LECTURE # 12

REFLEXIVE RELATION:

Let R be a relation on a set A. R is reflexive if, and only if, for all $a \in A$, $(a, a) \in R$. Or equivalently aRa.

That is, each element of A is related to itself.

REMARK

R is not reflexive iff there is an element "a" in A such that $(a, a) \notin R$. That is, some element "a" of A is not related to itself.

EXAMPLE:

Let A = {1, 2, 3, 4} and define relations R_1, R_2, R_3, R_4 on A as follows:

$R_1 = \{(1, 1), (3, 3), (2, 2), (4, 4)\}$
$R_2 = \{(1, 1), (1, 4), (2, 2), (3, 3), (4, 3)\}$
$R_3 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
$R_4 = \{(1, 3), (2, 2), (2, 4), (3, 1), (4, 4)\}$

Then,

R_1 is reflexive, since $(a, a) \in R_1$ for all $a \in A$.
R_2 is not reflexive, because $(4, 4) \notin R_2$.
R_3 is reflexive, since $(a, a) \in R_3$ for all $a \in A$.
R_4 is not reflexive, because $(1, 1) \notin R_4$, $(3, 3) \notin R_4$

DIRECTED GRAPH OF A REFLEXIVE RELATION:

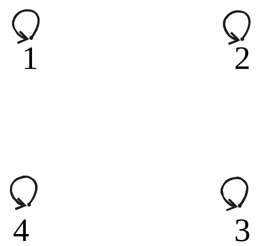
The directed graph of every reflexive relation includes an arrow from every point to the point itself (i.e., a loop).

EXAMPLE :

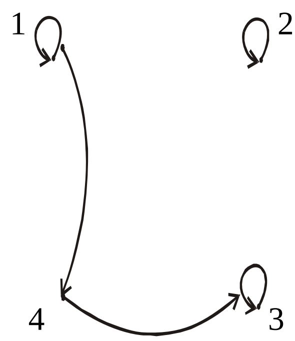
Let A = {1, 2, 3, 4} and define relations R_1, R_2, R_3, and R_4 on A by

R_1 = {(1, 1), (3, 3), (2, 2), (4, 4)}
R_2 = {(1, 1), (1, 4), (2, 2), (3, 3), (4, 3)}
R_3 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}
R_4 = {(1, 3), (2, 2), (2, 4), (3, 1), (4, 4)}

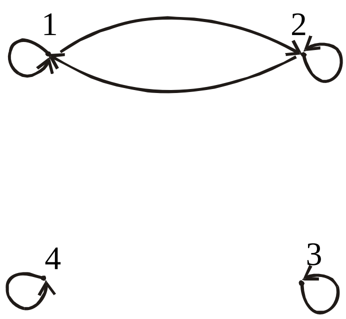
Then their directed graphs are



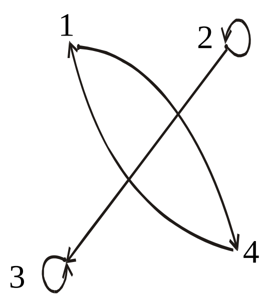
R_1 is reflexive because at every point of the set A we have a loop in the graph.



R_2 is not reflexive, as there is no loop at 4.



R_3 is reflexive



R_4 is not reflexive, as there are no loops at 1and 3.

MATRIX REPRESENTATION OF A REFLEXIVE RELATION:

Let A = {a_1, a_2, …, a_n}. A Relation R on A is reflexive if and only if $(a_i, a_j) \in R \ \forall$ i=1,2, …,n.

Accordingly, R is **reflexive** if all the elements on the **main diagonal** of the matrix **M** representing R are equal to 1.

EXAMPLE:

The relation R = {(1,1), (1,3), (2,2), (3,2), (3,3)} on A = {1,2,3} represented by the following matrix M, is reflexive.

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

SYMMETRIC RELATION

Let R be a relation on a set A. R is symmetric if, and only if,
for all a, b ∈ A, if (a, b) ∈R then (b, a) ∈R.
That is, if aRb then bRa.

REMARK

R is not symmetric iff there are elements *a* and *b* in A such that (a, b) ∈R but (b, a) ∉R.

EXAMPLE

Let A = {1, 2, 3, 4} and define relations R_1, R_2, R_3, and R_4on A as follows.

R_1 = {(1, 1), (1, 3), (2, 4), (3, 1), (4,2)}
R_2 = {(1, 1), (2, 2), (3, 3), (4, 4)}
R_3 = {(2, 2), (2, 3), (3, 4)}
R_4 = {(1, 1), (2, 2), (3, 3), (4, 3), (4, 4)}

Then R_1 is symmetric because for every order pair (a,b)in R_1awe have (b,a) in $R_1$for example we have (1,3)in R_1 the we have (3,1) in R_1 similarly all other ordered pairs can be cheacked.
R_2 is also symmetric symmetric we say it is vacuously true.
R_3 is not symmetric, because (2,3) ∈ R_3 but (3,2) ∉ R_3.
R_4 is not symmetric because (4,3) ∈ R_4 but (3,4) ∉ R_4.

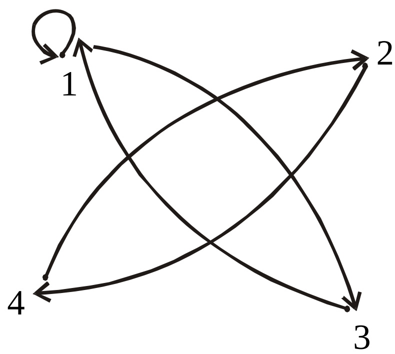
DIRECTED GRAPH OF A SYMMETRIC RELATION

For a symmetric directed graph whenever there is an arrow going from one point of the graph to a second, there is an arrow going from the second point back to the first.

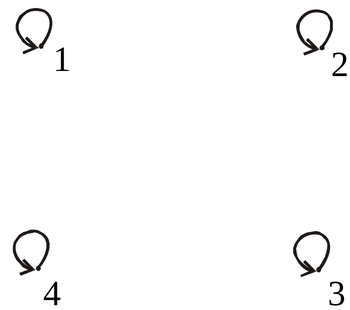
EXAMPLE

Let A = {1, 2, 3, 4} and define relations R_1, R_2, R_3, and R_4 on A by the directed graphs:

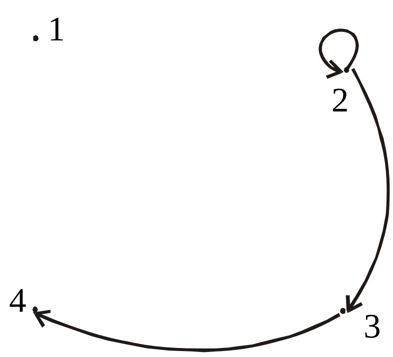
R_1 = {(1, 1), (1, 3), (2, 4), (3, 1), (4,2)}
R_2 = {(1, 1), (2, 2), (3, 3), (4, 4)}
R_3 = {(2, 2), (2, 3), (3, 4)}
R_4= {(1, 1), (2, 2), (3, 3), (4, 3), (4, 4)}



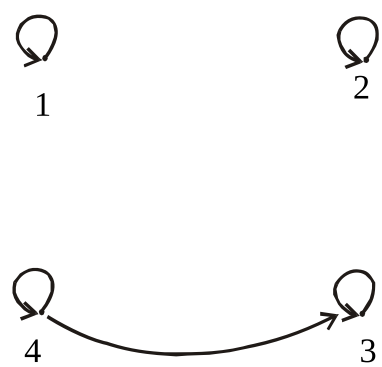
R_1 is symmetric



R_2 is symmetric



R_3 is not symmetric since there are arrows from 2 to 3 and from 3 to 4 but not conversely



R_4 is not symmetric since there is an arrow from 4 to 3 but no arrow from 3 to 4

MATRIX REPRESENTATION OF A SYMMETRIC RELATION

Let

$A = \{a_1, a_2, \ldots, a_n\}$.

A relation R on A is symmetric if and only if for all $a_i, a_j \in A$, if $(a_i, a_j) \in R$ then $(a_j, a_i) \in R$.

Accordingly, R is symmetric if the elements in the ith row are the same as the elements in the ith column of the matrix M representing R. More precisely, M is a symmetric matrix.i.e. $M = M^t$

EXAMPLE

The relation R = {(1,3), (2,2), (3,1), (3,3)} on A = {1,2,3} represented by the following matrix M is symmetric.

$$
\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ M = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}
$$

TRANSITIVE RELATION

Let R be a relation on a set A.R is transitive if and only if for all a, b, c $\in$A, if (a, b) $\in$R and (b, c) $\in$R then (a, c) $\in$R.

That is, if aRb and bRc then aRc.

In words, if any one element is related to a second and that second element is related to a third, then the first is related to the third. Note that the “first”, “second” and “third” elements need not to be distinct.

REMARK

R is not transitive iff there are elements a, b, c in A such that If (a,b) $\in$R and (b,c) $\in$R but (a,c) $\notin$R.

EXAMPLE

Let A = {1, 2, 3, 4} and define relations R_1, R_2 and R_3 on A as follows:

R_1 = {(1, 1), (1, 2), (1, 3), (2, 3)}

R_2 = {(1, 2), (1, 4), (2, 3), (3, 4)}

R_3 = {(2, 1), (2, 4), (2, 3), (3,4)}

Then R_1 is transitive because (1, 1), (1, 2) are in R then to be transitive relation (1,2) must be there and it belongs to R Similarly for other order pairs.

R_2 is not transitive since (1,2) and (2,3) $\in R_2$ but (1,3) $\notin R_2$.

R_3 is transitive.

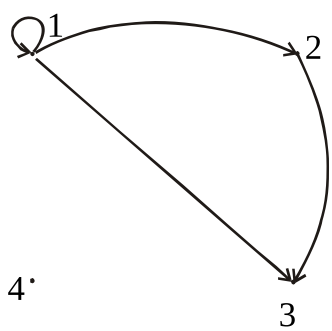
DIRECTED GRAPH OF A TRANSITIVE RELATION

For a transitive directed graph, whenever there is an arrow going from one point to the second, and from the second to the third, there is an arrow going directly from the first to the third.

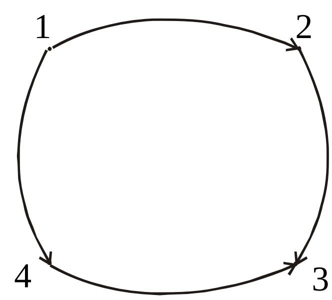
EXAMPLE

Let A = {1, 2, 3, 4} and define relations R_1, R_2 and R_3 on A by the directed graphs:

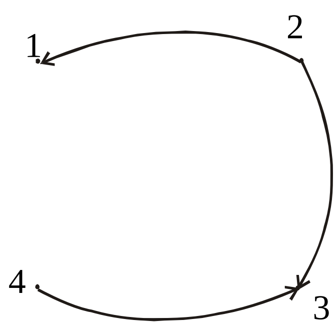
$R_1 = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$
$R_2 = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$
$R_3 = \{(2, 1), (2, 4), (2, 3), (3,4)\}$



R_1 is transitive



R_2 is not transitive since there is an arrow from 1 to 2 and from 2 to 3 but no arrow from 1 to 3 directly



R_3 is transitive

EXERCISE:

Let A = {1, 2, 3, 4} and define the null relation ϕ and universal relation $A \times A$ on A. Test these relations for reflexive, symmetric and transitive properties.

SOLUTION:

Reflexive:

(i) $\varnothing$ is not reflexive since $(1,1), (2,2), (3,3), (4,4) \notin \varnothing$.

(ii) $A \times A$ is reflexive since $(a,a) \in A \times A$ for all $a \in A$.

Symmetric

(i) For the null relation $\varnothing$ on A to be symmetric, it must satisfy the implication:

if $(a,b) \in \varnothing$ then $(a, b) \in \varnothing$.

Since $(a, b) \in \varnothing$ is never true, the implication is vacuously true or true by default. Hence $\varnothing$ is symmetric.

(ii) The universal relation $A \times A$ is symmetric, for it contains all ordered pairs of elements of A. Thus,

if $(a, b) \in A \times A$ then $(b, a) \in A \times A$ for all a, b in A.

Transitive

(i) The null relation $\varnothing$ on A is transitive, because the implication.

if $(a, b) \in \varnothing$ and $(b, c) \in \varnothing$ then $(a, c) \in \varnothing$ is true by default,

since the condition $(a, b) \in \varnothing$ is always false.

(i) The universal relation $A \times A$ is transitive for it contains all ordered pairs of elements of A.

Accordingly, if $(a, b) \in A \times A$ and $(b, c) \in A \times A$ then $(a, c) \in A \times A$ as well.

EXERCISE:

Let $A = \{0, 1, 2\}$ and

$R = \{(0,2), (1,1), (2,0)\}$ be a relation on A.

1. Is R reflexive? Symmetric? Transitive?
2. Which ordered pairs are needed in R to make it a reflexive and transitive relation.

SOLUTION:

1. R is not reflexive, since $0 \in A$ but $(0, 0) \notin R$ and also $2 \in A$ but $(2, 2) \notin R$.

R is clearly symmetric.

R is not transitive, since (0, 2) & $(2, 0) \in R$ but $(0, 0) \notin R$.

2. For R to be reflexive, it must contain ordered pairs (0,0) and (2,2).

For R to be transitive,

we note (0,2) and $(2,0) \in$ but $(0,0) \notin R$.

Also (2,0) and $(0,2) \in R$ but $(2,2) \notin R$.

Hence (0,0) and (2,2). Are needed in R to make it a transitive relation.

EXERCISE:

Define a relation L on the set of real numbers **R** be defined as follows:

for all $x, y \in R$, $x \, L \, y \Leftrightarrow x < y$.

a. Is L reflexive?
b. Is L symmetric?
c. Is L transitive?

SOLUTION:

a. L is not reflexive, because $x \nless x$ for any real number x.
   (e.g. $1 \nless 1$)
b. L is not symmetric, because for all $x, y \in R$, if
   $x < y$ then $y \nless x$
   (e.g. $0 < 1$ but $1 \nless 0$)
c. L is transitive, because for all, $x, y, z \in R$, if $x < y$ and $y < z$, then $x < z$.
   (by transitive law of order of real numbers).

EXERCISE:

Define a relation R on the set of positive integers Z^+ as follows:

for all $a, b \in Z^+$, a R b *iff* $a \times b$ is odd.

Determine whether the relation is

a. reflexive b. symmetric c. transitive

SOLUTION:

Firstly, recall that the product of two positive integers is odd if and only if both of them are odd.

a. reflexive

R is not reflexive, because $2 \in Z^+$ but 2 R 2 for $2 \times 2 = 4$ which is not odd.

b. symmetric

R is symmetric, because if a R b then $a \times b$ is odd or equivalently $b \times a$ is odd ($b \times a = a \times b$) $\Rightarrow$ b R a.

c. transitive

R is transitive, because if a R b then $a \times b$ is odd $\Rightarrow$ both "a" and "b" are odd. Also bRc means $b \times c$ is odd $\Rightarrow$ both "b" and "c" are odd.

Now if aRb and bRc, then all of a, b, c are odd and so $a \times c$ is odd. Consequently aRc.

<u>EXERCISE:</u>

Let "D" be the "divides" relation on Z defined as:

for all m, n $\in$Z, m D n$\Leftrightarrow$ m|n

Determine whether D is reflexive, symmetric or transitive. Justify your answer.

SOLUTION:

Reflexive

Let m $\in$Z, since every integer divides itself so m|m $\forall$ m $\in$Z therefore m D m $\forall$ m $\in$Z

Accordingly D is reflexive

Symmetric

Let m, n $\in$ Z and suppose m D n.

By definition of D, this means m|n (i.e.= an integer)

Clearly, then it is not necessary that = an integer.

Accordingly, if m D n then n D m, $\forall$ m, n $\in$Z

Hence D is not symmetric.

Transitive

Let m, n, p $\in$Z and suppose m D n and n D p.

Now m D n $\Rightarrow$ m|n $\Rightarrow$ = an integer.

Also n D p $\Rightarrow$ n|p $\Rightarrow$ = an integer.

We note $= \frac{p}{m}$ * $\left(\frac{p}{n}\right)$ = (an int) * $\left(\frac{n}{m}\right)$ (an int) = an int

$\Rightarrow$ m|p and so mDp

Thus if mDn and nDp then mDp $\forall$ m, n, p $\in$Z

Hence D is transitive.

