset of B, then there exist at least one $x \in A$ such that $x \notin B$.
3. Every set is a subset of itself.

EXAMPLES:

Let

$A = \{1, 3, 5\}$ $B = \{1, 2, 3, 4, 5\}$

$C = \{1, 2, 3, 4\}$ $D = \{3, 1, 5\}$

Then

$A \subseteq B$ (Because every element of A is in B)

$C \subseteq B$ (Because every element of C is also an element of B)

$A \subseteq D$ (Because every element of A is also an element of D and also note that every element of D is in A so $D \subseteq A$)

and A is not subset of C .

(Because there is an element 5 of A which is not in C)

EXAMPLE:

The set of integers "Z" is a subset of the set of Rational Number "Q", since every integer 'n' could be written as:

$$n = \frac{n}{1} \in Q$$

Hence $Z \subseteq Q$.

PROPER SUBSET

Let A and B be sets. A is a proper subset of B, if, and only if, every element of A is in B but there is at least one element of B that is not in A, and is denoted as $A \subset B$.

EXAMPLE:

Let $A = \{1, 3, 5\}$ $B = \{1, 2, 3, 5\}$

then $A \subset B$ (Because there is an element 2 of B which is not in A).

EQUAL SETS:

Two sets A and B are equal if, and only if, every element of A is in B and every element of B is in A and is denoted A = B.

Symbolically:

$$A = B \text{ iff } A \subseteq B \text{ and } B \subseteq A$$

EXAMPLE:

Let A = {1, 2, 3, 6} B = the set of positive divisors of 6

C = {3, 1, 6, 2} D = {1, 2, 2, 3, 6, 6, 6}

Then A, B, C, and D are all equal sets.

NULL SET:

A set which contains no element is called a **null set**, or an **empty set** or a **void set**. It is denoted by the Greek letter ∅ (phi) or { }.

EXAMPLE

A = {x | x is a person taller than 10 feet} = ∅ (Because there does not exist any human being which is taller then 10 feet)

B = {x | x2 = 4, x is odd} = ∅ (Because we know that there does not exist any odd whose square is 4)

REMARK

∅ is regarded as a subset of every set.

EXERCISE:

Determine whether each of the following statements is true or false.

a. $x \in \{x\}$ **TRUE**

(Because x is the member of the singleton set { x })

a. $\{x\} \subseteq \{x\}$ **TRUE**

(Because Every set is the subset of itself.

Note that every Set has necessarily tow subsets ∅ and the Set itself, these two subset are known as Improper subsets and any other subset is called Proper Subset)

a. $\{x\} \in \{x\}$ **FALSE**

(Because { x} is not the member of {x}) Similarly other

d. $\{x\} \in \{\{x\}\}$ **TRUE**

e. $\varnothing \subseteq \{x\}$ **TRUE**

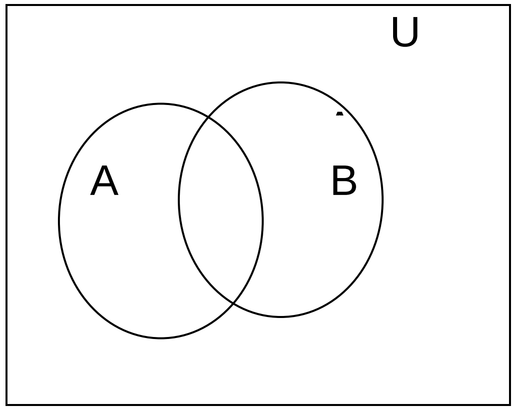
f. $\varnothing \in \{x\}$ **FALSE**

UNIVERSAL SET:

The set of all elements under consideration is called the Universal Set. The Universal Set is usually denoted by U.

VENN DIAGRAM:

A Venn diagram is a graphical representation of sets by regions in the plane. The Universal Set is represented by the interior of a rectangle, and the other sets are represented by disks lying within the rectangle.



FINITE AND INFINITE SETS:

A set **S** is said to be **finite** if it contains exactly *m* distinct elements where *m* denotes some non negative integer.

In such case we write $|S| = m$ or $n(S) = m$

A set is said to be **infinite** if it is not finite.

EXAMPLES:

1. The set S of letters of English alphabets is finite and $|S| = 26$
2. The null set $\varnothing$ has no elements, is finite and $|\varnothing| = 0$
3. The set of positive integers {1, 2, 3,…} is infinite.

EXERCISE:

Determine which of the following sets are finite/infinite.

1.	A = {month in the year}	**FINITE**
2.	B = {even integers}	**INFINITE**
3.	C = {positive integers less than 1}	**FINITE**
4.	D = {animals living on the earth}	**FINITE**
5.	E = {lines parallel to x-axis}	**INFINITE**
6.	$F = \{x \in \mathbf{R} \mid x^{100} + 29x^{50} - 1 = 0\}$	**FINITE**
7.	G = {circles through origin}	**INFINITE**

MEMBERSHIP TABLE:

A table displaying the membership of elements in sets. To indicate that an element is in a set, a 1 is used; to indicate that an element is not in a set, a 0 is used.

Membership tables can be used to prove set identities.

A	A^c
1	0
0	1

The above table is the Member ship table for Complement of A. now in the above table note that if an element is the member of A then it can`t be the member of A^c thus where in the table we have 1 for A in that row we have 0 in A^c.

LECTURE # 8

UNION:

Let A and B be subsets of a universal set U. The union of sets A and B is the set of all elements in U that belong to A or to B or to both, and is denoted $A \cup B$.

Symbolically:

$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$$

EMAMPLE:

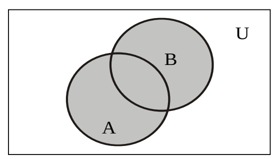
Let U = {a, b, c, d, e, f, g}

A = {a, c, e, g}, B = {d, e, f, g}

Then $A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$

$= \{a, c, d, e, f, g\}$

VENN DIAGRAM FOR UNION:



$A \cup B$ is shaded

REMARK:

1. $A \cup B = B \cup A$ that is union is commutative you can prove this very easily only by using definition.
2. $A \subseteq A \cup B$ and $B \subseteq A \cup B$

The above remark of subset is easily seen by the definition of union.

MEMBERSHIP TABLE FOR UNION:

A	B	$A \cup B$
1	1	1
1	0	1
0	1	1
0	0	0

REMARK:

This membership table is similar to the truth table for logical connective, disjunction ($\vee$).

INTERSECTION:

Let A and B subsets of a universal set U. The intersection of sets A and B is the set of all elements in U that belong to both A and B and is denoted $A \cap B$.

Symbolically:

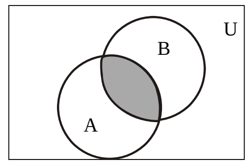
$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$$

EXMAPLE:

Let U = {a, b, c, d, e, f, g}

A = {a, c, e, g}, B = {d, e, f, g}

Then $A \cap B = \{e, g\}$



A ∩ B is shaded

VENN DIAGRAM FOR INTERSECTION:

REMARK:

1. $A \cap B = B \cap A$
2. $A \cap B \subseteq A$ and $A \cap B \subseteq B$
3. If $A \cap B = \phi$, then A & B are called disjoint sets.

MEMBERSHIP TABLE FOR INTERSECTION:

A	B	$A \cap B$
1	1	1
1	0	0
0	1	0
0	0	0

REMARK:

This membership table is similar to the truth table for logical connective, conjunction ($\wedge$).

DIFFERENCE:

Let A and B be subsets of a universal set U. The difference of "A and B" (or relative complement of B in A) is the set of all elements in U that belong to A but not to B, and is denoted A – B or A \ B.

Symbolically:

$A - B = \{x \in U \mid x \in A \text{ and } x \in B\}$

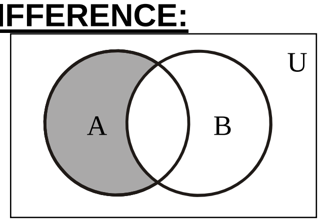
EXAMPLE:

Let $U = \{a, b, c, d, e, f, g\}$

$A = \{a, c, e, g\}$, $B = \{d, e, f, g\}$

Then $A - B = \{a, c\}$

VENN DIAGRAM FOR SET DIFFERENCE:



A-B is shaded

REMARK:

1. $A - B \neq B - A$ that is Set difference is not commutative.

2. $A - B \subseteq A$
3. $A - B$, $A \cap B$ and $B - A$ are mutually disjoint sets.

MEMBERSHIP TABLE FOR SET DIFFERENCE:

A	B	A – B
1	1	0
1	0	1
0	1	0
0	0	0

REMARK:

The membership table is similar to the truth table for $\sim (p \to q)$.

COMPLEMENT:

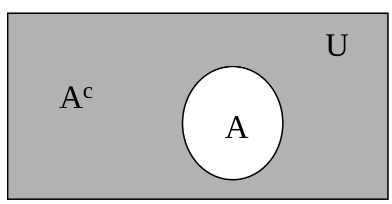
Let A be a subset of universal set U. The complement of A is the set of all element in U that do not belong to A, and is denoted AN, A or Ac
Symbolically:

$$A^c = \{x \in U \mid x \notin A\}$$

EXAMPLE:

Let $U = \{a, b, c, d, e, f, g]$
$A = \{a, c, e, g\}$
Then $A^c = \{b, d, f\}$

VENN DIAGRAM FOR COMPLEMENT:



A^c is shaded

REMARK :

1. $A^c = U - A$
2. $A \cap A^c = \phi$
3. $A \cup A^c = U$

MEMBERSHIP TABLE FOR COMPLEMENT:

A	A^c
1	0
0	1

REMARK

This membership table is similar to the truth table for logical connective negation (~)

EXERCISE:

Let $U = \{1, 2, 3, \ldots, 10\}$, $X = \{1, 2, 3, 4, 5\}$

$Y = \{y \mid y = 2x, x \in X\}, Z = \{z \mid z2 - 9z + 14 = 0\}$

Enumerate:

(1) $X \cap Y$ (2) $Y \cup Z$ (3) $X - Z$

(4) Y^c (5) $X^c - Z^c$ (6) $(X - Z)^c$

Firstly we enumerate the given sets.

Given

$U = \{1, 2, 3, \ldots, 10\}$,

$X = \{1, 2, 3, 4, 5\}$

$Y = \{y \mid y = 2x, x \in X\} = \{2, 4, 6, 8, 10\}$

$Z = \{z \mid z2 - 9z + 14 = 0\} = \{2, 7\}$

(1) $X \cap Y = \{1, 2, 3, 4, 5\} \cap \{2, 4, 6, 8, 10\}$
$= \{2, 4\}$

(2) $Y \cup Z = \{2, 4, 6, 8, 10\} \cup \{2, 7\}$
$= \{2, 4, 6, 7, 8, 10\}$

(3) $X - Z = \{1, 2, 3, 4, 5\} - \{2, 7\}$
$= \{1, 3, 4, 5\}$

(4) $Y^c = U - Y = \{1, 2, 3, \ldots, 10\} - \{2, 4, 6, 8, 10\}$
$= \{1, 3, 5, 7, 9$

(5) $X^c - Z^c = \{6, 7, 8, 9, 10\} - \{1, 3, 4, 5, 6, 8, 9, 10\}$
$= \{7\}$

(6) $(X - Z)c = U - (X - Z)$
$= \{1, 2, 3, \ldots, 10\} - \{1, 3, 4, 5\}$
$= \{2, 6, 7, 8, 9, 10\}$

NOTE $(X - Z)c \neq Xc - Zc$

EXERCISE:

Given the following universal set U and its two subsets P and Q, where

$U = \{x \mid x \in Z, 0 \le x \le 10\}$

$P = \{x \mid x \text{ is a prime number}\}$

$Q = \{x \mid x2 < 70\}$

(i) Draw a Venn diagram for the above

(ii) List the elements in $Pc \cap Q$

SOLUTION:

First we write the sets in Tabular form.

$U = \{x \mid x \in Z,\ 0 \le x \le 10\}$

Since it is the set of integers that are greater then or equal 0 and less or equal to 10. So we have

$U = \{0, 1, 2, 3, \ldots, 10\}$

$P = \{x \mid x \text{ is a prime number}\}$

It is the set of prime numbers between 0 and 10. Remember Prime numbers are those numbers which have only two distinct divisors.

$P = \{2, 3, 5, 7\}$

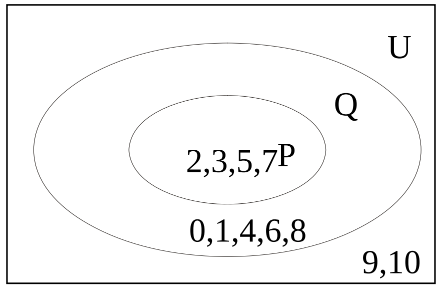
$Q = \{x \mid x2 < 70\}$

The set Q contains the elements between 0 and 10 which has their square less or equal to 70.

$Q = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

Thus we write the sets in Tabular form.

VENN DIAGRAM:



(i) $\mathbf{P^c \cap Q = ?}$

$P^c = U - P = \{0, 1, 2, 3, \ldots, 10\} - \{2, 3, 5, 7\}$
$= \{0, 1, 4, 6, 8, 9, 10\}$
and
$P^c \cap Q = \{0, 1, 4, 6, 8, 9, 10\} \cap \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$
$= \{0, 1, 4, 6, 8\}$

EXERCISE:

Let
$U = \{1, 2, 3, 4, 5\}$, $C = \{1, 3\}$
and A and B are non empty sets. Find A in each of the following:
(i) $A \cup B = U$, $A \cap B = \phi$ and $B = \{1\}$
(ii) $A \subset B$ and $A \cup B = \{4, 5\}$
(iii) $A \cap B = \{3\}$, $A \cup B = \{2, 3, 4\}$ and $B \cup C = \{1,2,3\}$
(iv) A and B are disjoint, B and C are disjoint, and the union of A and B is the set {1, 2}.

(i) $\mathbf{A \cup B = U}$, $\mathbf{A \cap B = \phi}$ **and** $\mathbf{B = \{1\}}$

SOLUTION

Since $A \cup B = U = \{1, 2, 3, 4, 5\}$
and $A \cap B = \phi$,
Therefore $\mathbf{A = B^c = \{1\}^c = \{2, 3, 4, 5\}}$

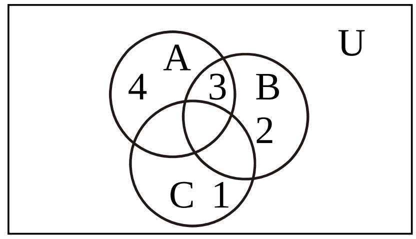
(i) $\mathbf{A \subset B}$ **and** $\mathbf{A \cup B = \{4, 5\}}$ **also** $\mathbf{C = \{1, 3\}}$

SOLUTION

When $A \subset B$, then $A \cup B = B = \{4, 5\}$
Also A being a proper subset of B implies
$\mathbf{A = \{4\}}$ or $\mathbf{A = \{5\}}$

(iii) $\mathbf{A \cap B = \{3\}, A \cup B = \{2, 3, 4\}}$ **and** $\mathbf{B \cup C = \{1,2,3\}}$
Also $\mathbf{C = \{1, 3\}}$

SOLUTION

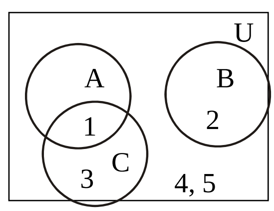


Since we have 3 in the intersection of A and B as well as in C so we place 3 in common part shared by the three sets in the Venn diagram. Now since 1 is in the union of B and C it means that 1 may be in C or may be in B, but 1cannot be in B because if 1 is in the B then it must be in $A \cup B$ but 1 is not there, thus we place 1 in the part of C which is not shared by any other set. Same is the reason for 4 and we place it in the set which is not shared by any other set. Now 2 will be in B, 2 cannot be in A because $A \cap B = \{3\}$, and is not in C.