<u>EXERCISE:</u>

Let A be the set of people living in the world today. A binary relation R is defined on A as follows:

for all p, q $\in$A, pRq $\Leftrightarrow$ p has the same first name as q.

Determine whether the relation R is reflexive, symmetric and/or transitive.

<u>SOLUTION:</u>

a. Reflexive

Since every person has the same first name as his/her self. Hence for all p $\in$ A, pRp. Thus, R is reflexive.

b. Symmetric:

Let p, q $\in$A and suppose pRq.

$\Leftrightarrow$ p has the same first name as q.

$\Leftrightarrow$ q has the same first name as p.

$\Leftrightarrow$ q R p

Thus if pRq then qRp $\forall$ p,q $\in$A.

$\Rightarrow$ R is symmetric.

a. Transitive

Let p, q, s $\in$A and suppose p R q and qRr.

Now pRq $\Leftrightarrow$p has the same first name as q

and qRr $\Leftrightarrow$ q has the same first name as r.

Consequently, p has the same first name as r.

$\Leftrightarrow$ pRr

Thus, if pRq and qRs then pRr, $\forall$ p, q, r $\in$A.

Hence R is transitive.

EQUIVALENCE RELATION:

Let A be a non-empty set and R a binary relation on A. R is an equivalence relation if, and only if, R is reflexive, symmetric, and transitive.

EXAMPLE:

Let A = {1, 2, 3, 4} and

R = {(1,1), (2,2), (2,4), (3,3), (4,2), (4,4)}

be a binary relation on A.

Note that R is reflexive, symmetric and transitive, hence an equivalence relation.

CONGRUENCES:

Let m and n be integers and d be a positive integer. The notation

$m \equiv n \pmod d$ means that

d | (m – n) {d divides m minus n}.There exists an integer k such that

(m – n) = d · k

EXAMPLE:

c. Is $22 \equiv 1 \pmod 3$? b. Is $-5 \equiv +10 \pmod 3$?

d. Is $7 \equiv 7 \pmod 3$? d. Is $14 \equiv 4 \pmod 3$?

SOLUTION

a. Since 22-1 = 21 = 3×7.

Hence 3|(22-1), and so $22 \equiv 1 \pmod 3$

b. Since – 5 – 10 = - 15 = 3 × (-5),

Hence 3|((-5)-10), and so $-5 \equiv 10 \pmod 3$

c. Since 7 – 7 = 0 = 3 × 0

Hence 3|(7-7), and so $7 \equiv 7 \pmod 3$

d. Since 14 – 4 = 10, and 3 / 10 because 10 $\neq$ 3· k for any integer k. Hence $14 \equiv 4 \pmod 3$.

EXERCISE:

Define a relation R on the set of all integers Z as follows:

for all integers m and n, m R n $\Leftrightarrow$ $m \equiv n \pmod 3$

Prove that R is an equivalence relation.

SOLUTION:

1. **R is reflexive.**

R is reflexive iff for all m $\in$Z, m R m.

By definition of R, this means that

For all m $\in$Z, $m \equiv m \pmod 3$

Since m – m = 0 = 3 ×0.

Hence 3|(m-m), and so $m \equiv m \pmod 3$

$\Leftrightarrow$ mRm

$\Rightarrow$ R is reflexive.

2. **R is symmetric.**

R is symmetric iff for all m, n $\in Z$
if m R n then n R m.

Now mRn $\Rightarrow$ $m \equiv n \pmod 3$
$\Rightarrow$ $3|(m-n)$
$\Rightarrow$ $m-n = 3k$, for some integer k.
$\Rightarrow$ $n - m = 3(-k)$, $-k \in Z$
$\Rightarrow$ $3|(n-m)$
$\Rightarrow$ $n \equiv m \pmod 3$
$\Rightarrow$ nRm

Hence R is symmetric.

1. **R is transitive**

R is transitive iff for all m, n, p $\in Z$,
if mRn and nRp then mRp

Now mRn and nRp means $m \equiv n \pmod 3$ and $n \equiv p \pmod 3$
$\Rightarrow$ $3|(m-n)$ and $3|(n-p)$
$\Rightarrow$ $(m-n) = 3r$ and $(n-p) = 3s$ for some r, s $\in Z$

Adding these two equations, we get,
$(m - n) + (n - p) = 3r + 3s$
$\Rightarrow$ $m - p = 3(r + s)$, where $r + s \in Z$
$\Rightarrow$ $3|(m - p)$
$\Rightarrow$ $m \equiv p \pmod 3 \Leftrightarrow$ m Rp

Hence R is transitive. R being reflexive, symmetric and transitive, is an equivalence relation.

LECTURE # 13

EXERCISE:

Suppose R and S are binary relations on a set A.

a. If R and S are reflexive, is $R \cap S$ reflexive?
b. If R and S are symmetric, is $R \cap S$ symmetric?
c. If R and S are transitive, is $R \cap S$ transitive?

SOLUTION:

a. $R \cap S$ is reflexive:

Suppose R and S are reflexive.

Then by definition of reflexive relation

$\forall$ $a \in A$ $(a,a) \in R$ and $(a,a) \in S$

$\Rightarrow$ $\forall a \in A$ $(a,a) \in R \cap S$

(by definition of intersection)

Accordingly, $R \cap S$ is reflexive.

b. $R \cap S$ is symmetric.

Suppose R and S are symmetric.

To prove $R \cap S$ is symmetric we need to show that

$\forall$ $a, b \in A$, if $(a,b) \in R \cap S$ then $(b,a) \in R \cap S$.

Suppose $(a,b) \in R \cap S$.

$\Rightarrow$ $(a,b) \in R$ and $(a,b) \in S$

(by the definition of Intersection of two sets)

Since R is symmetric, therefore if $(a,b) \in R$ then $(b,a) \in R$. Similarly S is symmetric, so if $(a,b) \in S$ then $(b,a) \in S$.

Thus $(b,a) \in R$ and $(b,a) \in S$

$\Rightarrow$ $(b,a) \in R \cap S$ (by definition of intersection)

Accordingly, $R \cap S$ is symmetric.

c. $R \cap S$ is transitive.

Suppose R and S are transitive.

To prove $R \cap S$ is transitive we must show that

$\forall$ $a,b,c, \in A$, if $(a,b) \in R \cap S$ and $(b,c) \in R \cap S$ then $(a,c) \in R \cap S$.

Suppose $(a,b) \in R \cap S$ and $(b,c) \in R \cap S$

$\Rightarrow$ $(a,b) \in R$ and $(a,b) \in S$ and $(b,c) \in R$ and $(b,c) \in S$

Since R is transitive, therefore

if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$.

Also S is transitive, so $(a,c) \in S$

Hence we conclude that $(a,c) \in R$ and $(a,c) \in S$

and so $(a,c) \in R \cap S$ (by definition of intersection)

Accordingly, $R \cap S$ is transitive.

EXAMPLE:

Let A = {1,2,3,4}

and let R and S be transitive binary relations on A defined as:

R = {(1,2), (1,3), (2,2), (3,3), (4,2), (4,3)}

and S = {(2,1), (2,4),(3,3)}

Then $R \cup S$ = {(1,2), (1,3), (2,1), (2,2), (2,4), (3,3), (4,2), (4,3)}

We note (1,2) and (2,1) $\in R \cup S$, but $(1,1) \notin R \cup S$

Hence $R \cup S$ is not transitive.

IRREFLEXIVE RELATION:

Let R be a binary relation on a set A. R is irreflexive iff for all $a \in A, (a,a) \notin R$.
That is, R is irreflexive if no element in A is related to itself by R.

REMARK:

R is not irreflexive iff there is an element $a \in A$ such that $(a,a) \in R$.

EXAMPLE:

Let A = {1,2,3,4} and define the following relations on A:

R_1 = {(1,3), (1,4), (2,3), (2,4), (3,1), (3,4)}
R_2 = {(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)}
R_3 = {(1,2), (2,3), (3,3), (3,4)}

Then R_1 is irreflexive since no element of A is related to itself in R_1. i.e.
$(1,1) \notin R_1$, $(2,2) \notin R_1$, $(3,3) \notin R_1$, $(4,4) \notin R_1$
R_2 is not irreflexive, since all elements of A are related to themselves in R_2
R_3 is not irreflexive since $(3,3) \in R_3$. Note that R_3 is not reflexive.

NOTE:

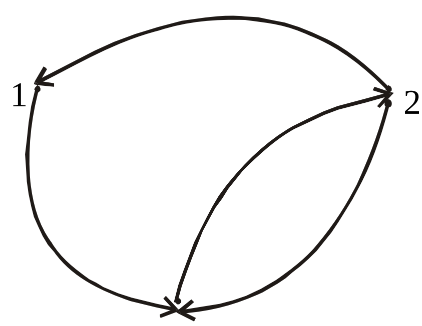
A relation may be neither reflexive nor irreflexive.

DIRECTED GRAPH OF AN IRREFLEXIVE RELATION:

Let R be an **irreflexive** relation on a set A. Then by definition, no element of A is related to itself by R. Accordingly, there is no loop at each point of A in the directed graph of R.

EXAMPLE:

Let A = {1,2,3}
and R = {(1,3), (2,1), (2,3), (3,2)} be represented by the directed graph.



MATRIX REPRESENTATION OF AN IRREFLEXIVE RELATION

Let R be an irreflexive relation on a set A. Then by definition, no element of A is related to itself by R.
Since the self related elements are represented by 1's on the main diagonal of the matrix representation of the relation, so for irreflexive relation R, the matrix will contain all 0's in its main diagonal.
It means that a relation is irreflexive if in its matrix representation the diagonal elements are all zero, if one of them is not zero the we will say that the relation is not irreflexive.

EXAMPLE:

Let A = {1,2,3} and R = {(1,3), (2,1), (2,3), (3,2)} be represented by the matrix

$$\begin{array}{c} \\ M = \end{array}\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Then R is irreflexive, since all elements in the main diagonal are 0's.

EXERCISE:

Let R be the relation on the set of integers Z defined as: for all a,b $\in$Z, (a,b) $\in$R $\Leftrightarrow$ a > b.
Is R irreflexive?

SOLUTION:

R is irreflexive if for all a $\in$Z, (a,a) $\notin$R.
Now by the definition of given relation R,
for all a $\in$Z, (a,a) $\notin$R since a $\not>$ a.
Hence R is irreflexive.

ANTISYMMETRIC RELATION:

Let R be a binary relation on a set A.R is a**nti-symmetric** iff
$\forall$a, b $\in$A if (a,b) $\in$R and (b,a) $\in$R then a = b.

REMARK:

1. R is not **anti-symmetric** iff there are elements a and b in A such that (a,b) $\in$R and (b,a) $\in$R but a $\neq$ b.
2. The properties of being **symmetric** and being **anti-symmetric** are not negative of each other.

EXAMPLE:

Let A = {1,2,3,4} and define the following relations on A.

R_1 = {(1,1),(2,2),(3,3)}
R_2 = {(1,2),(2,2), (2,3), (3,4), (4,1)}
R_3={(1,3),(2,2), (2,4), (3,1), (4,2)}
R_4={(1,3),(2,4), (3,1), (4,3)}

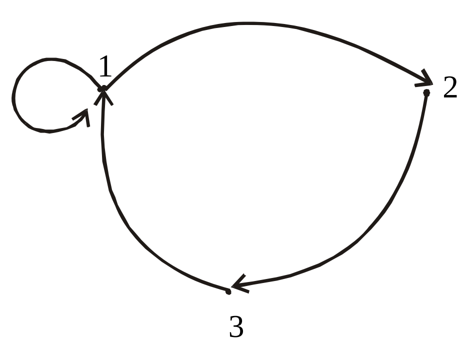
R_1 is anti-symmetric and symmetric .
R_2 is anti-symmetric but not symmetric because (1,2) $\in$ R_2but (2,1) $\notin$ R_2.
R_3 is not anti-symmetric since (1,3) & (3,1) $\in$ R_3 but 1 $\neq$ 3.
Note that R_3 is symmetric.
$R_4$is neither anti-symmetric because (1,3) & (3,1) $\in$ R_4 but 1 $\neq$ 3 nor symmetric because (2,4) $\in$ R_4 but (4,2) $\notin$$R_4$

DIRECTED GRAPH OF AN ANTISYMMETRIC RELATION:

Let R be an anti-symmetric relation on a set A. Then by definition, no two distinct elements of A are related to each other.
Accordingly, there is no pair of arrows between two distinct elements of A in the directed graph of R.

EXAMPLE:

Let A = {1,2,3} And R be the relation defined on A is
R ={(1,1), (1,2), (2,3), (3,1)}.Thus R is represented by the directed graph as



R is anti-symmetric, since there is no pair of arrows between two distinct points in A.

MATRIX REPRESENTATION OF AN ANTISYMMETRIC RELATION:

Let R be an anti-symmetric relation on a set

$A = \{a_1, a_2, \ldots, a_n\}$. Then if $(a_i, a_j) \in R$ for $i \neq j$ then $(a_i, a_j) \notin R$.

Thus in the matrix representation of R there is a 1 in the ith row and jth column iff the jth row and ith column contains 0 vice versa.

EXAMPLE:

Let A = {1,2,3} and a relation R = {(1,1), (1,2), (2,3), (3,1)}on A be represented by the matrix.

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Then R is anti-symmetric as clear by the form of matrix M

PARTIAL ORDER RELATION:

Let R be a binary relation defined on a set A. R is a partial order relation,if and only if, R is **reflexive, antisymmetric**, and **transitive**. The set A together with a partial ordering R is called a **partially ordered** set or **poset.**

EXAMPLE:

Let R be the set of real numbers and define the"less than or equal to" , on R as follows:

for all real numbers x and y in R.$x \leq y \Leftrightarrow x < y$ or $x = y$

Show that $\leq$ is a partial order relation.

SOLUTION:

$\leq$ is reflexive

For $\leq$ to be reflexive means that $x \leq x$ for all $x \in R$

But $x \leq x$ means that $x < x$ or $x = x$ and $x = x$ is always true.

Hence under this relation every element is related to itself.

$\leq$ is anti-symmetric.

For $\leq$ to be anti-symmetric means that

$\forall x,y \in R$, if $x \leq y$ and $y \leq x$, then $x = y$.

This follows from the definition of $\leq$ and the trichotomy property, which says that

"given any real numbers x and y, exactly one of thefollowing holds:

$x < y$ or $x = y$ or $x > y$"

$\leq$ is transitive

For $\leq$ to be transitive means that

$\forall x,y,z \in R$, if $x\leq y$ and $y \leq z$ then $x \leq z$.

This follows from the definition of $\leq$ and the transitive property of order of real numbers, which says that "given any real numbers x, y and z,

if $x < y$ and $y < z$ then $x < z$"

Thus $\leq$ being reflexive, anti-symmetric and transitive is a partial order relation on R.

EXERCISE:

Let A be a non-empty set and P(A) the power set of A.

Define the "subset" relation, $\subseteq$, as follows:

for all $X,Y \in P(A)$, $X \subseteq Y \Leftrightarrow \forall x$, iff $x \in X$ then $x \in Y$.

Show that $\subseteq$ is a partial order relation.

SOLUTION:

1. $\subseteq$ is reflexive

Let $X \in P(A)$. Since every set is a subset of itself, therefore

$X \subseteq X, \forall X \in P(A)$.

Accordingly $\subseteq$ is reflexive.

2. $\subseteq$ is anti-symmetric

Let X, Y $\in$P(A) and suppose X $\subseteq$ Y and Y $\subseteq$ X.Then by definition of equality of two sets it follows that X = Y.

Accordingly, $\subseteq$ is anti-symmetric.

3. $\subseteq$ is transitive

Let X, Y, Z $\in$P(A) and suppose X $\subseteq$ Y and Y $\subseteq$ Z. Then by the transitive property of subsets "if U $\subseteq$ V and V $\subseteq$ W then U $\subseteq$ W"it follows X $\subseteq$ Z.

Accordingly $\subseteq$ is transitive.

EXERCISE:

Let "|" be the "divides" relation on a set A of positive integers. That is, for all a, b $\in$A, a|b $\Leftrightarrow$ b = k ·a for some integer k.

Prove that | is a partial order relation on A.