So A = {3, 4} and B = {2, 3}

(i) **$A \cap B = \phi$, $B \cap C = \phi$, $A \cup B = \{1, 2\}$.**
Also C = {1, 3}

SOLUTION

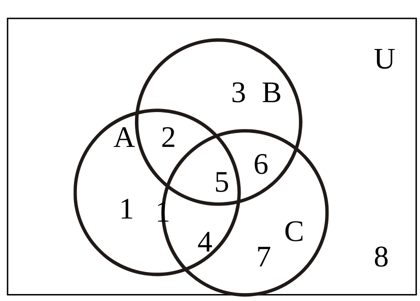


A = {1}

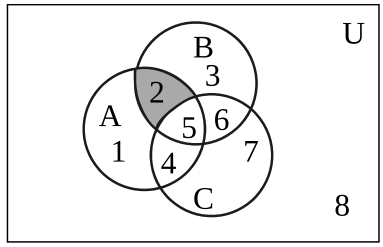
EXERCISE:

Use a Venn diagram to represent the following:

(i) $(A \cap B) \cap C^c$
(ii) $A^c \cup (B \cup C)$
(iii) $(A - B) \cap C$
(iv) $(A \cap B^c) \cup C^c$

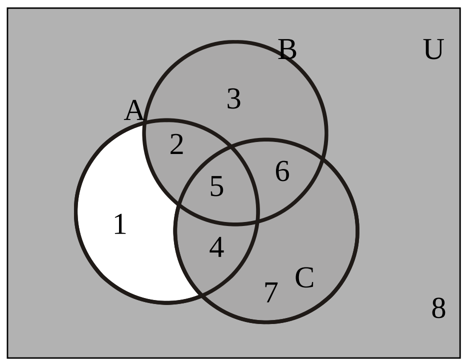


(1) **$(A \cap B) \cap C^c$**

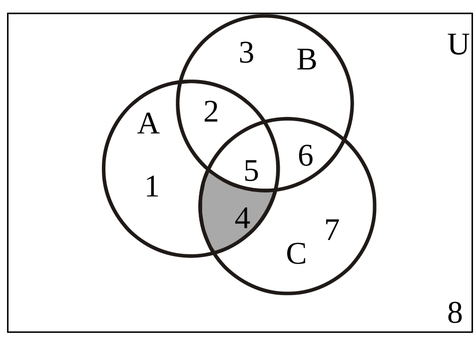


$(A \cap B) \cap C^c$ is shaded

(ii) $\mathbf{A^c \cup (B \cup C)}$ **is shaded.**

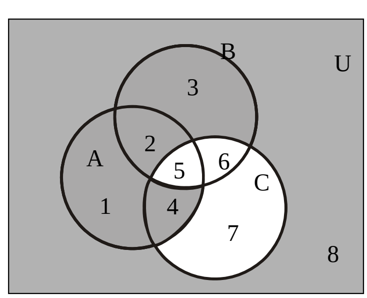


(iii) $\mathbf{(A - B) \cap C}$



$(A - B) \cap C$ is shaded

(iii) $\mathbf{(A \cap B^c) \cup C^c}$ **is shaded.**



PROVING SET IDENTITIES BY VENN DIAGRAMS:

Prove the following using Venn Diagrams:

(i) $A - (A - B) = A \cap B$

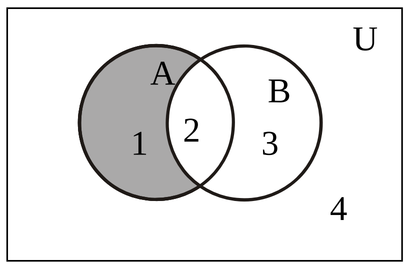
(ii) $(A \cap B)^c = A^c \cup B^c$

(iii) $A - B = A \cap B^c$

SOLUTION (i)

$A - (A - B) = A \cap B$

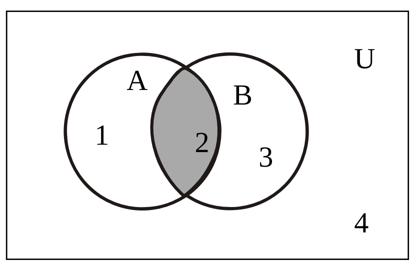
(a)



$A = \{ 1, 2 \}$
$B = \{ 2, 3 \}$
$A - B = \{ 1 \}$

A – B is shaded

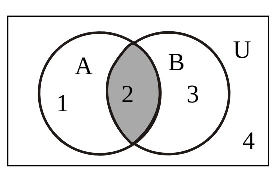
(b)



$A = \{ 1, 2 \}$
$A - B = \{ 1 \}$
$A - (A - B) = \{ 2 \}$

A – (A – B) is shaded

(c)



$A \cap B$ is shaded

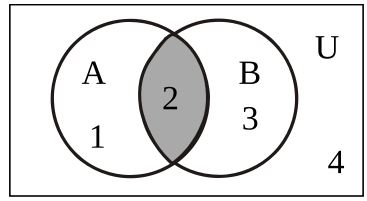
$A = \{ 1, 2 \}$
$B = \{ 2, 3 \}$
$A \cap B = \{2\}$

RESULT: $A - (A - B) = A \cap B$

SOLUTION (ii)

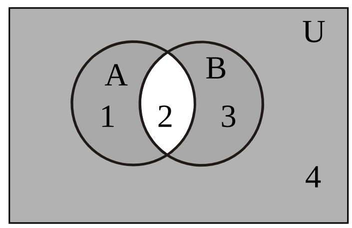
$(A \cap B)^c = A^c \cup B^c$

(a)



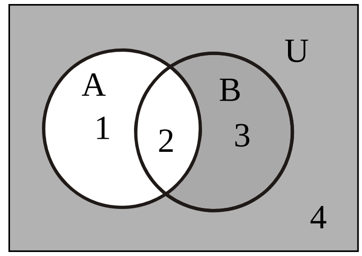
$A \cap B$

(b)



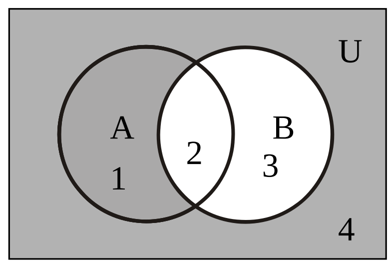
$(A \cap B)^c$

(c)



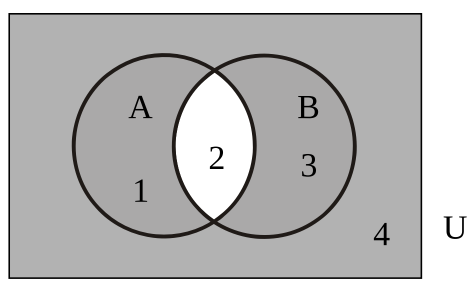
A^c is shaded.

(d)



B^c is shaded.

(e)



$A^c \cup B^c$ is shaded.

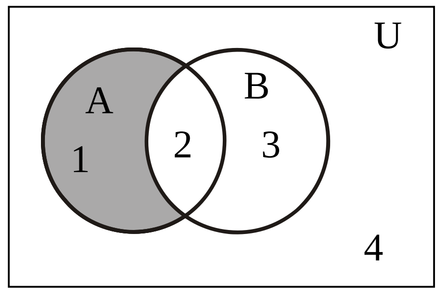
Now diagrams (b) and (e) are same hence

RESULT: $(A \cap B)^c = A^c \cup B^c$

SOLUTION (iii)

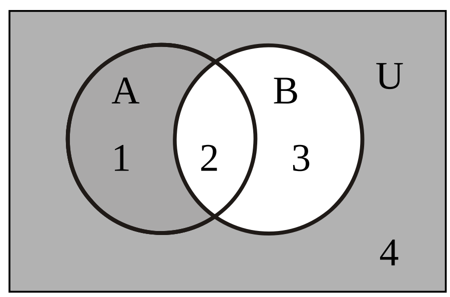
$A - B = A \cap B^c$

(a)



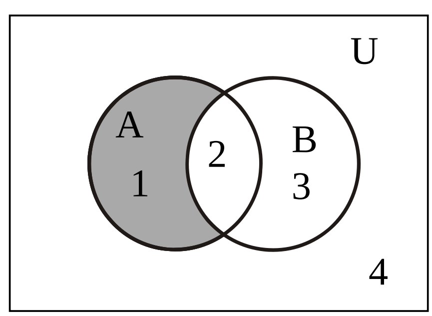
A – B is shaded.

(b)



B^c is shaded.

(c)



$A \cap B^c$ is shaded

From diagrams (a) and (b) we can say

RESULT: $A - B = A \cap B^c$

<u>PROVING SET IDENTITIES BY MEMBERSHIP TABLE:</u>

Prove the following using Membership Table:

(i) $A - (A - B) = A \cap B$

(ii) $(A \cap B)^c = A^c \cup B^c$

(iii) $A - B = A \cap B^c$

SOLUTION (i)

$$A - (A - B) = A \cap B$$

A	B	A-B	A-(A-B)	A∩B
1	1	0	1	1
1	0	1	0	0
0	1	0	0	0
0	0	0	0	0

Since the last two columns of the above table are same hence the corresponding set expressions are same. That is

$$A - (A - B) = A \cap B$$

SOLUTION (ii)

$$(A \cap B)^c = A^c \cup B^c$$

A	B	A∩B	$(A\cap B)^c$	A^c	B^c	$A^c \cup B^c$
1	1	1	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	0	1	1	1	1

Since the fourth and last columns of the above table are same hence the corresponding set expressions are same. That is

$$(A \cap B)^c = A^c \cup B^c$$

SOLUTION (iii)

A	B	A – B	B^c	$A \cap B^c$
1	1	0	0	0
1	0	1	1	1
0	1	0	0	0
0	0	0	1	0

LECTURE # 9

SET IDENTITIES:

Let A, B, C be subsets of a universal set U.

1. Idempotent Laws
   a. $A \cup A = A$ b. $A \cap A = A$
2. Commutative Laws
   a. $A \cup B = B \cup A$ b. $A \cap B = B \cap A$
3. Associative Laws
   a. $A \cup (B \cup C) = (A \cup B) \cup C$
   b. $A \cap (B \cap C) = (A \cap B) \cap C$
4. Distributive Laws
   a. $A \cup (B \cap C) = (A \cup B) \cap (A \cup B)$
   b. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
5. Identity Laws
   a. $A \cup \varnothing = A$ b. $A \cap \varnothing = \varnothing$
   c. $A \cup U = U$ d. $A \cap U = A$
6. Complement Laws
   a. $A \cup A^c = U$ b. $A \cap A^c = \varnothing$
   c. $U^c = \varnothing$ d. $\varnothing^c = U$
8. Double Complement Law
   $(A^c)^c = A$
9. DeMorgan's Laws
   a. $(A \cup B)c = Ac \cap Bc$ b. $(A \cap B)c = Ac \cup Bc$
10. Alternative Representation for Set Difference
   $A – B = A \cap Bc$
11. Subset Laws
   a. $A \cup B \subseteq C$ *iff* $A \subseteq C$ and $B \subseteq C$
   b. $C \subseteq A \cap B$ *iff* $C \subseteq A$ and $C \subseteq B$
12. Absorption Laws
   a. $A \cup (A \cup B) = A$ b. $A \cap (A \cup B) = A$

EXERCISE:

1. $A \subseteq A \cup B$
2. $A – B \subseteq A$
3. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$
4. $A \subseteq B$ if, and only if, $B^c \subseteq A^c$

1. Prove that $A \subseteq A \cup B$

SOLUTION

Here in order to prove the identity you should remember the definition of Subset of a set. We will take the arbitrary element of a set then show that, that element is the member of the other then the first set is the subset of the other. So

Let x be an arbitrary element of A, that is $x \in A$.

$\Rightarrow$ $x \in A$ or $x \in B$

$\Rightarrow$ $x \in A \cup B$

But x is an arbitrary element of A.

$\therefore$ $A \subseteq A \cup B$ (proved)

1. Prove that $A – B \subseteq A$

SOLUTION:

Let $x \in A - B$

$\Rightarrow$ $x \in A$ and $x \notin B$ (by definition of A – B)

$\Rightarrow$ $x \in A$ (in particular)

But x is an arbitrary element of A – B

$\therefore$ $A - B \subseteq A$ (proved)

1. **Prove that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$**

SOLUTION

Suppose that $A \subseteq B$ and $B \subseteq C$

Consider $x \in A$

$\Rightarrow$ $x \in B$ (as $A \subseteq B$)

$\Rightarrow$ $x \in C$ (as $B \subseteq C$)

But x is an arbitrary element of A

$\therefore$ $A \subseteq C$ (proved)

1. **Prove that $A \subseteq B$ iff $B^c \subseteq A^c$**

SOLUTION:

Suppose $A \subseteq B$ {To prove $B^c \subseteq A^c$}

Let $x \in B^c$

$\Rightarrow$ $x \notin B$ (by definition of B^c)

$\Rightarrow$ $x \notin A$

$\Rightarrow$ $x \in A^c$ (by definition of A^c)

Now we know that implication and its contrapositivity are logically equivalent and the contrapositive statement of if $x \in A$ then $x \in B$ is: if $x \notin B$ then $x \notin A$ which is the definition of the $A \subseteq B$. Thus if we show for any two sets A and B, if $x \notin B$ then $x \notin A$ it means that

$A \subseteq B$. Hence

But x is an arbitrary element of B^c

$\therefore B^c \subseteq A^c$

Conversely,

Suppose $B^c \subseteq A^c$ {To prove $A \subseteq B$}

Let $x \in A$

$\Rightarrow$ $x \notin A^c$ (by definition of A^c)

$\Rightarrow$ $x \notin B^c$ ($\therefore B^c \subseteq A^c$)

$\Rightarrow$ $x \in B$ (by definition of B^c)

But x is an arbitrary element of A.

$\therefore A \subseteq B$ (proved)

EXERCISE:

Let A and B be subsets of a universal set U.

Prove that $A - B = A \cap B^c$.

SOLUTION:

Let $x \in A - B$

$\Rightarrow$ $x \in A$ and $x \notin B$ (definition of set difference)

$\Rightarrow$ $x \in A$ and $x \in B^c$ (definition of complement)

$\Rightarrow$ $x \in A \cap B^c$ (definition of intersection)

But x is an arbitrary element of A – B so we can write

$\therefore A - B \subseteq A \cap B^c$…………(1)

Conversely,

let $y \in A \cap B^c$

$\Rightarrow$ $y \in A$ and $y \in B^c$ (definition of intersection)

$\Rightarrow$ $y \in A$ and $y \notin B$ (definition of complement)

$\Rightarrow$ $y \in A - B$ (definition of set difference)

But y is an arbitrary element of $A \cap B^c$

$\therefore A \cap B^c \subseteq A - B$………… (2)

From (1) and (2) it follows that

$A - B = A \cap B^c$ (as required)

EXERCISE:

Prove the DeMorgan's Law: $(A \cup B)c = Ac \cap Bc$

PROOF

Let $x \in (A \cup B)^c$

$\Rightarrow$ $x \notin A \cup B$ (definition of complement)

$x \notin A$ and $x \notin B$ (DeMorgan's Law of Logic)

$\Rightarrow$ $x \in A^c$ and $x \in B^c$ (definition of complement)

$\Rightarrow$ $x \in A^c \cap B^c$ (definition of intersection)

But x is an **arbitrary** element of $(A \cup B)^c$ so we have proved that

$\therefore (A \cup B)^c \subseteq A^c \cap B^c$………(1)

Conversely

let $y \in Ac \cap B^c$

$\Rightarrow$ $y \in A^c$ and $y \in B^c$ (definition of intersection)

$\Rightarrow$ $y \notin A$ and $y \notin B$ (definition of complement)

$\Rightarrow$ $y \notin A \cup B$ (DeMorgan's Law of Logic)

$\Rightarrow$ $y \in (A \cup B)^c$ (definition of complement)

But y is an arbitrary element of $A^c \cap B^c$

$\therefore$ $A^c \cap B^c \subseteq (A \cup B)^c$………………(2)

From (1) and (2) we have

$(A \cup B)^c = A^c \cap B^c$

Which is the Demorgan`s Law.

EXERCISE:

Prove the associative law: $A \cap (B \cap C) = (A \cap B) \cap C$

PROOF:

Consider $x \in A \cap (B \cap C)$

$\Rightarrow$ $x \in A$ and $x \in B \cap C$ (definition of intersection)

$\Rightarrow$ $x \in A$ and $x \in B$ and $x \in C$ (definition of intersection)

$\Rightarrow$ $x \in A \cap B$ and $x \in C$ (definition of intersection)

$\Rightarrow$ $x \in (A \cap B) \cap C$ (definition of intersection)

But x is an arbitrary element of $A \cap (B \cap C)$

$\therefore A \cap (B \cap C) \subseteq (A \cap B) \cap C$……(1)

Conversely

let $y \in (A \cap B) \cap C$

$\Rightarrow$ $y \in A \cap B$ and $y \in C$ (definition of intersection)

$\Rightarrow$ $y \in A$ and $y \in B$ and $y \in C$ (definition of intersection)

$\Rightarrow$ $y \in A$ and $y \in B \cap C$ (definition of intersection)

$\Rightarrow$ $y \in A \cap (B \cap C)$ (definition of intersection)

But y is an arbitrary element of $(A \cap B) \cap C$

$\therefore$ $(A \cap B) \cap C \subseteq A \cap (B \cap C)$……..(2)

From (1) & (2), we conclude that

$$A \cap (B \cap C) = (A \cap B) \cap C \qquad \text{(proved)}$$

EXERCISE:

Prove the distributive law: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

PROOF:

Let $x \in A \cup (B \cap C)$

$\Rightarrow$ $x \in A$ or $x \in B \cap C$ (definition of union)

Now since we have $x \in A$ or $x \in B \cap C$ it means that either x is in A or in $A \cap B$ it is in the $A \cup (B \cap C)$ so in order to show that

$A \cup (B \cap C)$ is the subset of $(A \cup B) \cap (A \cup C)$ we will consider both the cases when x is iu A or x is in $B \cap C$ hence we will consider the two cases.