SOLUTION:

1. **"|" is reflexive.** [We must show that, $\forall$ a $\in$A, a|a]

Suppose a $\in$A. Then a = 1·a and so a|a by definition of divisibility.

2. "|" is anti-symmetric

[We must show that for all a, b $\in$A, if a|b and b|a then a=b]

Suppose a|b and b|a

By definition of divides there are integers k_1, and k_2 such that

$$b = k_1 \cdot a \quad \text{and} \quad a = k_2 \cdot b$$

Now $b = k_1 \cdot a$

$= k_1 \cdot (k_2 \cdot b)$ (by substitution)

$= (k_1 \cdot k_2) \cdot b$

Dividing both sides by b gives

$$1 = k_1 \cdot k_2$$

Since a, b $\in$A, where A is the set of positive integers, so the equations

$$b = k_1 \cdot a \quad \text{and} \quad a = k_2 \cdot b$$

implies that k_1 and k_2 are both positive integers. Now the equation

$$k_1 \cdot k_2 = 1$$

can hold only when $k_1 = k_2 = 1$

Thus $a = k_2 \cdot b = 1 \cdot b = b$ i.e., a = b

3. "|" is transitive

[We must show that $\forall$a,b,c$\in$A if a|b and b|c than a|c]

Suppose a|b and b|c

By definition of divides, there are integers k_1 and k_2 such that

$$b = k_1 \cdot a \quad \text{and} \quad c = k_2 \cdot b$$

Now $c = k_2 \cdot b$

$= k_2 \cdot (k_1 \cdot a)$ (by substitution)

$= (k_2 \cdot k_1) \cdot a$ (by associative law under multiplication)

$= k_3 \cdot a$ where $k_3 = k_2 \cdot k_1$ is an integer

$\Rightarrow$ a|c by definition of divides

Thus "|" is a partial order relation on A.

EXERCISE:

Let "R" be the relation defined on the set of integers Z as follows:

for all a, b $\in$Z, aRb iff $b=a^r$ for some positive integer r.

Show that R is a partial order on Z.

SOLUTION:

Let a, b $\in$Z and suppose aRb and bRa. Then there are positive integers r and s such that

$b = a^r$ and $a = b^s$

Now, $a = b^s$

$= (a^r)^s$ by substitution

$= a^{rs}$

$\Rightarrow$ rs =1

Since r and s are positive integers, so this equation can hold if, and only if, r =1 and s = 1

and then $a = b^s = b^1 = b$ i.e., a = b

Thus R is anti-symmetric.

3. Let a, b, c $\in$Z and suppose aRb and bRc**.**

Then there are positive integers r and s such that

$b = a^r$ and $c = b^s$

Now $c = b^r$

$= (a^r)^s$ (by substitution)

$= a^{rs} = a^t$ (where t = rs is also a positive integer)

Hence by definition of R, aRc. Therefore, R is transitive.

Accordingly, R is a **partial order** relation on Z.

LECTURE # 14

INVERSE OF A RELATION:

Let R be a relation from A to B. The inverse relation R^{-1} from B to A is defined as:

$R^{-1} = \{(b,a) \in B \times A \mid (a,b) \in R\}$

More simply, the inverse relation R^{-1} of R is obtained by interchanging the elements of all the ordered pairs in R.

EXAMPLE:

Let A = {2, 3, 4} and B = {2,6,8}and let R be the “divides” relation from A to Bi.e. for all $(a,b) \in A \times B$, a R b $\Leftrightarrow$ a | b (a divides b)

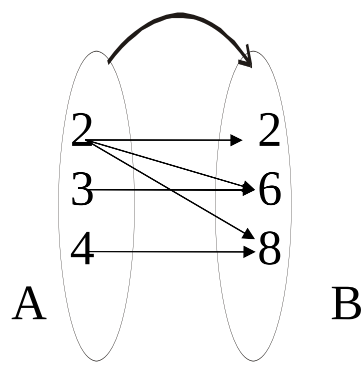
Then R = {(2,2), (2,6), (2,8), (3,6), (4,8)}and R^{-1}= {(2,2), (6,2), (8,2), (6,3), (8,4)}

In words, R^{-1}may be defined as:

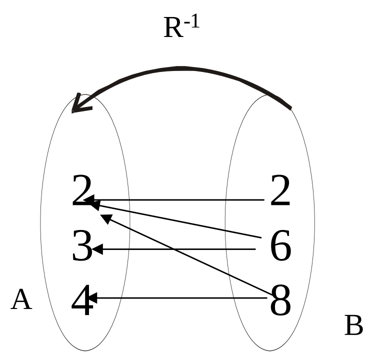
for all $(b,a) \in B \times A$, b R a $\Leftrightarrow$ b is a multiple of a.

ARROW DIAGRAM OF AN INVERSE RELATION:

The relation R = {(2,2), (2,6), (2,8), (3,6), (4,8)} is represented by the arrow diagram.



Then inverse of the above relation can be obtained simply changing the directions of the arrows and hence the diagram is



MATRIX REPRESENTATION OF INVERSE RELATION:

The relation R = {(2, 2), (2, 6), (2, 8), (3, 6), (4, 8)}from A = {2, 3,4} to B = {2, 6, 8} is defined by the matrix M below:

$$M = \begin{array}{c} \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{matrix} 2 & 6 & 8 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \qquad M^t = \begin{array}{c} \\ 2 \\ 6 \\ 8 \end{array}\begin{array}{c} \begin{matrix} 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

The matrix representation of inverse relation R^{-1} is obtained by simply taking its transpose. (i.e., changing rows by columns and columns by rows). Hence R-1 is represented by Mt as shown.

EXERCISE:

Let R be a binary relation on a set A. Prove that:

(i) If R is reflexive, then R^{-1} is reflexive.
(ii) If R is symmetric, then R^{-1} is symmetric.
(iii) If R is transitive, then R^{-1} is transitive.
(iv) If R is antisymmetric, then R^{-1} is antisymmetric.

SOLUTION (i)

if R is reflexive, then R^{-1} is reflexive.

Suppose that the relation R on A is reflexive. By definition, $\forall$ a $\in$ A, (a, a) $\in$R. Since $\mathbf{R^{-1}}$ consists of exactly those ordered pairs which are obtained by interchanging the first and second element of ordered pairs in R, therefore, if (a, a) $\in$ R then (a, a) $\in$ R^{-1}. Accordingly, $\forall$ a $\in$ A, (a, a) $\in$ R^{-1}.
Hence $R^{-1}$is reflexive as well.

SOLUTION (ii)

Suppose that the relation R on A is symmetric.
Let (a, b) $\in$ $R^{-1}$for a,b $\in$A. By definition of R^{-1}, (b, a) $\in$R. Since R is symmetric, therefore (a, b) $\in$R. But then by definition of R^{-1}, (b, a) $\in$R.
We have thus shown that for all a, b $\in$ A, if (a, b) $\in$R-1 then (b, a) $\in$ R^{-1}.
Accordingly $R^{-1}$is symmetric.

SOLUTION (iii)

Prove that if R is transitive, then R-1 is transitive.

Suppose that the relation R on A is transitive. Let (a, b) $\in$ $R^{-1}$and (b, c) $\in$ R^{-1}. Then by definition of R^{-1}, (b, a) $\in$R and (c, b) $\in$R. Now R is transitive, therefore

if (c, b) $\in$R and (b, a) $\in$R then (c, a) $\in$R.

Again by definition of R^{-1}, we have (a, c) $\in$ R^{-1}. We have thus shown that for all a, b, c $\in$ A, if (a, b) $\in$ $R^{-1}$and (b, c) $\in$ R^{-1}then (a, c) $\in$ R^{-1}.
Accordingly R^{-1} is transitive.

SOLUTION (iv)

Prove that if R is anti-symmetric. Then R^{-1} **is anti-symmetric**.

Suppose that relation R on A is anti-symmetric. Let (a,b)$\in$ R^{-1} 1 and (b,a)$\in$ R^{-1}
Then by definition of R^{-1}(b,a) $\in$R and (a,b) $\in$R. Since R is antisymmetric, so if (a,b)$\in$R and (b,a)$\in$R then a = b.Thus we have shown that if (a,b) $\in$ R^{-1} and (b,a) $\in$ R^{-1} then a=b.

Accordingly R^{-1} is antisymmetric.

EXERCISE:

Show that the relation R on a set A is symmetric if, and only if,
R= R^{-1}.

SOLUTION:

Suppose the relation R on A is symmetric.
Let (a,b)$\in$R. Since R is symmetric, so (b,a) $\in$ R. But by definition of R^{-1} if (b,a) $\in$ R then (a,b) $\in$ R^{-1}. Since (a,b) is an arbitrary element of R, so

$R \subseteq R^{-1}$ …………(1)

Next, let (c,d) $\in$ R^{-1}. By definition of R^{-1} (d,c) $\in$R. Since R is symmetric, so (c,d) $\in$R. Thus we have shown that if (c,d) $\in$ R^{-1} then (c,d) $\in$R.Hence

$R^{-1} \subseteq R$…………..(2)

By (1) and (2) it follows that R= R^{-1}.
Conversely

suppose R = R^{-1}.

We have to show that R is symmetric. Let (a,b)∈R.
Now by definition of R^{-1} (b,a) ∈ R^{-1} .Since R = R^{-1}, so (b,a) ∈ R^{-1}= R
Thus we have shown that if (a,b)∈R then (b,a)∈R
Accordingly R is symmetric.

COMPLEMENTRY RELATION:

Let R be a relation from a set A to a set B. The complementry relation $\overline{R}$ of R is the set of all those ordered pairs in A×B that do not belong to R.
Symbolically:

$$\overline{R} = A\times B - R = \{(a,b) \in A\times B \mid (a,b)\notin R\}$$

EXAMPLE:

Let A = {1,2,3} and
R = {(1,1), (1,3), (2,2), (2,3), (3,1)} be a relation on A
Then $\overline{R}$ = {(1,2), (2,1), (3,2), (3,3)}

EXERCISE:

Let R be the relation R = {(a,b)| a<b} on the set of integers. Find
a) $\overline{R}$ b) R^{-1}

SOLUTION:

a) R = Z×Z - R = {(a,b)| a ≮ b}
= {(a,b)| a ≥ b}
b) R^{-1}= {(a,b) | a > b}

EXERCISE:

Let R be a relation on a set A. Prove that R is reflexive iff R is irreflexive

SOLUTION:

Suppose R is reflexive. Then by definition, for all a∈A, (a,a) ∈R
But then by definition of the complementry relation (a,a) ∉R, ∀ a∈A.
Accordingly R is irreflexive.
Conversely
if R is irreflexive, then (a,a) ∉R, ∀ a∈A.
Hence by definition of R, it follows that (a,a) ∈R, ∀ a∈A
Accordingly R is reflexive.

EXERCISE:

Suppose that R is a symmetric relation on a set A. Is R also symmetric.

SOLUTION:

Let (a,b)∈R. Then by definition of R, (a,b) ∉R. Since R is symmetric, so
if (a,b) ∉R then (b,a) ∉ R.
{for (b,a) ∈ R and (a,b) ∉ R will contradict the symmetry property of R}
Now (b,a) ∉R ⇒(b,a) ∈ R.Hence if (a,b) ∈R then (b,a) ∈R
Thus R is also symmetric.

COMPOSITE RELATION:

Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S denoted SoR is the relation from A to C, consisting of ordered pairs (a,c) where a ∈A, c ∈C, and for which there exists an element b ∈B such that (a,b) ∈R and (b,c) ∈S.
Symbolically:

$$SoR = \{(a,c) \mid a \in A,\ c \in C, \exists\ b \in B, (a,b) \in R \text{ and } (b,c) \in S\}$$

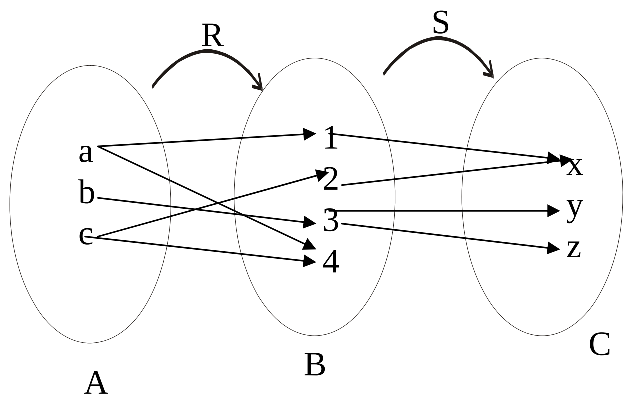
EXAMPLE:

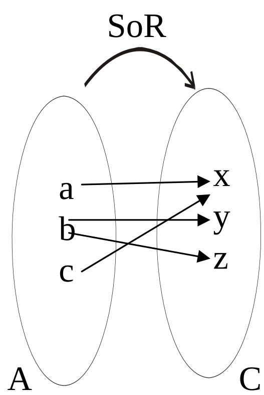
Define R = {(a,1), (a,4), (b,3),(c,1), (c,4)} as a relation from A to B and S = {(1,x),(2,x), (3,y), (3,z)} be a relation from B to C.
Hence

SoR = {(a,x), (b,y), (b,z), (c,x)}

COMPOSITE RELATION FROM ARROW DIAGRAM:

Let A = {a,b,c},B = {1,2,3,4} and C = {x,y,z}. Define relation R from A to B and S from B to C by the following arrow diagram.





MATRIX REPRESENTATION OF COMPOSITE RELATION:

The matrix representation of the composite relation can be found using the Boolean product of the matrices for the relations. Thus if MR and MS are the matrices for relations R (from A to B) and S (from B to C), then

$$M_{SoR} = M_R \, O M_S$$

is the matrix for the composite relation SoR from A to C.

	BOOLEAN ADDITION		BOOLEAN MULTIPLICATION
a.	1 + 1 = 1	a.	1 . 1 = 1
b.	1 + 0 = 1	b.	1 . 0 = 0
c.	0 + 0 = 0	c.	0 . 0 = 0

EXERCISE:

Find the matrix representing the relations SoR and RoS where the matrices representing R and S are

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

SOLUTION:

The matrix representation for SoR is

$$M_{SOR} = M_R O M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix representation for RoS is

$$M_{ROS} = M_S O M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

EXERCISE:

Let R and S be reflexive relations on a set A. Prove SoR is reflexive.

SOLUTION:

Since R and S are reflexive relations on A, so

$\forall$ a $\in$A, (a,a) $\in$R and (a,a) $\in$S

and by definition of the composite relation SoR, it is clear that

(a,a) $\in$SoR $\forall$ a $\in$A.

Accordingly SoR is also reflexive.

LECTURE # 15

RELATIONS AND FUNCTIONS:

A function **F** from a set X to a set Y is a relation from X to Y that satisfies the following two properties

1.For every element x in X, there is an element y in Y such that $(x,y) \in F$.
In other words every element of X is the first element of some ordered pair of F.

2.For all elements x in X and y and z in Y, if $(x,y) \in F$ and $(x,z) \in F$, then $y = z$
In other words no two distinct ordered pairs in F have the same first element.

EXERCISE:

Which of the relations define functions from X = {2,4,5} to Y={1,2,4,6}.

a. $R_1 = \{(2,4), (4,1)\}$
b. $R_2 = \{(2,4), (4,1), (4,2), (5,6)\}$
c. $R_3 = \{(2,4), (4,1), (5,6)\}$

SOLUTION :

a. R_1 is not a function, because $5 \in X$ does not appear as the first element in any ordered pair in R_1.

b. R_2 is not a function, because the ordered pairs (4,1) and (4,2) have the same first element but different second elements.

c. R_3 defines a function because it satisfy both the conditions of the function that is every element of X is the first element of some order pair and there is no pair which has the same first order pair but different second order pair.

EXERCISE:

Let A = {4,5,6} and B = {5,6} and define binary relations R and S from A to B as follows:

for all $(x,y) \in A \times B$, $(x,y) \in R \Leftrightarrow x \geq y$
for all $(x,y) \in A \times B$, $xSy \Leftrightarrow 2|(x-y)$

a. Represent R and S as a set of ordered pairs.
b. Indicate whether R or S is a function

SOLUTION:

Since we are given the relation R contains those order pairs of $A \times B$ which has their first element greater or equal to the second Hence R contains the order pairs.
R = {(5,5), (6,5), (6,6)}
Similarly S is such a relation which consists of those order pairs for which the difference of first and second elements difference divisible by 2.
Hence S = {(4,6), (5,5), (6,6)}

b. R is not a function because $4 \in A$ is not related to any element of B.
S clearly defines a function since each element of A is related to a unique element of B.