CASE I:

(when $x \in A$)

$\Rightarrow x \in A \cup B$ and $x \in A \cup C$ (definition of union)

Hence,

$x \in (A \cup B) \cap (A \cup C)$(definition of intersection)

CASE II:

(when $x \in B \cap C$)

We have $x \in B$ and $x \in C$ (definition of intersection)

Now $x \in B \Rightarrow x \in A \cup B$ (definition of union)

and $x \in C \Rightarrow x \in A \cup C$ (definition of union)

Thus $x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow$ $x \in (A \cup B) \cap (A \cup C)$

In both of the cases $x \in (A \cup B) \cap (A \cup C)$

Accordingly,

$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$……..(1)

Conversely,

Suppose $x \in (A \cup B) \cap (A \cup C)$

$\Rightarrow$ $x \in (A \cup B)$ and $x \in (A \cup C)$ (definition of intersection)

Consider the two cases $x \in A$ and $x \notin A$

CASE I: (when $x \in A$)

We have $x \in A \cup (B \cap C)$ (definition of union)

CASE II: (when $x \notin A$)

Since $x \in A \cup B$ and $x \notin A$, therefore $x \in B$

Also, since $x \in A \cup C$ and $x \notin A$, therefore $x \in C$. Thus $x \in B$ and $x \in C$

That is, $x \in B \cap C$

$\Rightarrow$ $x \in A \cup (B \cap C)$ (definition of union)

Hence in both cases

$x \in A \cup (B \cap C)$

$\therefore (A \cup B) \cap C\ (A \cup C) \subseteq A \cup (B \cap C)$……..(2)

By (1) and (2), it follows that

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (proved)

EXERCISE:

For any sets A and B if $A \subseteq B$ then

(a) $A \cap B = A$ (b) $A \cup B = B$

SOLUTION:

(a) Let $x \in A \cap B$

$\Rightarrow$ x $\in$A and x $\in$B
$\Rightarrow$ x $\in$A (in particular)
Hence A $\cap$ B $\subseteq$ A…………..(1)

Conversely,
let x $\in$A.
Then x $\in$B (since A $\subseteq$ B)
Now x $\in$A and x $\in$B, therefore x $\in$A $\cap$ B
Hence, A $\subseteq$ A $\cap$ B…………..(2)
From (1) and (2) it follows that
A = A $\cap$ B (proved)

(b) Prove that A $\cup$ B = B when A $\subseteq$ B

SOLUTION:
Suppose that A $\subseteq$ B. Consider x $\in$A $\cup$ B.
CASE I (when x $\in$A)
Since A $\subseteq$ B, x $\in$A $\Rightarrow$ x $\in$B
CASE II (when x $\notin$A)
Since x $\in$A $\cup$ B, we have x $\in$B
Thus x $\in$B in both the cases, and we have
A $\cup$ B $\subseteq$ B……………(1)
Conversely
let x $\in$B. Then clearly, x $\in$A $\cup$ B
Hence B $\subseteq$ A $\cup$ B…………….(2)
Combining (1) and (2), we deduce that
A $\cup$ B = B (proved)

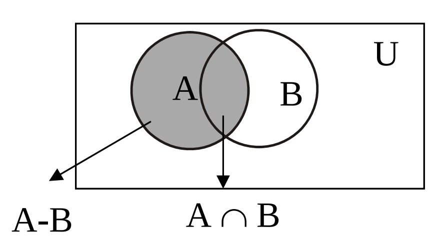
USING SET IDENTITIES:

For all subsets A and B of a universal set U, prove that

$$(A - B) \cup (A \cap B) = A$$

PROOF:
LHS = (A – B) $\cup$ (A $\cap$ B)
= (A $\cap$ B^c) $\cup$ (A $\cap$ B) (Alternative representation for set difference)
= A $\cap$ (B^c $\cup$ B) Distributive Law
= A $\cap$ U Complement Law
= A Identity Law
= RHS (proved)

The result can also be seen by Venn diagram.



EXERCISE:

For any two sets A and B prove that A – (A – B) = A $\cap$ B

SOLUTION

LHS = $A - (A - B)$
$= A - (A \cap B^c)$ Alternative representation for set difference
$= A \cap (A \cap B^c)^c$ Alternative representation for set difference
$= A \cap (A^c \cup (B^c)^c)$ DeMorgan's Law
$= A \cap (A^c \cup B)$ Double Complement Law
$= (A \cap A^c) \cup (A \cap B)$ Distributive Law
$= \varnothing \cup (A \cap B)$ Complement Law
$= A \cap B$ Identity Law
= RHS (proved)

EXERCISE:

For all set A, B, and C prove that $(A - B) - C = (A - C) - B$

SOLUTION

LHS = $(A - B) - C$
$= (A \cap B^c) - C$ Alternative representation of set difference
$= (A \cap B^c) \cap C^c$ Alternative representation of set difference
$= A \cap (B^c \cap C^c)$ Associative Law
$= A \cap (C^c \cap B^c)$ Commutative Law
$= (A \cap C^c) \cap B^c$ Associative Law
$= (A - C) \cap B^c$ Alternative representation of set difference
$= (A - C) - B$ Alternative representation of set difference
= RHS (proved)

EXERCISE:

Simplify $(B^c \cup (B^c - A))^c$

SOLUTION

$(B^c \cup (B^c - A))^c = (B^c \cup (B^c \cap A^c))^c$
Alternative representation for set difference
$= (B^c)^c \cap (B^c \cap A^c)^c$ DeMorgan's Law
$= B \cap ((Bc)c \cup (Ac)c)$ DeMorgan's Law
$= B \cap (B \cup A)$ Double Complement Law
$= B$ Absorption Law

is the simplified form of the given expression.

PROVING SET IDENTITIES BY MEMBERSHIP TABLE:

Prove the following using Membership Table:

(i) $A - (A - B) = A \cap B$
(ii) $(A \cap B)^c = A^c \cup B^c$
(iii) $A - B = A \cap B^c$

$A - (A - B) = A \cap B$

A	B	A-B	A-(A-B)	A∩B
1	1	0	1	1
1	0	1	0	0
0	1	0	0	0
0	0	0	0	0

$(A \cap B)^c = A^c \cup B^c$

A	B	$A \cap B$	$(A \cap B)^c$	A^c	B^c	$A^c \cup B^c$
1	1	1	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	0	1	1	1	1

SOLUTION (iii):

A	B	$A - B$	B^c	$A \cap B^c$
1	1	0	0	0
1	0	1	1	1
0	1	0	0	0
0	0	0	1	0

LECTURE # 10

Exercise:

A number of computer users are surveyed to find out if they have a printer, modem or scanner. Draw separate Venn diagrams and shade the areas, which represent the following configurations.

(i) modem and printer but no scanner
(ii) scanner but no printer and no modem
(iii) scanner or printer but no modem.
(iv) no modem and no printer.

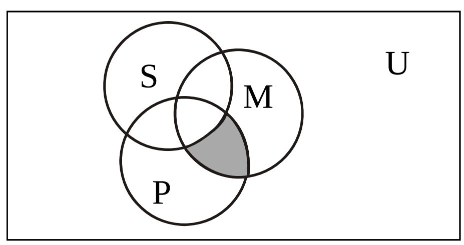
SOLUTION

Let

P represents the set of computer users having printer.
M represent the set of computer users having modem.
S represents the set of computer users having scanner.

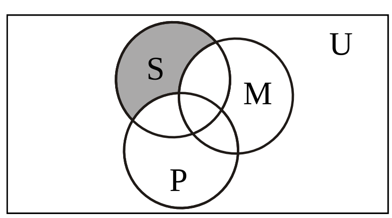
SOLUTION (i)

Modem and printer but no Scanner is shaded.