FUNCTION:

A function ***f*** from a set X to a set Y is a ***relationship*** between elements of X and elements of Y such that ***each*** element of X is related to a ***unique*** element of Y, and is denoted $f : X \rightarrow Y$.
The set X is called the domain of *f* and Y is called the co-domain of *f*.

NOTE: The unique element y of Y that is related to x by *f* is denoted *f*(x) and is called *f* of x, or the value of *f* at x, or the image of x under *f*

ARROW DIAGRAM OF A FUNCTION:

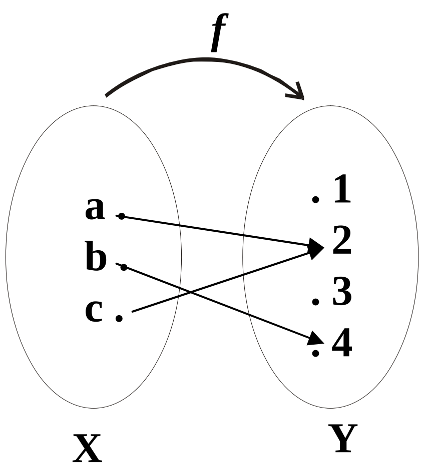
The definition of a function implies that the arrow diagram for a function f has the following two properties:

1. Every element of X has an arrow coming out of it

2. No two elements of X has two arrows coming out of it that point to two different elements of Y.

EXAMPLE:

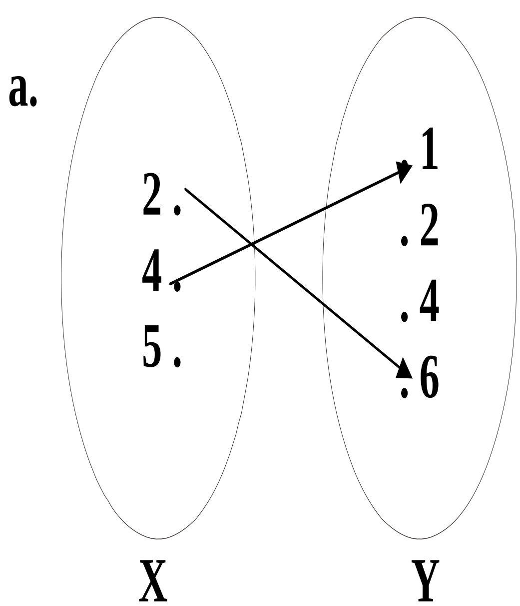
Let X = {a,b,c} and Y={1,2,3,4}.
Define a function *f* from X to Y by the arrow diagram.

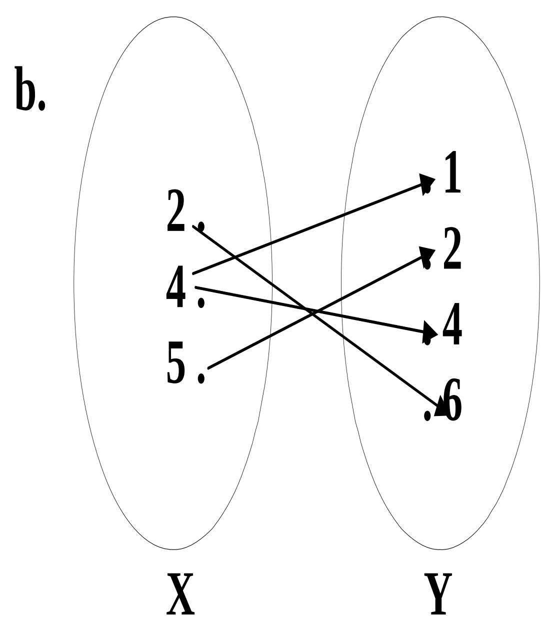


You can easily note that the above diagram satisfy the two conditions of a function hence a graph of the function.
Note that f(a) = 2, f(b) = 4,and f(c) = 2

FUNCTIONS AND NONFUNCTIONS:

Which of the arrow diagrams define functions from X = {2,4,5}to Y = {1,2,4,6}.





The relation given in the diagram (a) is **Not a function** because there is no arrow coming out of of 5∈X to any element of Y.
The relation in the diagram (b) is **Not a function**, because there are two arrows coming out of 4∈X. i.e.,4∈X is not related to a unique element of Y.

RANGE OF A FUNCTION:

Let f: X→Y. The range of f consists of those elements of Y that are image of elements of X.
Symbolically: **Range** of f = {y ∈Y| y = f(x), for some x ∈X}

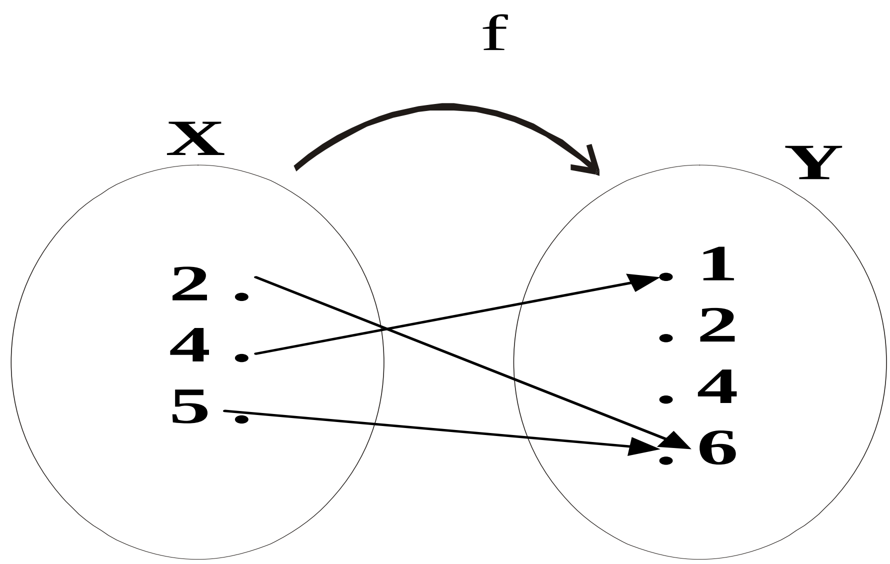
NOTE:

1. The range of a function f is always a subset of the co-domain of f.
2. The range of f: X →Y is also called the image of X under f.
3. When y = f(x), then x is called the pre-image of y.
4. The set of all elements of X, that are related to some y ∈Y is called the inverse image of y.

EXERCISE:

Determine the range of the functions f, g, h from X = {2,4,5} to Y = {1,2,4,6} defined as:

1.



2. g = {(2,6), (4,2), (5,1)}
3. h(2) = 4, h (4) = 4, h(5) = 1

SOLUTION:

1. Range of f = {1, 6}
2. Range of g = {1, 2, 6}
3. Range of h = {1, 4}

GRAPH OF A FUNCTION:

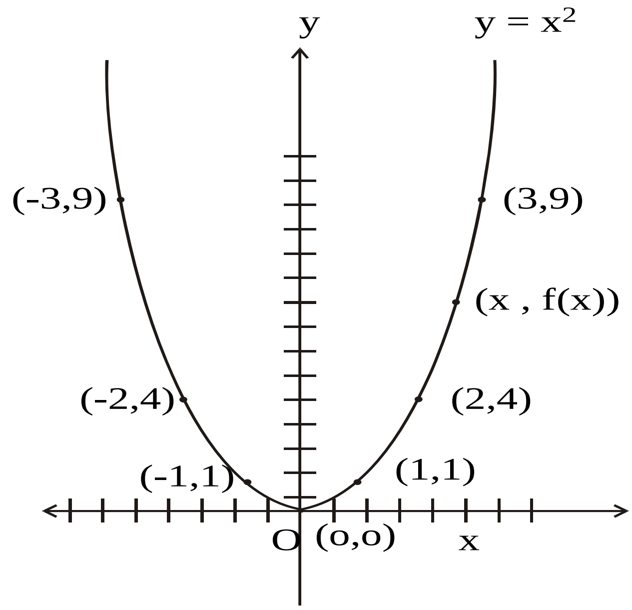
Let f be a real-valued function of a real variable. i.e. f:R →R. The graph of f is the set of all points (x,y) in the Cartesian coordinate plane with the property that x is in the domain of f and y = f(x).

EXAMPLE:

We have to draw the graph of the function f given by the relation $y=x^2$ in order to draw the graph of the function we will first take some elements from the domain will see the image of them and then plot then on the graph as follows

Graph of $y = x^2$

x	y=f(x)
-3	9
-2	4
-1	1
0	0
+1	1
+2	4
+3	9

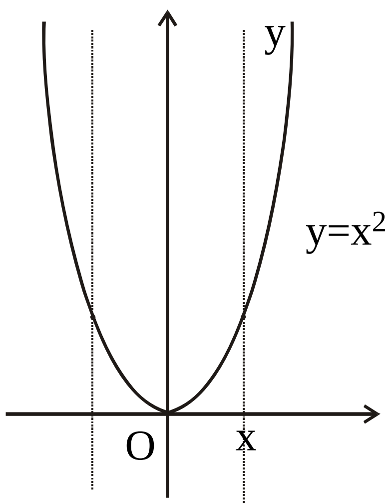


VERTICAL LINE TEST FOR THE GRAPH OF A FUNCTION:

For a graph to be the graph of a function, any given vertical line in its domain intersects the graph in at most one point.

EXAMPLE:

The graph of the relation $y = x^2$ on R defines a function by vertical line test.



EXERCISE:

Define a binary relation P from R to R as follows:

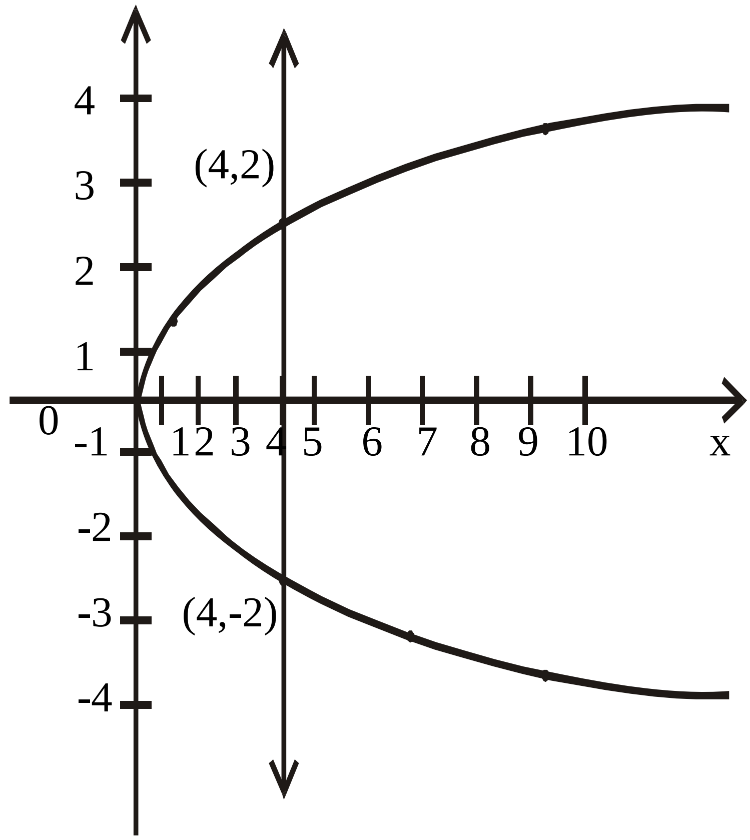
for all real numbers x and y $(x, y) \in P \Leftrightarrow x = y^2$

Is P a function? Explain.

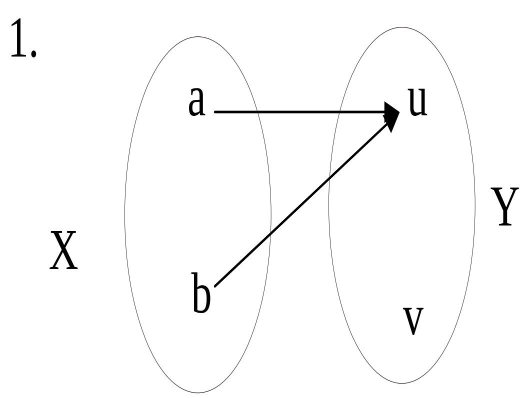
SOLUTION:

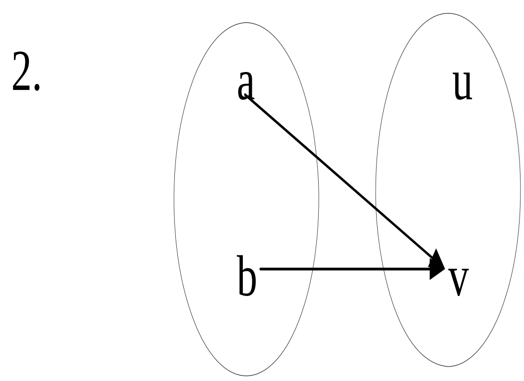
The graph of the relation $x = y^2$ is shown below. Since a vertical line intersects the graph at two points; the graph does not define a function.

x	Y
9	-3
4	-2
1	-1
0	0
1	1
4	2
9	3

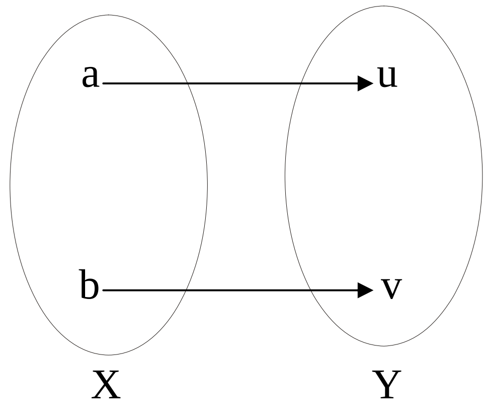


EXERCISE:
Find all functions from X = {a,b} to Y = {u,v}
SOLUTION:

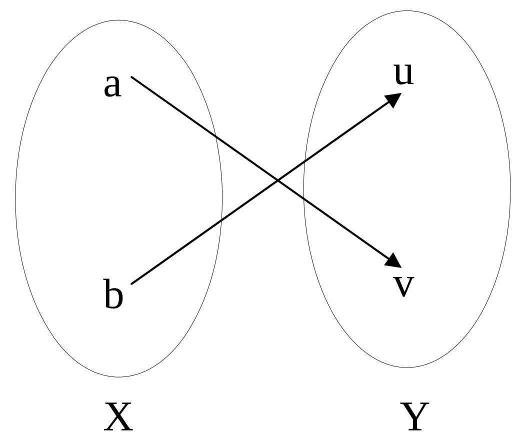




3.

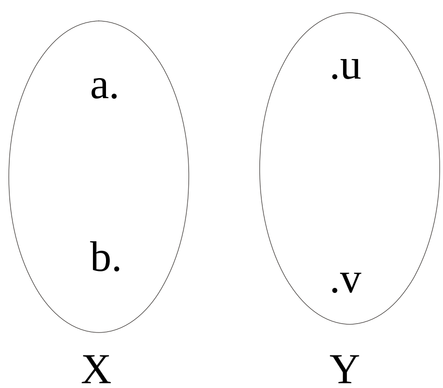


4.

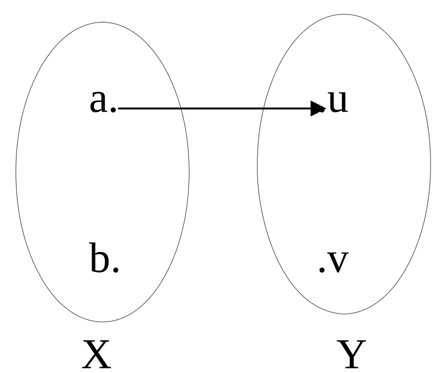


EXERCISE:
Find four binary relations from X = {a,b}to Y = {u,v}that are not functions.
SOLUTION:
The four relations are

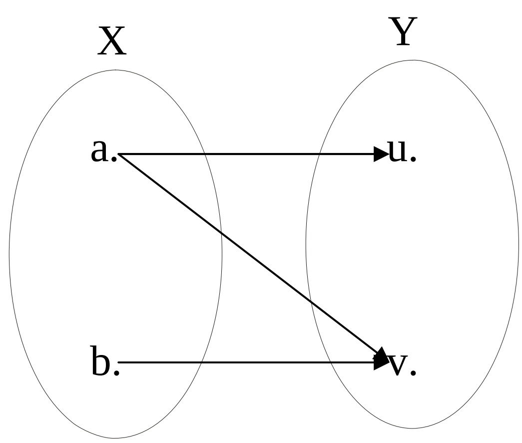
1.