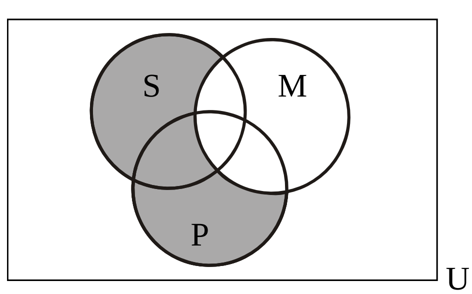
SOLUTION (ii)

Scanner but no printer and no modem is shaded.



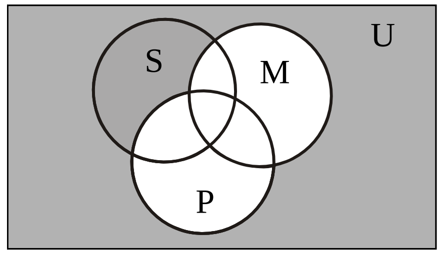
SOLUTION (iii)

scanner or printer but no modem is shaded.



SOLUTION (iv)

no modem and no printer is shaded.

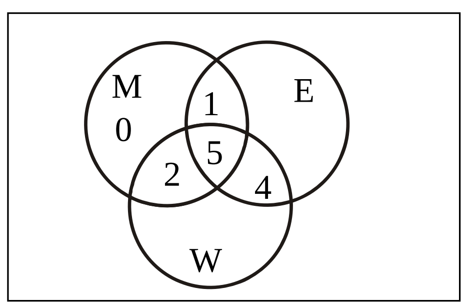


EXERCISE:

Of **21 typists** in an office, **5** use all **manual typewriters (M), electronic typewriters (E) and word processors (W); 9** use **E** and **W**; **7** use **M** and **W**; **6** use **M** and **E**; but **no one** uses **M** only.

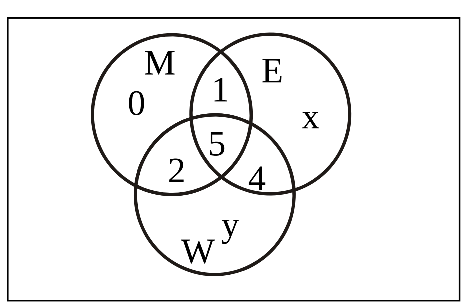
(i) Represent this information in a Venn Diagram.
(ii) If the same number of typists use electronic as use word processors, then
1. (a) How many use word processors only,
2. (b) How many use electronic typewriters?

SOLUTION (i)



SOLUTION (ii-a)

Let the number of typists using electronic typewriters (E) only be x, and the number of typists using word processors (W) only be y.



Total number of typists using E = Total Number of typists using W

$1 + 5 + 4 + x = 2 + 5 + 4 + y$

or, $x – y = 1$ …………(1)

Also, total number of typists = 21

$\Rightarrow \quad 0 + x + y + 1 + 2 + 4 + 5 = 21$

or, $\quad x + y = 9 \qquad \ldots\ldots\ldots\ldots (2)$

Solving (1) & (2), we get

$x = 5, \quad y = 4$

$\therefore$ Number of typists using word processor only is y = 4

(ii)-(b) How many typists use electronic typewriters?

SOLUTION:

Typists using electronic typewriters = No. of elements in E

$= 1 + 5 + 4 + x$

$= 1 + 5 + 4 + 5$

$= 15$

EXERCISE

In a school, 100 students have access to three software packages, A, B and C

28 did not use any software

8 used only packages A

26 used only packages B

7 used only packages C

10 used all three packages

13 used both A and B

(i) Draw a Venn diagram with all sets enumerated as for as possible. Label the two subsets which cannot be enumerated as x and y, in any order.

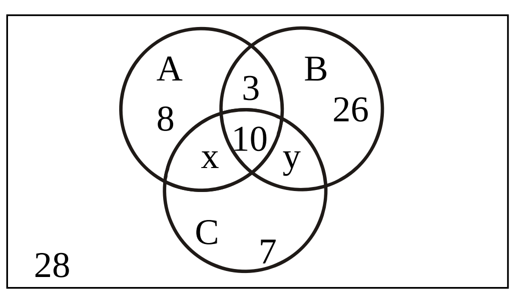
(ii) If twice as many students used package B as package A, write down a pair of simultaneous equations in x and y.

(iii) Solve these equations to find x and y.

(iv) How many students used package C?

SOLUTION(i)

Venn Diagram with all sets enumerated.



(ii) If twice as many students used package B as package A, write down a pair of simultaneous equations in x and y.

SOLUTION:

We are given

students using package B = 2 (# students using package A)

Now the number of students which used package B and A are clear from the diagrams given below. So we have the following equation

$\Rightarrow$ $3 + 10 + 26 + y = 2 (8 + 3 + 10 + x)$

$\Rightarrow$ $39 + y = 42 + 2x$

or $y = 2x + 3$ ....(1)

Also, total number of students = 100.

Hence, $8 + 3 + 26 + 10 + 7 + 28 + x + y = 100$

or $82 + x + y = 100$

or $x + y = 18$ ....(2)

(iii) Solving simultaneous equations for x and y.

SOLUTION:

$y = 2x + 3$ ....(1)

$x + y = 18$ ....(2)

Using (1) in (2), we get,

$x + (2 x + 3) =18$

or $3x + 3=18$

or $3x=15$

$\Rightarrow$ $x=5$

Consequently $y= 13$

How many students used package C?

SOLUTION:

No. of students using package C

$= x + y + 10 + 7$

$= 5 + 13 + 10 + 7$

$= 35$

EXAMPLE:

Use diagrams to show the validity of the following argument:

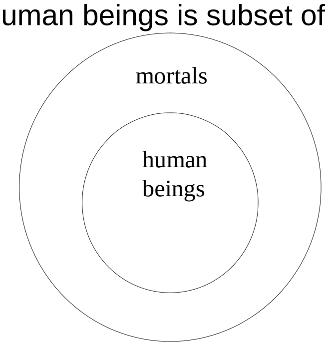
All human beings are mortal

Zeus is not mortal

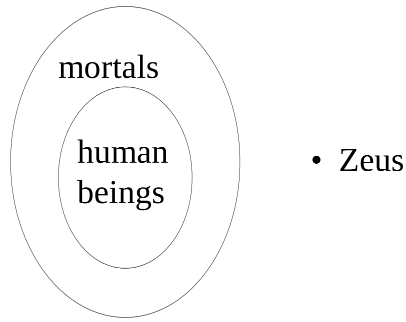
$\therefore$ Zeus is not a human being

SOLUTION:

The premise "All human beings are mortal is pictured by placing a disk labeled "human beings" inside a disk labeled "mortals". We place the disk of human Beings in side the Disk of mortals because there are things which are mortal but not Human beings so the set of human beings is subset of set of Mortals.



The second premise "Zeus is not mortal" could be pictured by placing a dot labeled "Zeus" outside the disk labeled "mortals"



Argument is valid.

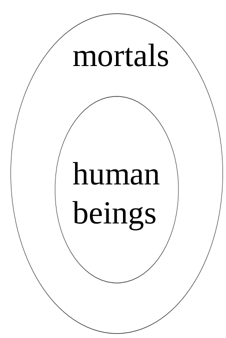
EXAMPLE:

Use a diagram to show the invalidity of the following argument:

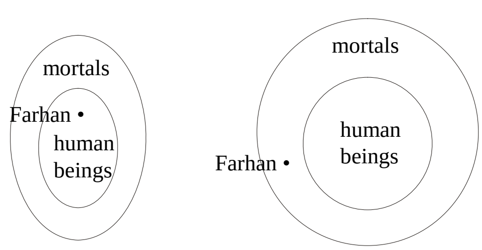
All human beings are mortal.
Farhan is mortal
∴ Farhan is a human being

SOLUTION:

The first premise “All human beings are mortal” is pictured as:



The second premise “Farhan is mortal” is represented by a dot labeled “Farhan” inside the mortal disk in either of the following two ways:



argument is invalid.

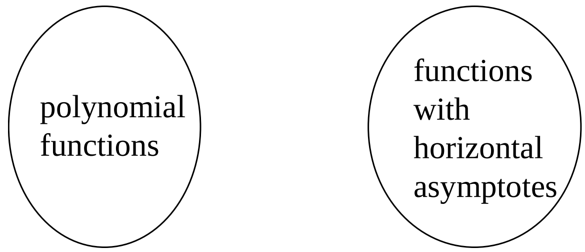
EXAMPLE

Use diagrams to test the following argument for validity:

No polynomial functions have horizontal asymptotes.
This function has a horizontal asymptote.
∴ This function is not polynomial.

SOLUTION

The premise “No polynomial functions have horizontal asymptotes” can be represented diagrammatically by two**disjoint** disks labeled “polynomial functions” and “functions with horizontal tangents.



The argument is valid.

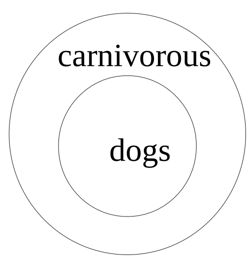
EXERCISE:

Use a diagram to show that the following argument can have true premises and a false conclusion.

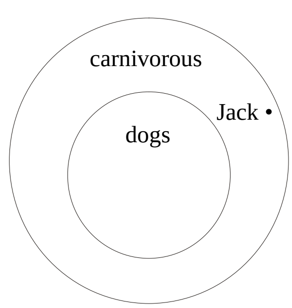
All dogs are carnivorous.
Jack is not a dog.
∴ Jack is not carnivorous

SOLUTION:

The premise “All dogs are carnivorous” is pictured by placing a disk labeled “dogs” inside a disk labeled “carnivorous”. :



The second premise “Jack is not a dog” could be represented by placing a dot outside the disk labeled “dogs” but inside the disk labeled “carnivorous” to make the conclusion “Jack is not carnivorous” false.



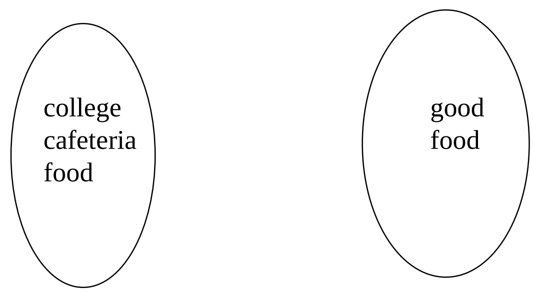
EXERCISE:

Indicate by drawing diagrams, whether the argument is valid or invalid.

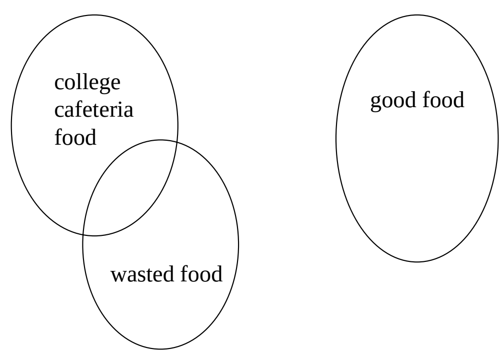
No college cafeteria food is good.
No good food is wasted.
∴ No college cafeteria food is wasted.

SOLUTION

The premise "No college food is good" could be represented by two disjoint disks shown below.



The next premise "No good food is wasted" introduces another disk labeled "wasted food" that does not overlap the disk labeled "good food", but may intersect with the disk labeled "college cafeteria food."



Argument is **invalid**

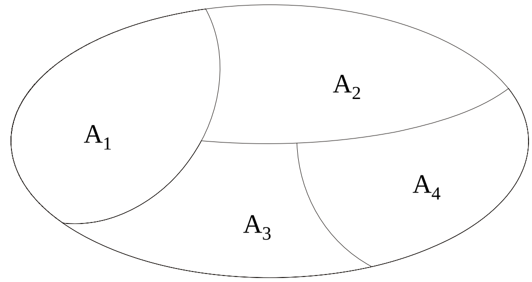
PARTITION OF A SET

A set may be divided up into its disjoint subsets. Such division is called a partition.

More precisely,

A partition of a set **A** is a collection of non- empty subsets $\{A_1, A_2, \ldots A_n\}$ of A, such that

1. $A = A_1 \cup, A_2 \cup \ldots \cup A_n$
2. $A_1, A_2, \ldots, A_n$ are mutually disjoint (or pair wise disjoint), i.e., $\forall$ i, j = 1, 2, …, n $A_i \cap A_j = \emptyset$ whenever $i \neq j$



A partition of a set

POWER SET:

The power set of a set A is the set of all subsets of A, denoted *P*(A).

EXAMPLE:

Let A = {1, 2}, then
P(A) = {∅, {1}, {2}, {1, 2}}

REMARK:

If A has n elements then *P*(A) has 2^n elements.

EXERCISE

a. Find *P*(∅) b. Find *P*(*P*(∅)) c. Find *P*(*P*(*P*(∅)))

SOLUTION:

a. Since ∅ contains no element, therefore *P*(∅) will contain 2^0=1 element.

P(∅) = {∅}

a. Since *P*(∅) contains one element, namely φ, therefore *P*(∅) will contain 2^1 = 2 elements

P(*P*(∅)) = {∅,{∅}}

a. Since *P*(*P*(∅)) contains two elements, namely ∅ and {∅}, so *P*(*P*(*P*(∅))) will contain 2^2 = 4 elements.

P(*P*(*P*(∅)))= {∅,{∅},{{∅}},{{∅}}, {∅,{∅}}}

LECTURE # 11

ORDERED PAIR:

An ordered pair (a, b) consists of two elements "a" and "b" in which "a" is the first element and "b" is the second element.

The ordered pairs (a, b) and (c, d) are equal if, and only if, a= c and b = d.

Note that (a, b) and (b, a) are not equal unless a = b.

EXERCISE:

Find x and y given (2x, x + y) = (6, 2)

SOLUTION:

Two ordered pairs are equal if and only if the corresponding components are equal. Hence, we obtain the equations:

$2x = 6$ ………………(1)

and $x + y = 2$ ……………..(2)

Solving equation (1) we get x = 3 and when substituted in (2) we get y = -1.

ORDERED n-TUPLE:

The ordered n-tuple, $(a_1, a_2, \ldots, a_n)$ consists of elements $a_1, a_2, ..a_n$ together with the ordering: first a_1, second a_2, and so forth up to a_n. In particular, an ordered 2-tuple is called an ordered pair, and an ordered 3-tuple is called an ordered triple.

Two ordered n-tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are equal if and only if each corresponding pair of their elements is equal, i.e., $a_i = b_j$, for all i = 1, 2… n.

CARTESIAN PRODUCT OF TWO SETS:

Let A and B be sets. The Cartesian product of A and B, denoted A × B (read "A cross B") is the set of all ordered pairs (a, b), where a is in A and b is in B.

Symbolically:

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

NOTE

If set A has ***m*** elements and set B has ***n*** elements then A ×B has ***m*** × ***n*** elements.

EXAMPLE:

Let A = {1, 2}, B = {a, b, c} then

A ×B = {(1,a), (1,b), (1,c), (2,a), (2, b), (2, c)}

B ×A = {(a,1), (a,2), (b, 1), (b, 2), (c, 1), (c, 2)}

A ×A = {(1, 1), (1,2), (2, 1), (2, 2)}

B ×B = {(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b),(c, c)}

REMARK:

1. $A \times B \neq B \times A$ for non-empty and unequal sets A and B.
2. $A \times \phi = \phi \times A = \phi$
3. $|A \times B| = |A| \times |B|$

CARTESIAN PRODUCT OF MORE THAN TWO SETS:

The Cartesian product of sets $A_1, A_2, \ldots, A_n$, denoted $A_1 \times A_2 \times \ldots \times A_n$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$.

Symbolically:

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i, \text{ for } i=1, 2, \ldots, n\}$$

BINARY RELATION:

Let A and B be sets. A (binary) relation R from A to B is a subset of $A \times B$.
When $(a, b) \in R$, we say a is related to b by R, written a R b.
Otherwise if $(a, b) \notin R$, we write $a \not R b$.

EXAMPLE:

Let A = {1, 2}, B = {1, 2, 3}

Then $A \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$

Let

$R_1=\{(1,1), (1, 3), (2, 2)\}$

$R_2=\{(1, 2), (2, 1), (2, 2), (2, 3)\}$

$R_3=\{(1, 1)\}$

$R_4= A \times B$

$R_5= \varnothing$

All being subsets of $A \times B$ are relations from A to B.

DOMAIN OF A RELATION:

The domain of a relation R from A to B is the set of all first elements of the ordered pairs which belong to R denoted Dom(R).

Symbolically:

$$\text{Dom}(R) = \{a \in A \mid (a,b) \in R\}$$

RANGE OF A RELATION:

The range of A relation R from A to B is the set of all second elements of the ordered pairs which belong to R denoted Ran(R).