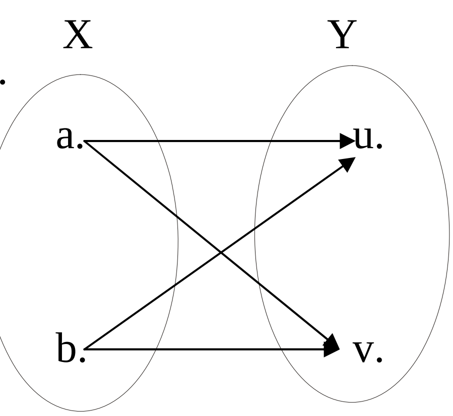
2.



3.



4.



EXERCISE:
How many functions are there from a set with three elements to a set with four elements.
SOLUTION:
Let X = $\{x_1, x_2, x_3\}$ and Y= $\{y_1, y_2, y_3, y_4\}$

Then x_1 may be related to any of the four elements y_1, y_2, y_3, y_4 of Y. Hence there are 4 ways to relate x_1 in Y. Similarly x_2 may also be related to any one of the 4 elements in Y. Thus the total number of different ways to relate x_1 and x_2 to elements of Y are $4 \times 4 = 16$. Finally x_3 must also has its image in Y and again any one of the 4 elements y_1, or y_2 or y_3 or y_4 could be its image.
Therefore the total number of functions from X to Y are

$$4 \times 4 \times 4 = 4^3 = 64.$$

EXERCISE:

Suppose A is a set with m elements and B is a set with n elements.
1.How many binary relations are there from A to B?
2.How many functions are there from A to B?
3.What fraction of the binary relations from A to B are functions?

SOLUTION:

1.Number of elements in $A \times B = m.n$
Therefore, number of binary relations from A to B =
Number of all subsets of $A \times B = 2^{mn}$
2.Number of functions from A to B = n.n.n. … .n (m times)
$= n^m$
3.Fraction of binary relations that are functions = $n^m / 2^{mn}$

FUNCTIONS NOT WELL DEFINED:

Determine whether f is a function from Z to R if

$a.\quad f(n) = \pm n \qquad b.\quad f(n) = \dfrac{1}{n^2 - 4}$

$c.\quad f(n) = \sqrt{n} \qquad d.\quad f(n) = \sqrt{n^2 + 1}$

SOLUTION:

a. f is not well defined since each integer n has two images +n and -n
b. f is not well defined since f(2) and f(-2) are not defined.
c. f is not defined for $n < 0$ since f then results in imaginary values (not real)
d. f is well defined because each integer has unique (one and only one) image in R under f.

EXERCISE:

Student C tries to define a function $h : Q \to Q$ by the rule. $h\left(\dfrac{m}{n}\right) = \dfrac{m^2}{n}$
for all integers m and n with $n \neq 0$
Students D claims that h is not well defined. Justify students D's claim.

SOLUTION:

The function h is well defined if each rational number has a unique (one and only one) image.

Consider $\dfrac{1}{2} \in Q$

$$h\left(\frac{1}{2}\right) = \frac{1^2}{2} = \frac{1}{2}$$

Now $\dfrac{1}{2} = \dfrac{2}{4}$ and

$$h\left(\frac{2}{4}\right) = \frac{2^2}{4} = \frac{4}{4} = 1$$

Hence an element of Q has more than one images under h. Accordingly h is not well defined.

REMARK:

A function f: $X \to Y$ is well defined *iff* $\forall x_1, x_2 \in X$, if $x_1 = x_2$ then $f(x_1) = f(x_2)$

EXERCISE:

Let g: $R \to R+$ be defined by $g(x) = x^2 + 1$

1. Show that g is well defined.
2. Determine the domain, co-domain and range of g.

SOLUTION:

1. g is well defined:

Let $x_1, x_2 \in R$ and suppose $x_1 = x_2$

$\Rightarrow$ $x_1^2 = x_2^2$ (squaring both sides)

$\Rightarrow$ $x_1^2 + 1 = x_2^2 + 1$ (adding 1 on both sides)

$\Rightarrow$ $g(x_1) = g(x_2)$ (by definition of g)

Thus if $x_1 = x_2$ then $g(x_1) = g(x_2)$. According $g: R \to R+$ is well defined.

2. $g: R \to R^+$ defined by $g(x) = x^2 + 1$.

Domain of g = R (set of real numbers)

Co-domain of g = R^+ (set of positive real numbers)

The range of g consists of those elements of R^+ that appear as image points.

Since $x^2 \geq 0$ $\forall x \in R$

$x^2 + 1 \geq 1$ $\forall x \in R$

i.e. $g(x) = x^2 + 1 \geq 1$ $\forall x \in R$

Hence the range of g is all real number greater than or equal to 1, i.e., the internal $[1,\propto)$

IMAGE OF A SET:

Let $f : X \to Y$ is function and $A \subseteq X$.

The image of A under f is denoted and defined as:

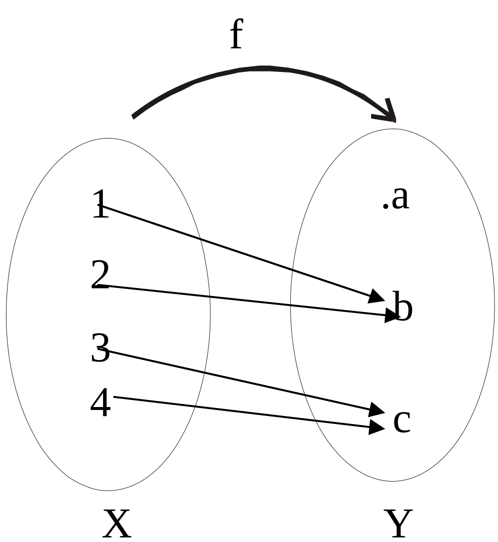
$f(A) = \{y \in Y \mid y=f(x), \quad \text{for some x in A}\}$

EXAMPLE:

Let f: $X \to Y$ be defined by the arrow diagram

Let A = {1,2}and B = {2,3} then

f(A)={b} and f(B) = {b,c} under the function defined in the Diagram then we say that image set of A is {b} and I mage set of B is {b,c}.

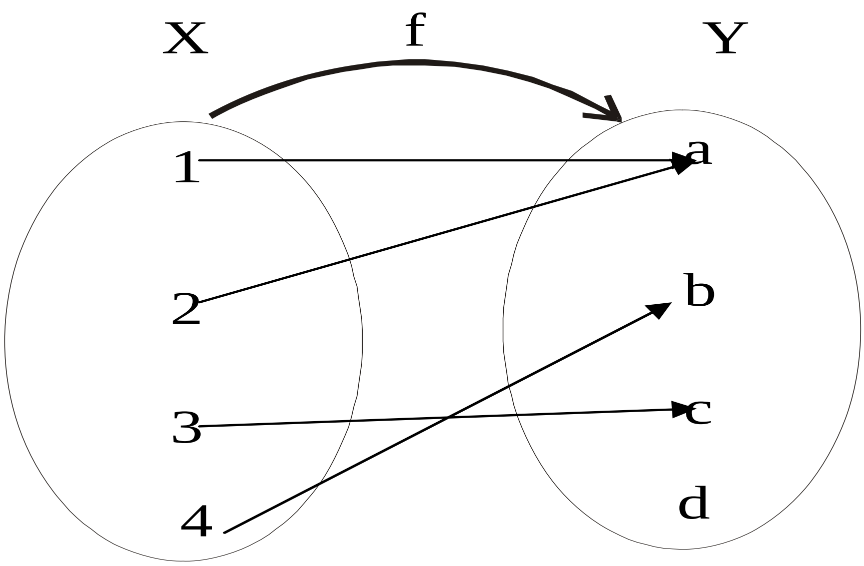


INVERSE IMAGE OF A SET:

Let f: $X \to Y$ is a function and $C \subseteq Y$. The inverse image of C under f is denoted and defined as:

$f^{-1}(C)=\{x \in X \mid f(x) \in C\}$

EXAMPLE:
Let f: X →Y be defined by the arrow diagram.
Let C = {a},D = {b,c},E = {d} then
$f^{-1}(C)=\{1,2\}$,
$f^{-1}(D) = \{3,4\}$, and
$f^{-1}(E) =\varnothing$



SOME RESULTS:
Let f: X →Y is a function. Let A and B be subsets of X and C and D be subsets of Y.
1. if $A\subseteq B$ then $f(A) \subseteq f(B)$
2. $f(A\cup B) = f(A) \cup f(B)$
3. $f(A\cap B) \subseteq f(A) \cap f(B)$
4. $f(A-B) \supset f(A) - f(B)$
5. if $C \subseteq D$, then $f^{-1}(C) \subseteq f^{-1}(D)$
6. $f^{-1}(C\cup D) = f^{-1}(C) \cup f^{-1}(D)$
7. $f^{-1}(C\cap D) = f^{-1}(C) \cap f^{-1}(D)$
8. $f^{-1}(C-D) = f^{-1}(C) - f^{-1}(D)$

BINARY OPERATIONS:
A binary operation "*" defined on a set A assigns to each ordered pair (a,b) of elements of A, a uniquely determined element a*b of A.
That is, a binary operation takes two elements of A and maps them to a third element of A.

EXAMPLE:
1. "+" and "." are binary operations on the set of natural numbers N.
2. "-" is not a binary operation on N.
3. "-" is a binary operation on Z, the set of integers.
4. "÷" is a binary operation on the set of non-zero rational numbers Q-{0}, but not a binary operation on Z.

BINARY OPERATION AS FUNCTION:
A binary operation "*" on a set A is a function from A * A to A.

i.e. $*: A \times A \to A$.

Hence $*(a,b) = c$, where $a, b, c \in A$.

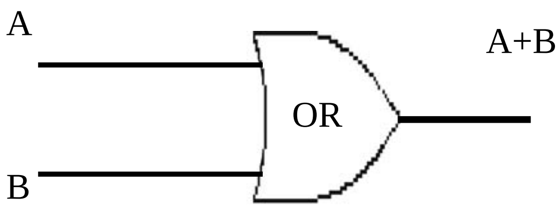
NOTE
*(a,b) is more commonly written as a*b.

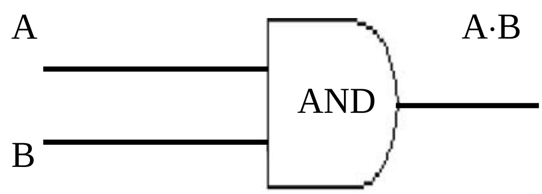
EXAMPLES:

1.The set operations union ∪, intersection ∩ and set difference -, are binary operators on the power set P(A) of any set A.

2.The logical connectives ∨, ∧, →, ↔ are binary operations on the set {T, F}

3. The logic gates OR and AND are binary operations on {0,1}

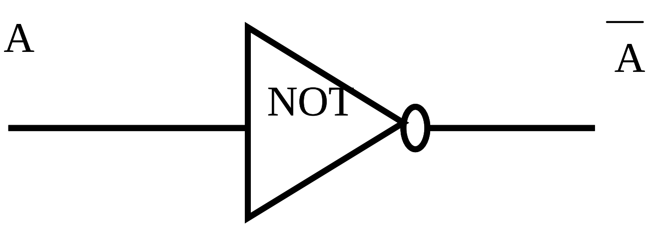




A	B	A+B
1	1	1
1	0	1
0	1	1
0	0	0

A	B	A·B
1	1	1
1	0	0
0	1	0
0	0	0

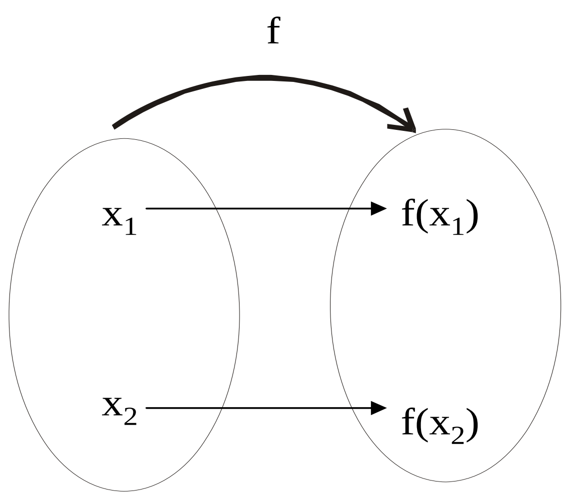
4. The logic gate NOT is a uniary operation on {0,1}



A	$\overline{A}$
1	0
0	1

LECTURE # 16
INJECTIVE FUNCTION
or
ONE-TO-ONE FUNCTION

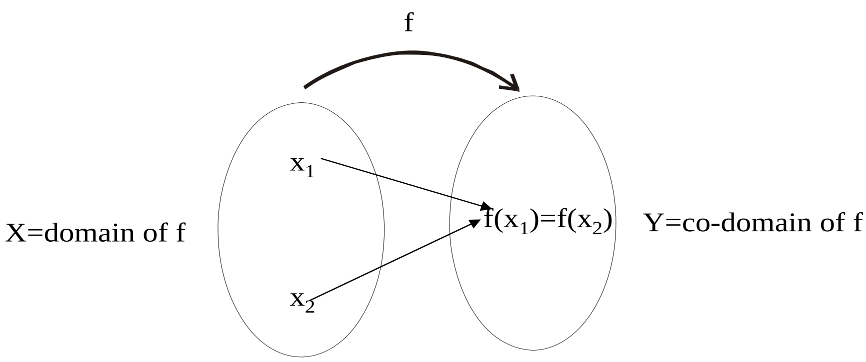
Let f: X →Y be a function. f is injective or one-to-one if, and only if, ∀ x1, x2 ∈X, if x1 ≠ x2 then f(x1) ≠ f(x2)That is, f is one-to-one if it maps distinct points of the domain into the distinct points of the co-domain.



A one-to-one function separates points.

FUNCTION NOT ONE-TO-ONE:

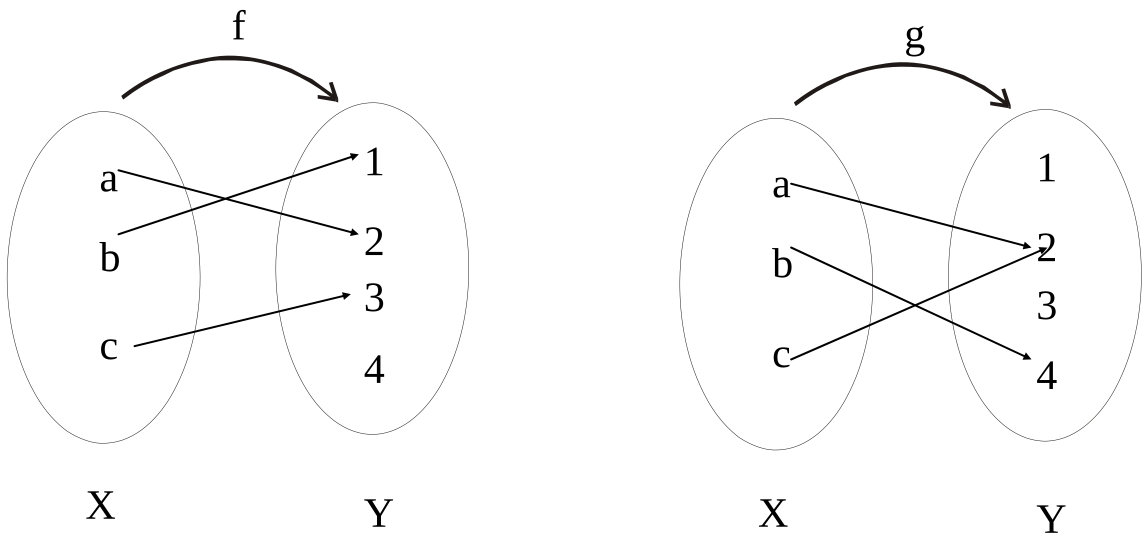
A function f: X →Y is not one-to-one iff there exist elements x_1 and x_2 in such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$.That is, if distinct elements x_1 and x_2 can found in domain of f that have the same function value.



A function that is not one-to-one collapses points together.

EXAMPLE:

Which of the arrow diagrams define one-to-one functions?



SOLUTION:

f is clearly one-to-one function, because no two different elements of Xare mapped onto the same element of Y.
g is not one-to-one because the elements a and c are mapped onto the same element 2 of Y.

ALTERNATIVE DEFINITION FOR ONE-TO-ONE FUNCTION:

A function $f: X \to Y$ is one-to-one (1-1) iff $\forall\, x_1, x_2 \in X$, if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$
(i.e distinct elements of 1st set have their distinct images in 2nd set)

The equivalent contra-positive statement for this implication is $\forall\, x_1, x_2 \in X$,
if $f(x_1) = f(x_2)$, then $x_1 = x_2$

REMARK:

$f: X \to Y$ is not one-to-one iff $\exists\, x_1, x_2 \in X$ with $f(x_1) = f(x_2)$ but $x_1 \neq x_2$

EXAMPLE:

Define $f: R \to R$ by the rule $f(x) = 4x - 1$ for all $x \in R$
Is f one-to-one? Prove or give a counter example.

SOLUTION:

Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$\Rightarrow \quad 4x_1 - 1 = 4x_2 - 1$ (by definition of f)

$\Rightarrow \quad 4x_1 = 4x_2$ (adding 1 to both sides)

$\Rightarrow \quad x_1 = x_2$ (dividing both sides by 4)

Thus we have shown that if $f(x_1) = f(x_2)$ then $x_1=x_2$
Therefore, f is one-to-one

EXAMPLE:

Define $g : Z \to Z$ by the rule $g(n)=n^2$ for all $n \in Z$
Is g one-to-one? Prove or give a counter example.

SOLUTION:

Let $n_1, n_2 \in Z$ and suppose $g(n_1) = g(n_2)$

$\Rightarrow \quad n_1^2 = n_2^2 \quad$ (by definition of g)

$\Rightarrow \quad$ either $n_1 = + n_2$ or $n_1 = - n_2$

Thus $g(n_1) = g(n_2)$ does not imply $n_1 = n_2$ always.

As a counter example, let $n_1 = 2$ and $n_2 = -2$.

Then

$g(n_1) = g(2) = 2^2 = 4$ and also $g(n_2) = g(-2) = (-2)^2 = 4$

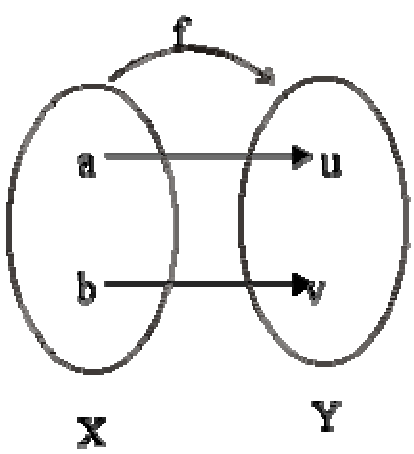
Hence $g(2) = g(-2)$ where as $2 \neq -2$ and so g is not one-to-one.

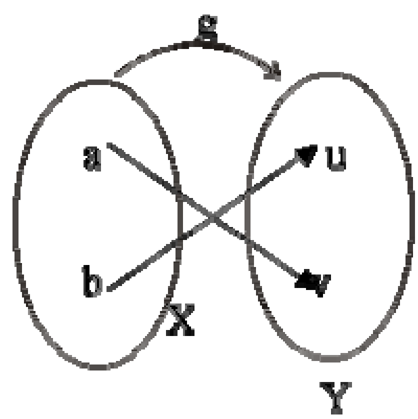
EXERCISE:

Find all one-to-one functions from X = {a,b} to Y = {u,v}

SOLUTION:

There are two one-to-one functions from X to Y defined by the arrow diagrams.



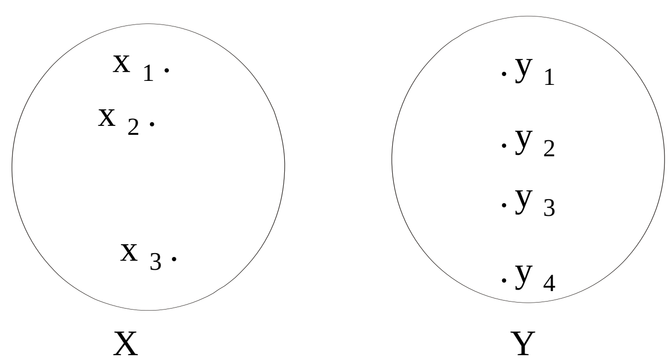


EXERCISE:

How many one-to-one functions are there from a set with three elements to a set with four elements.

SOLUTION:

Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3, y_4\}$



x_1 may be mapped to any of the 4 elements of Y. Then x_2 may be mapped to any of the remaining 3 elements of Y & finally x_3 may be mapped to any of the remaining 2 elements of Y.

Hence, total no. of one-to-one functions from X to Y are

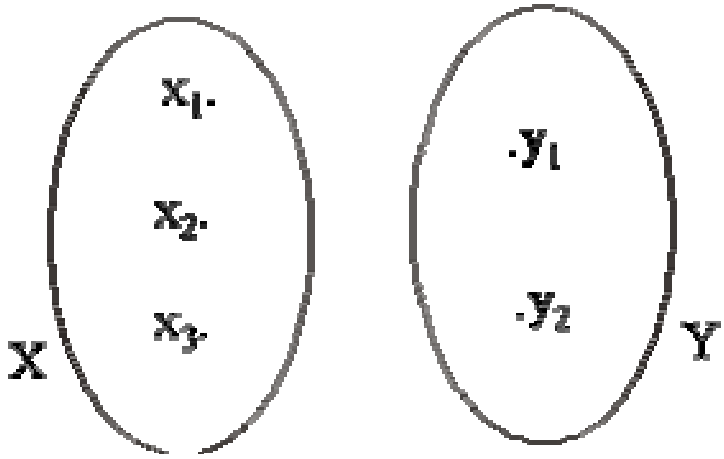
$$4 \times 3 \times 2 = 24$$

EXERCISE:

How many one-to-one functions are there from a set with three elements to a set with two elements.

SOLUTION:

Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$

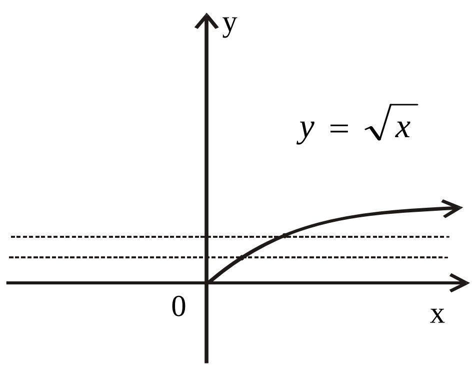


Two elements in X could be mapped to the two elements in Y separately. But there is no new element in Y to which the third element in X could be mapped. Accordingly there is no one-to-one function from a set with three elements to a set with two elements.

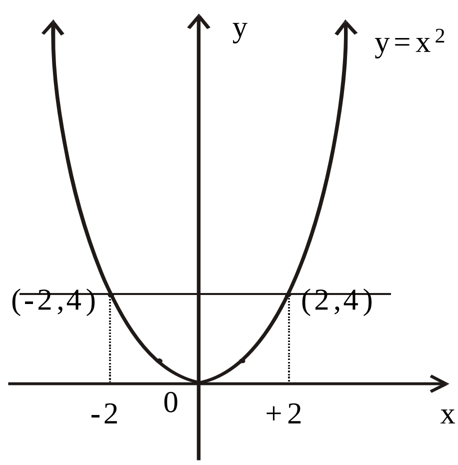
GRAPH OF ONE-TO-ONE FUNCTION:

A graph of a function f is one-to-one iff every horizontal line intersects the graph in at most one point.

EXAMPLE:



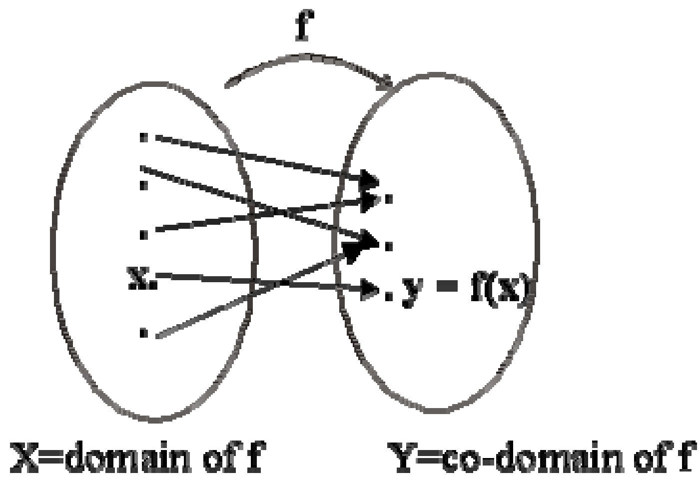
ONE-TO-ONE FUNCTION
from R^+ to R



NOT ONE-TO-ONE FUNCTION
From R to R^+

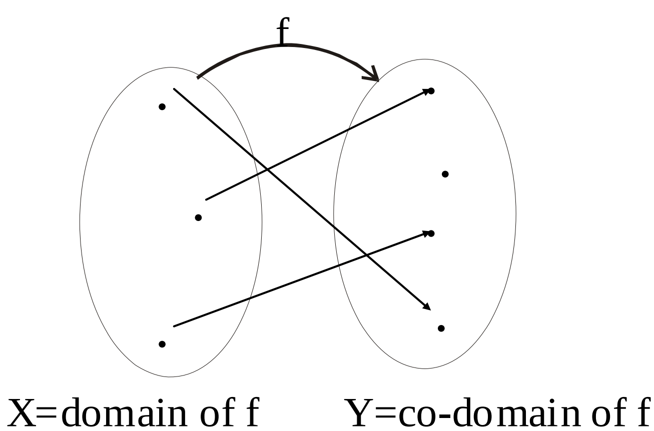
SURJECTIVE FUNCTION or ONTO FUNCTION:

Let f: X→Y be a function. f is surjective or onto if, and only if, "∀ y ε Y, ∃ x εX such that f(x) = y.
That is, f is onto if every element of its co-domain is the image of some element(s) of its domain.i.e., co-domain of f = range of f

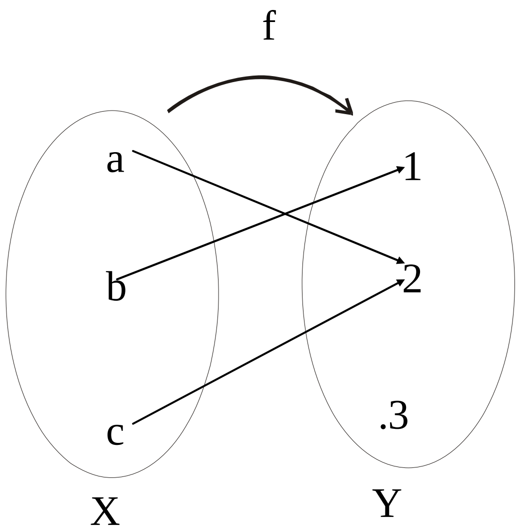


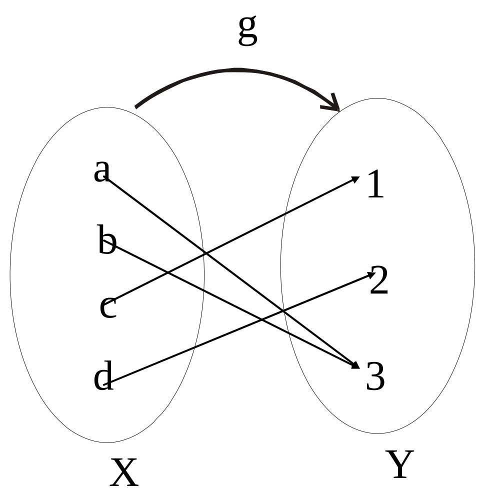
Each element y in Y equals f(x) for at least one x in X

FUNCTION NOT ONTO:
A function f:X→Y is not onto iff there exists yε Y such that ∀x εX, f(x) ≠y.
That is, there is some element in Y that is not the image of any element in X.



EXAMPLE:
Which of the arrow diagrams define onto functions?





SOLUTION:

f is not onto because 3 ≠ f(x) for any x in X. g is clearly onto because each element of Y equals g(x) for some x in X.
as 1 = g(c);,2 = g(d);3 = g(a) = g(b)

EXAMPLE:

Define f: R →R by the rule

f(x) = 4x-1 for all x ∈R

Is f onto? Prove or give a counter example.

SOLUTION:

Let y ∈R. We search for an x ∈ R such that

f(x) = y

or 4x-1 = y (by definition of f)

Solving it for x, we find x=y+1 $x = \frac{y+1}{4} \in R$. Hence for every y ∈R, there exists $x = \frac{y+1}{4} \in R$ such that

$$f(x) = f\left(\frac{y+1}{4}\right)$$

$$= 4.\left(\frac{y+1}{4}\right) - 1 = (y+1) - 1 = y$$

Hence f is onto.

EXAMPLE:

Define h: Z →Z by the rule

h(n) = 4n - 1 for all n ∈ Z

Is h onto? Prove or give a counter example.

SOLUTION:

Let m ∈Z. We search for an n ∈Z such that h(n) = m.

or 4n - 1 = m (by definition of h)

Solving it for n, we find $n = \frac{m+1}{4}$

But $n = \frac{m+1}{4}$ is not always an integer for all m $\in$Z.

As a **counter example**, let m = 0$\in$ Z, then

h(n) = 0

$\Rightarrow$ 4n-1 = 0

$\Rightarrow$ 4n = 1

$\Rightarrow$ $n = \frac{1}{4} \notin \mathrm{Z}$

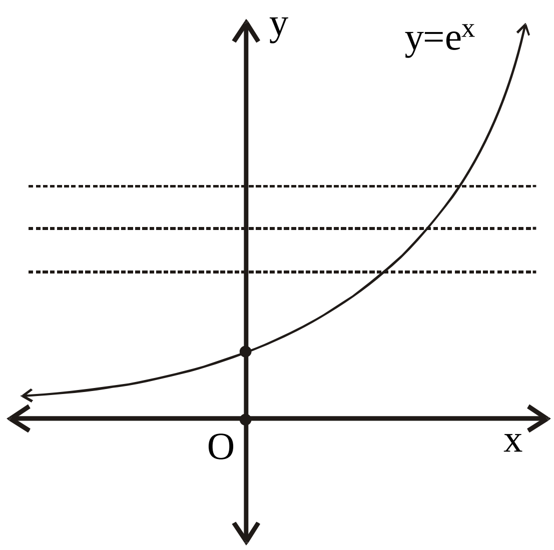
Hence there is no integer n for which h(n) = 0.

Accordingly, h is not onto.

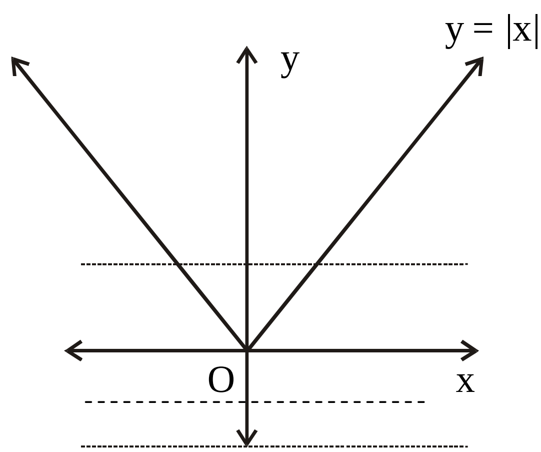
GRAPH OF ONTO FUNCTION:

A graph of a function f is onto iff every horizontal line intersects the graph in at least one point.