Symbolically:

$$\text{Ran}(R) = \{b \in B \mid (a,b) \in R\}$$

EXERCISE:

Let A = {1, 2}, B = {1, 2, 3},

Define a binary relation R from A to B as follows:

$$R = \{(a, b) \in A \times B \mid a < b\}$$

Then

a. Find the ordered pairs in R.
b. Find the Domain and Range of R.
c. Is 1R3, 2R2?

SOLUTION:

Given A = {1, 2}, B = {1, 2, 3},

$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$

a. $R = \{(a, b) \in A \times B \mid a < b\}$
   $R = \{(1,2), (1,3), (2,3)\}$
b. Dom(R) = {1,2} and Ran(R) = {2, 3}
a. Since $(1,3) \in R$ so 1R3
   But $(2, 2) \notin R$ so 2 is not related with3.

EXAMPLE:

Let A = {eggs, milk, corn} and B = {cows, goats, hens}

Define a relation R from A to B by $(a, b) \in R$ *iff* a is produced by b.

Then R = {(eggs, hens), (milk, cows), (milk, goats)}

Thus, with respect to this relation eggs R hens , milk R cows, etc.

EXERCISE :

Find all binary relations from {0,1} to {1}

SOLUTION:

Let $A = \{0,1\}$ & $B = \{1\}$

Then $A \times B = \{(0,1), (1,1)\}$

All binary relations from A to B are in fact all subsets of $A \times B$, which are:

$R_1 = \emptyset$

$R_2 = \{(0,1)\}$

$R_3 = \{(1,1)\}$

$R_4 = \{(0,1), (1,1)\} = A \times B$

REMARK:

If $|A| = m$ and $|B| = n$

Then as we know that the number of elements in $A \times B$ are $m \times n$. Now as we know that the total number of and the total number of relations from A to B are $2^{m \times n}$.

RELATION ON A SET:

A relation on the set A is a relation from A to A.

In other words, a relation on a set A is a subset of $A \times A$.

EXAMPLE: :

Let $A = \{1, 2, 3, 4\}$

Define a relation R on A as

$(a,b) \in R$ *iff* a divides b {symbolically written as $a \mid b$}

Then $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

REMARK:

For any set A

1. $A \times A$ is known as the universal relation.
2. $\emptyset$ is known as the empty relation.

EXERCISE:

Define a binary relation E on the set of the integers Z, as follows:

for all $m,n \in Z$, $m \, E \, n \Leftrightarrow m - n$ is even.

a. Is 0E0? Is 5E2? Is $(6,6) \in E$? Is $(-1,7) \in E$?
b. Prove that for any even integer n, nE0.

SOLUTION

$E = \{(m,n) \in Z \times Z \mid m - n \text{ is even}\}$

a. (i) $(0,0) \in Z \times Z$ and $0-0 = 0$ is even

Therefore 0E0.

(ii) $(5,2) \in Z \times Z$ but $5-2 = 3$ is not even

so $5 \not{E} 2$

(iii) $(6,6) \in E$ since $6-6 = 0$ is an even integer.

(iv) $(-1,7) \in E$ since $(-1) - 7 = -8$ is an even integer.

a. For any even integer, n, we have

$n - 0 = n$, an even integer

so $(n, 0) \in E$ or equivalently $n \, E \, 0$

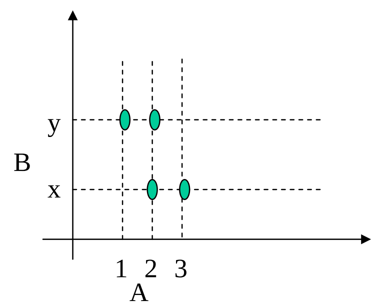
COORDINATE DIAGRAM (GRAPH) OF A RELATION:

Let $A = \{1, 2, 3\}$ and $B = \{x, y\}$

Let R be a relation from A to B defined as

$R = \{(1, y), (2, x), (2, y), (3, x)\}$

The relation may be represented in a coordinate diagram as follows:



EXAMPLE:

Draw the graph of the binary relation C from R to R defined as follows:

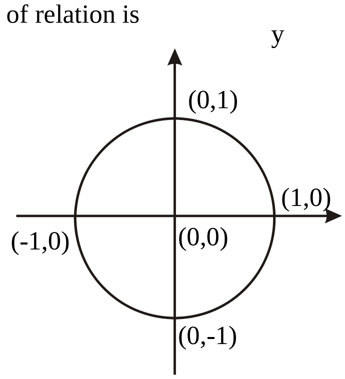
for all $(x, y) \in R \times R$, $(x, y) \in C \Leftrightarrow x^2 + y^2 = 1$

SOLUTION

All ordered pairs (x, y) in relation C satisfies the equation $x^2+y^2=1$, which when solved for y gives

Clearly y is real, whenever $-1 \le x \le 1$
Similarly x is real, whenever $-1 \le y \le 1$
Hence the graph is limited in the range $-1 \le x \le 1$ and $-1 \le y \le 1$

The graph of relation is



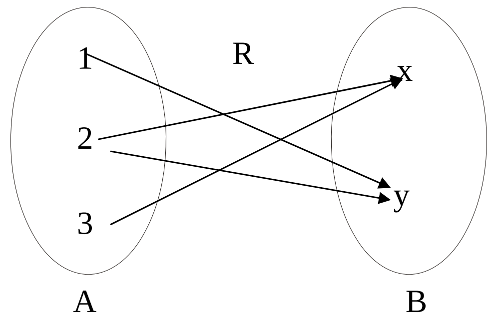
ARROW DIAGRAM OF A RELATION:

Let

A = {1, 2, 3}, B = {x, y} and
R = {1,y), (2,x), (2,y), (3,x)}

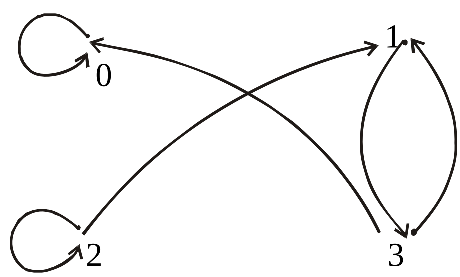
be a relation from A to B.

The arrow diagram of R is:



DIRECTED GRAPH OF A RELATION:

Let A = {0, 1, 2, 3}
and R = {(0,0), (1,3), (2,1), (2,2), (3,0), (3,1)}
be a binary relation on A.



DIRECTED GRAPH

MATRIX REPRESENTATION OF A RELATION

Let A = {$a_1, a_2, \ldots, a_n$} and B = {$b_1, b_2, \ldots, b_m$}. Let R be a relation from A to B. Define the n × m order matrix M by

$$m(i,\, j) = \begin{cases} 1 \text{ if } \ (a_i, b_i) \in R \\ 0 \text{ if } \ (a_i, b_i) \notin R \end{cases}$$

for i=1,2,…,n and j=1,2,…,m

EXAMPLE:

Let A = {1, 2, 3} and B = {x, y}
Let R be a relation from A to B defined as
R ={(1,y), (2,x), (2,y), (3,x)}

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}_{3\times 2} \end{array}$$

EXAMPLE:

For the relation matrix.

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

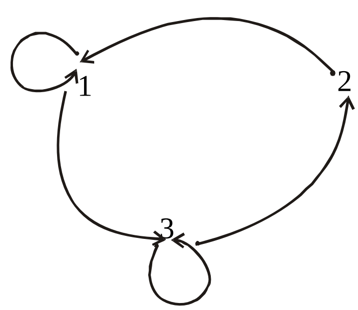
1. List the set of ordered pairs represented by M.
2. Draw the directed graph of the relation.

SOLUTION:

The relation corresponding to the given Matrix is

- R = {(1,1), (1,3), (2,1), (3,1), (3,2), (3,3)}

And its Directed graph is given below



EXERCISE:

Let A = {2, 4} and B = {6, 8, 10} and define relations R and S from A to B as follows:

for all (x,y) $\in$ A $\times$ B, x R y $\Leftrightarrow$ x | y

for all (x,y) $\in$ A $\times$ B, x S y $\Leftrightarrow$ y – 4 = x

State explicitly which ordered pairs are in A $\times$ B, R, S, R$\cup$S and R$\cap$S.

SOLUTION

$A \times B = \{(2,6), (2,8), (2,10), (4,6), (4,8), (4,10)\}$
$R = \{(2,6), (2,8), (2,10), (4,8)\}$
$S = \{(2,6), (4,8)\}$
$R \cup S = \{(2,6), (2,8), (2,10), (4,8)\}= R$
$R \cap S = \{(2,6), (4,8)\}= S$