EXAMPLE:



ONTO FUNCTION
from R to R^+



NOT ONTO FUNCTION FROM
R to R

EXERCISE:

Let X = {1,5,9} and Y = {3,4,7}.Define g: X $\rightarrow$Y by specifying that

g(1) = 7, g(5) = 3, g(9) = 4

Is g one-to-one? Is g onto?

SOLUTION:

g is one-to-one because each of the three elements of X are mapped to a different elements of Y by g.

g(1) $\neq$ g(5), g(1) $\neq$ g(a), g(5) $\neq$ g(a)

g is onto as well, because each of the three elements of co-domain Y of g is the image of some element of the domain of g.

3 = g(5), 4 = g(9), 7 = g(1)

EXERCISE:

Define f: P({a,b,c})→Z as follows:

for all A∈P ({a,b,c}), f(A)= the number of elements in A.

a. Is f one-to-one? Justify.

b. Is f onto? Justify.

SOLUTION:

a. f is not one-to-one because f({a}) = 1 and f({b}) = 1 but {a}≠ {b}

b. f is not onto because, there is no element of P({a,b,c}) that is mapped to 4 ∈Z.

EXERCISE:

Determine if each of the functions is injective or surjective.

a. f: $Z \to Z^+$ define as f(x) = |x|

b. g: $Z^+ \to Z^+ \times Z^+$ defined as g(x) = (x,x+1)

SOLUTION:

a. f is not injective, because

f(1) = |1| = 1 and f(-1) = |-1| = 1

i.e., f(1) = f(-1) but 1 ≠ -1

f is onto, because for every $a \in Z^+$, there exist –a and +a in Z such that

f(-a) = |-a| = a and f(a) = |a| = a

b. g: $Z^+ \to Z^+ \times Z^+$ defined as g(x) = (x,x+1)

Let $g(x_1) = g(x_2)$ for $x_1, x_2 \in Z^+$

$\Rightarrow$ $(x_1, x_1 + 1) = (x_2, x_2 + 1)$ (by definition of g)

$\Rightarrow$ $x_1 = x_2$ and $x_1 + 1 = x_2 + 1$

(by equality of ordered pairs)

$\Rightarrow$ $x_1 = x_2$

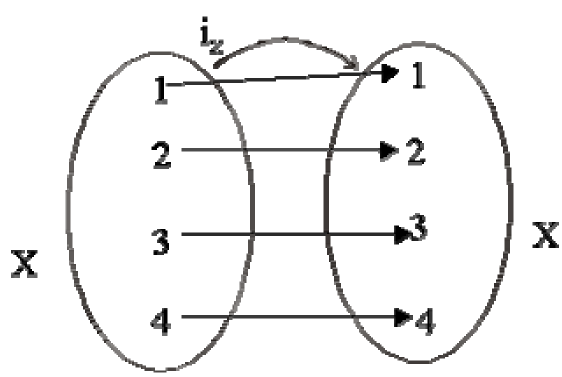
Thus if $g(x_1) = g(x_2)$ then $x_1 = x_2$

Hence **g is one-to-one**.

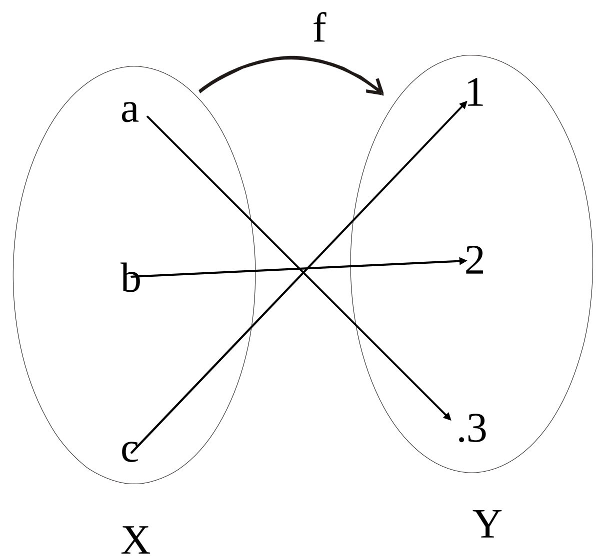
g is not onto because $(1,1) \in Z^+ \times Z^+$ is not the image of any element of Z^+.

BIJECTIVE FUNCTION
or
ONE-TO-ONE CORRESPONDENCE

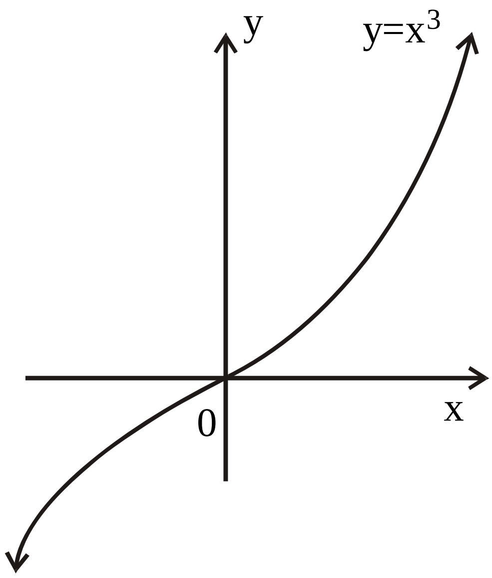
A function f: X→Y that is both one-to-one (injective) and onto (surjective) is called a bijective function or a one-to-one correspondence.



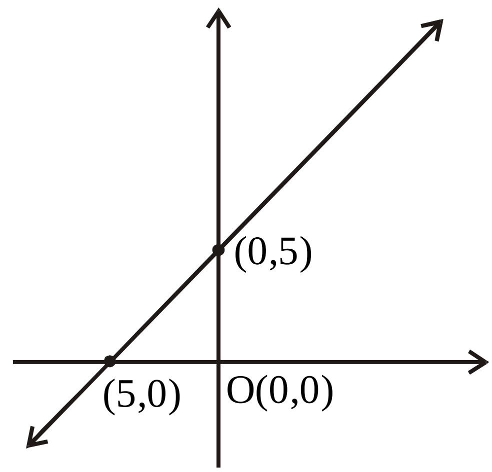
EXAMPLE:
The function f: X→Y defined by the arrow diagram is both one-to-one and onto; hence a bijective function.



EXERCISE:
Let f: R →R be defined by the rule $f(x) = x^3$.Show that f is a bijective.
SOLUTION:
f is one-to-one
Let $f(x_1) = f(x_2)$ for $x_1, x_2 \in R$
$\Rightarrow x_1^3 = x_2^3$
$\Rightarrow x_1^3 - x_2^3 = 0$
$\Rightarrow (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$
$\Rightarrow x_1 - x_2 = 0$ or $x_1^2 + x_1x_2 + x_2^2 = 0$
$\Rightarrow x_1 = x_2$ (the second equation gives no real solution)
Accordingly f is one-to-one.
f is onto
Let $y \in R$. We search for a $x \in R$ such that
$f(x)=y$
$\Rightarrow x^3 = y$ (by definition of f)
or $x = (y)^{1/3}$
Hence for $y \in R$, there exists $x = (y)^{1/3} \in R$ such that
$f(x) = f((y)^{1/3})$
$= ((y)^{1/3})^3 = y$
Accordingly f is onto.
Thus, f is a bijective.
GRAPH OF BIJECTIVE FUNCTION:
A graph of a function f is bijective iff every horizontal line intersects the graph at exactly one point.



BIJECTIVE FUNCTION
from R to R

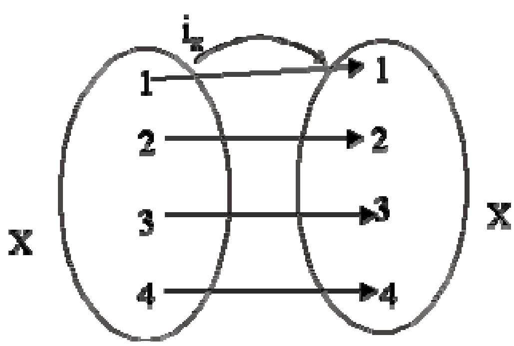


BIJECTIVE FUNCTION
from R to R

IDENTITY FUNCTION ON A SET:

Given a set X, define a function i_X from X to X by $i_X(x) = x$ from all $x \in X$.
The function i_X is called the identity function on X because it sends each element of X to itself.

EXAMPLE:
Let X = {1,2,3,4}. The identity function i_X on X is represented by the arrow diagram



EXERCISE:
Let X be a non-empty set. Prove that the identity function on X is bijective.

SOLUTION:
Let $i_X: X \to X$ be the identity function defined as $i_X(x) = x \ \forall \in X$

1. **i_X is injective (one-to-one)**

   Let $i_X(x_1) = i_X(x_2)$ for $x_1, x_2 \in X$
   $\Rightarrow x_1 = x_2$ (by definition of i_X)
   Hence i_X is one-to-one.

2. **i_X is surjective (onto)**

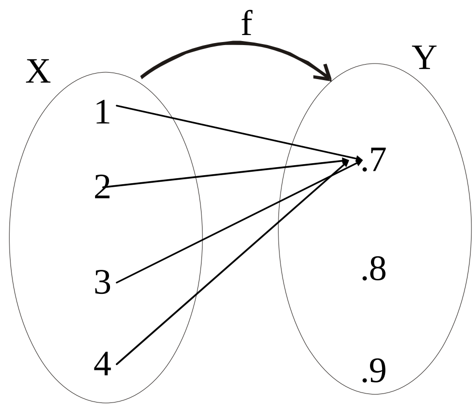
Let $y \in X$ (co-domain of i_X) Then there exists $y \in X$ (domain of i_X) such that $i_X(y) = y$
Hence i_X is onto. Thus, i_X being injective and surjective is bijective.

CONSTANT FUNCTION:

A function $f:X \rightarrow Y$ is a constant function if it maps (sends) all elements of X to one element of Y i.e. $\forall x \in X$, $f(x) = c$, for some $c \in Y$

EXAMPLE:

The function f defined by the arrow diagram is constant.



REMARK:

1. A constant function is one-to-one iff its domain is a singleton.
2. A constant function is onto iff its co-domain is a singleton.

LECTURE # 17

EQUALITY OF FUNCTIONS

Suppose f and g are functions from X to Y. Then f equals g, written f = g, if, and only if, f(x)=g(x) for all x εX

EXAMPLE:

Define f: R →R and g: R→R by formulas:

f(x) = |x| for all x εR

g(x)= $\sqrt{x^2}$ for all x εR

Since the absolute value of a real number equals to square root of its square

i.e., |x| = $\sqrt{x^2}$ for all x εR

Therefore f(x) = g(x) for all x εR

Hence f = g

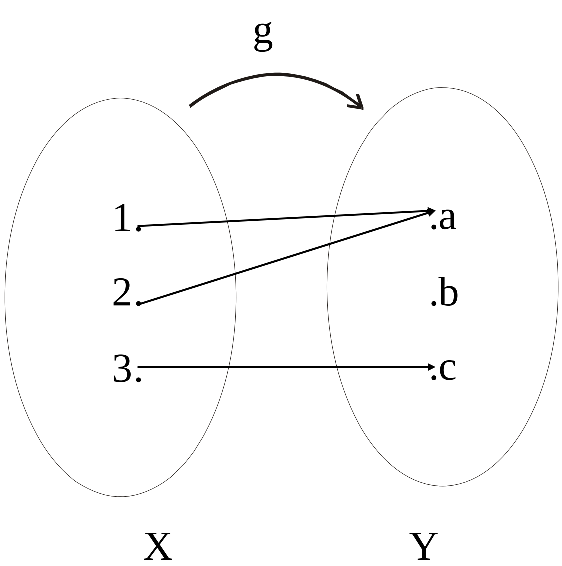
EXERCISE:

Define functions f and g from R to R by formulas:f(x) = 2x and $g(x) = \frac{2x^3+2x}{x^2+1}$ for all x εR. Show that f = g

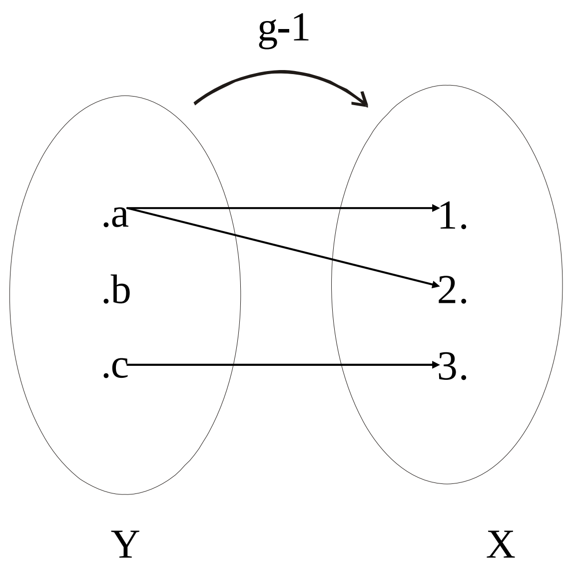
SOLUTION:

$$\begin{aligned} g(x) &= \frac{2x^3+2x}{x^2+1} \\ &= \frac{2x(x^2+1)}{(x^2+1)} \\ &= 2x \qquad [\because x^2+1 \neq 0] \\ &= f(x) \qquad \text{for all } x \in R \end{aligned}$$

INVERSE OF A FUNCTION:

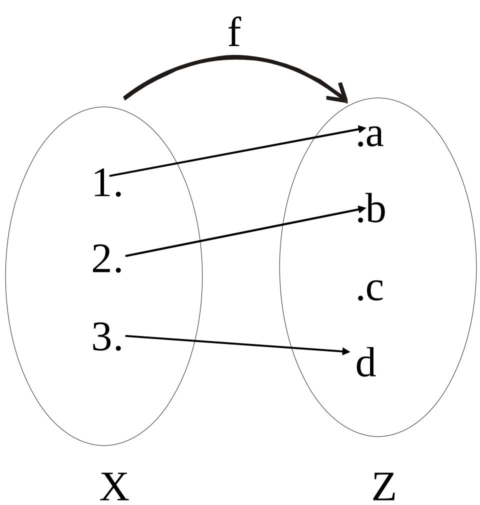


FUNCTION

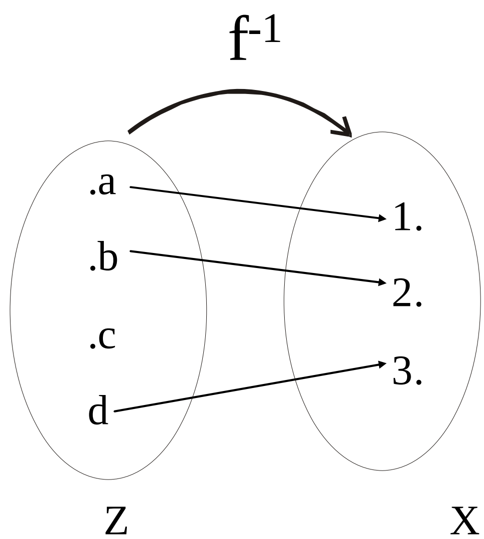


INVERSE

Remark: ***Inverse of a function may not be a function.***

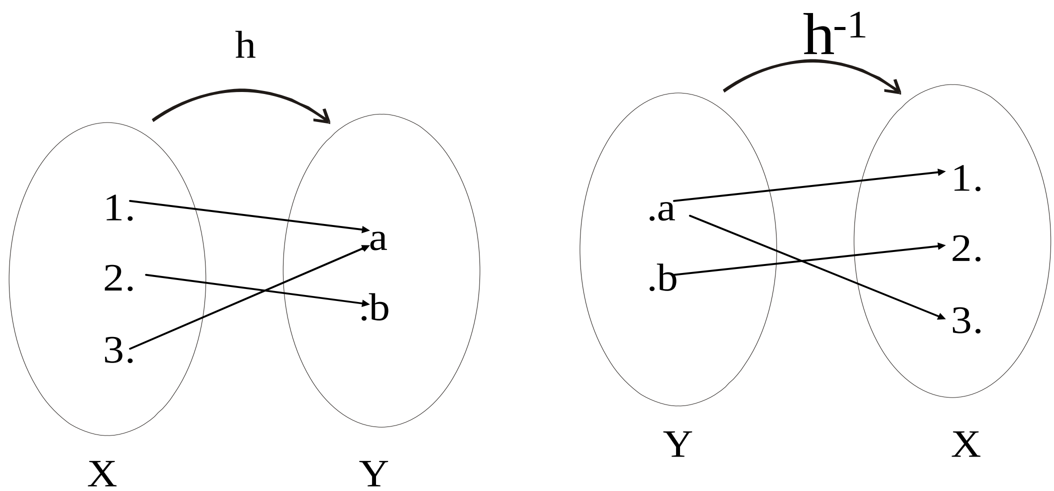


INJECTIVE FUNCTION



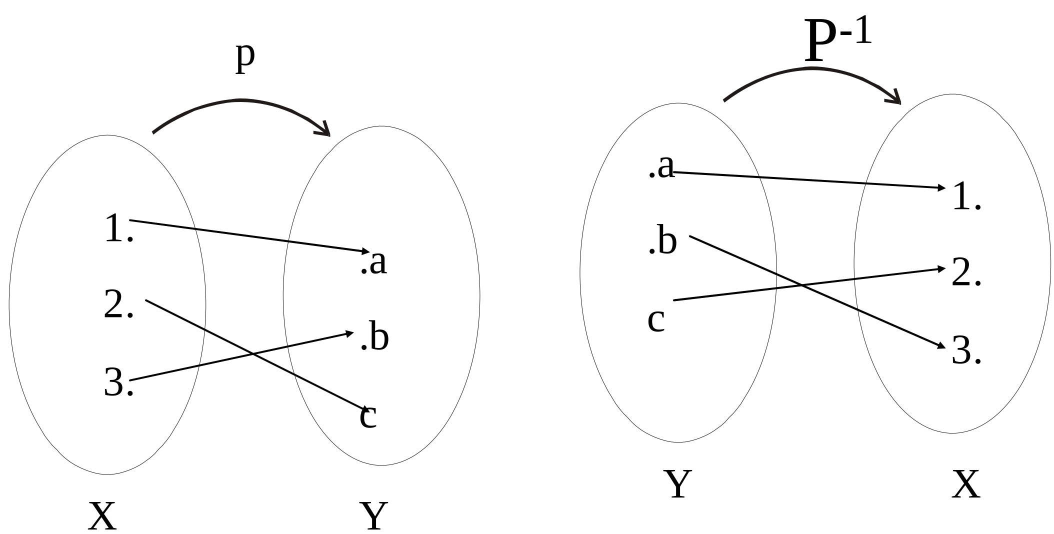
INVERSE

Note:Inverse of an injective function may not be a function.



SURJECTIVE FUNCTION **INVERSE**

Note:Inverse of a surjective function may not be a function.



BIJECTIVE FUNCTION **INVERSE**

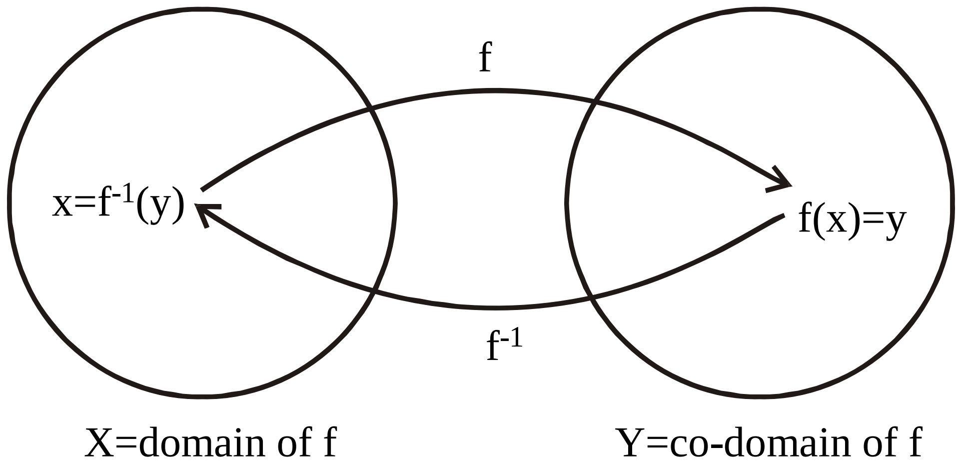
Note:Inverse of a surjective function may not be a function.

<u>INVERSE FUNCTION:</u>

Suppose $f:X\rightarrow Y$ is a bijective function. Then the inverse function $f^{-1}: Y\rightarrow X$ is defined as:

$\forall\ y\in Y, f^{-1}(y) = x \Leftrightarrow y = f(x)$

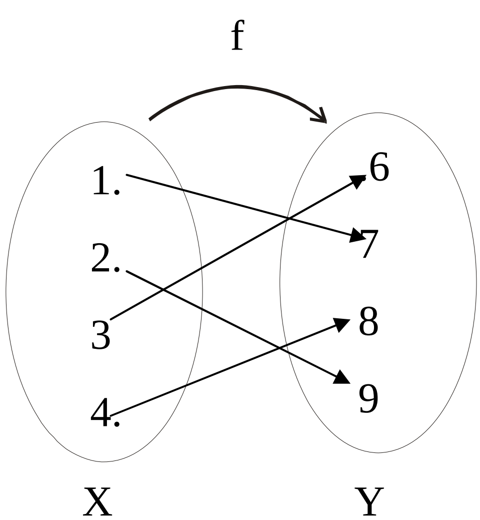
That is, f^{-1} sends each element of Y back to the element of X that it came from under f.



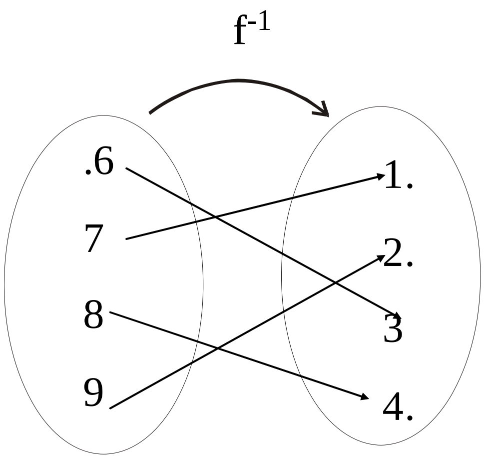
REMARK:
A function whose inverse function exists is called an invertible function.

INVERSE FUNCTION FROM AN ARROW DIAGRAM:

Let the bijection f:X→Y be defined by the arrow diagram.



The inverse function f^{-1}:Y→X is represented below by the arrow diagram.



INVERSE FUNCTION FROM A FORMULA:

Let f:R→R be defined by the formula f(x) = 4x-1 ∀x∈R

Then f is bijective, therefore f^{-1} exists. By definition of f^{-1},

$$f^{-1}(y) = x \Leftrightarrow f(x)=y$$

Now solving f(x) = y for x

⇔ 4x-1 = y (by definition of f)

⇔ 4x = y + 1

⇔ $x = \frac{y+1}{4}$

Hence, $f^{-1}(y) = \frac{y+1}{4}$ is the inverse of f(x)=4x-1 which defines f^{-1} : R→R.

WORKING RULE TO FIND INVERSE FUNCTION:

Let f: X →Y be a one-to-one correspondence defined by the formula f(x)=y.

1. Solve the equation f(x) = y for x in terms of y.
2. f^{-1} (y) equals the right hand side of the equation found in step 1.

EXAMPLE:

Let a function f be defined on a set of real numbers as

$f(x) = \frac{x+1}{x-1}$ for all real numbers x ≠1.

1. Show that f is a bijective function on R-{1}.
2. Find the inverse function f^{-1}

SOLUTION:

1. To show:**f is injective**

Let $x_1, x_2 \in$ R-{1} and suppose

$f(x_1) = f(x_2)$ we have to show that $x_1=x_2$

$\Rightarrow \frac{x_1+1}{x_1-1} = \frac{x_2+1}{x_2-1}$ (by definition of f)

$\Rightarrow (x_1 + 1)(x_2 - 1) = (x_2 + 1)(x_1 - 1)$

$\Rightarrow x_1x_2 - x_1 + x_2 - 1 = x_1x_2 - x_2 + x_1 - 1$

$\Rightarrow - x_1 + x_2 = - x_2 + x_1$

$\Rightarrow x_2 + x_2 = x_1 + x_1$

$\Rightarrow 2x_2 = 2x_1$

$\Rightarrow x_2 = x_1$

Hence f is injective.

b.Next to show:f is surjective

Let $y \in R - \{1\}$. We look for an $x \in R - \{1\}$ such that $f(x) = y$

$\Rightarrow$ $x + 1 = y(x-1)$

$\Rightarrow$ $1 + y = xy - x$

$\Rightarrow$ $1 + y = x(y-1)$

$\Rightarrow$ $x = \dfrac{y+1}{y-1}$

Thus for each $y \in R - \{1\}$, there exists $x = \dfrac{y+1}{y-1} \in R - \{1\}$

such that $f(x) = f\left(\dfrac{y+1}{y-1}\right) = y$

Accordingly f is surjective

2. inverse function of f

The given function f is defined by the rule

$$f(x) = \frac{x+1}{x-1} = y \qquad \text{(say)}$$

$\Rightarrow$ $x + 1 = y(x-1)$

$\Rightarrow$ $x + 1 = yx - y$

$\Rightarrow$ $y + 1 = yx - x$

$\Rightarrow$ $y + 1 = x(y-1)$

$\Rightarrow$ $x = \dfrac{y+1}{y-1}$

Hence $f^{-1}(y) = \dfrac{y+1}{y-1}$; $\qquad y \neq 1$

EXERCISE:

Let $f:R \rightarrow R$ be defined by

$$f(x) = x^3 + 5$$

Show that f is one-to-one and onto. Find a formula that defines the inverse function f^{-1}.

SOLUTION:

1. f is one-to-one

Let $f(x_1) = f(x_2)$ for $x_1, x_2 \in R$

$\Rightarrow$ $x_1^3 + 5 = x_2^3 + 5$ (by definition of f)

$\Rightarrow$ $x_1^3 = x_2^3$ (subtracting 5 on both sides)

$\Rightarrow$ $x_1 = x_2$ .Hence f is one-to-one.

2. f is onto

Let $y \in R$. We search for an $x \in R$ such that $f(x) = y$.

$\Rightarrow$ $x^3 + 5 = y$ (by definition of f)

$\Rightarrow$ $x^3 = y - 5$

$\Rightarrow$ $x = \sqrt[3]{y-5}$

Thus for each $y \in R$, there exists $x = \sqrt[3]{y-5} \in R$

such that

$$\begin{aligned} f(x) &= f\left(\sqrt[3]{y-5}\right) \\ &= \left(\sqrt[3]{y-5}\right)^3 + 5 \qquad \text{(by definition of f)} \\ &= (y-5)+5 = y \end{aligned}$$

Hence f is onto.

3. formula for f^{-1}

f is defined by $y = f(x) = x^3 + 5$

$\Rightarrow$ $y-5 = x^3$

or $x = \sqrt[3]{y-5}$

Hence $f^{-1}(y) = \sqrt[3]{y-5}$

which defines the inverse function

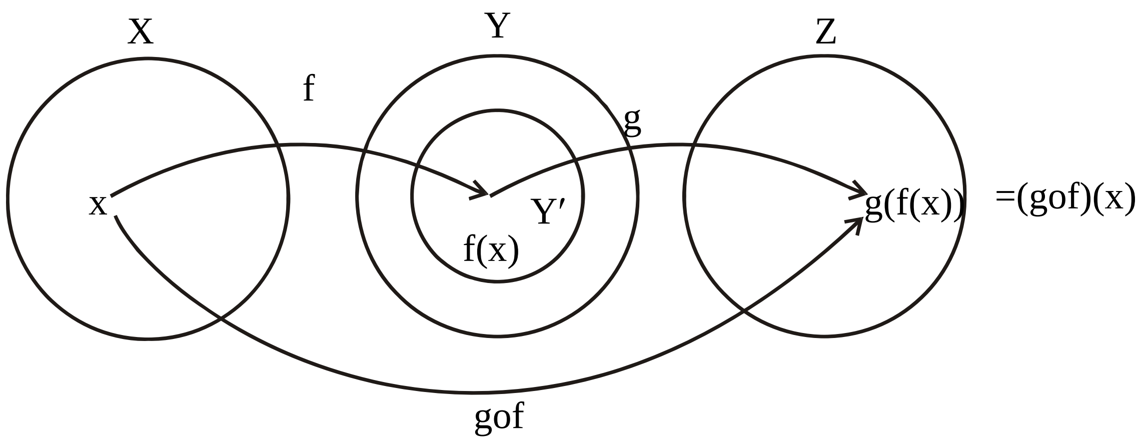
COMPOSITION OF FUNCTIONS:

Let f: X $\rightarrow$Y′ and g: Y $\rightarrow$Z be functions with the property that the range of f is a subset of the domain of g i.e. $f(X) \subseteq Y$.

Define a new function gof:X $\rightarrow$Z as follows:

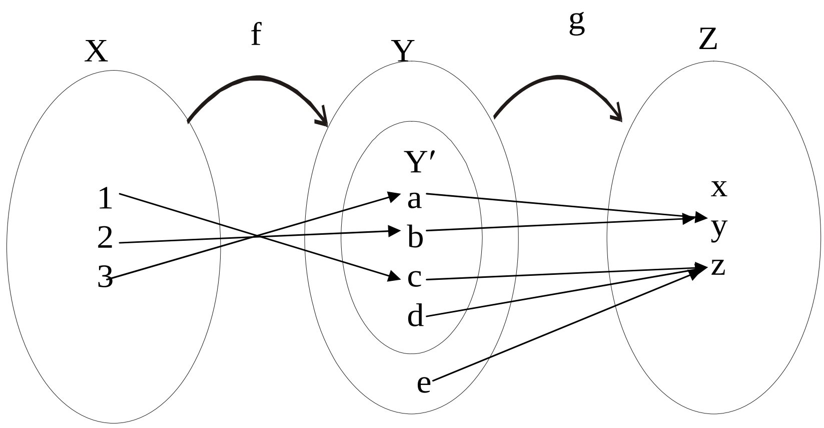
(gof)(x) = g(f(x)) for all $x \in X$

The function gof is called the composition of f and g.

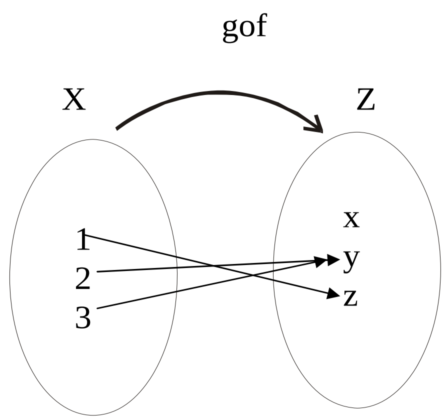


COMPOSITION OF FUNCTIONS DEFINED BY ARROW DIAGRAMS:

Let X = {1,2,3},Y′={a,b,c,d}, Y={a,b,c,d,e} and Z ={x,y,z}. Define functions f:X$\rightarrow$Y′ and g:X $\rightarrow$Z by the arrow diagrams:



Then gof f: X →Z is represented by the arrow diagram.



EXERCISE:

Let A = {1,2,3,4,5} and we define functions f:A →A and then g:A→A :
f(1)=3, f(2)=5, f(3)=3, f(4)=1, f(5)=2
g(1)=4, g(2)=1, g(3)=1, g(4)=2, g(5)=3
Find the composition functions fog and gof.

SOLUTION:

We are the definition of the composition of functions and compute:

(fog) (1) = f(g(1)) = f(4) = 1
(fog) (2) = f(g(2)) = f(1) = 3
(fog) (3) = f(g(3)) = f(1) = 3
(fog) (4) = f(g(4)) = f(2) = 5
(fog) (5) = f(g(5)) = f(3) = 3

Also

(gof) (1) = g(f(1)) = g(3) = 1
(gof) (2) = g(f(2)) = g(5) = 3
(gof) (3) = g(f(3)) = g(3) = 1
(gof) (4) = g(f(4)) = g(1) = 4
(gof) (5) = g(f(5)) = g(2) = 1

REMARK: The functions fog and gof are not equal.

COMPOSITION OF FUNCTIONS DEFINED BY FORMULAS:

Let f: Z →Z and g:Z → Z be defined by

f(n) = n+1 for n∈Z
and g(n) = n^2 for n∈Z

a. Find the compositions gof and fog.
b. Is gof = fog?

SOLUTION:

a. By definition of the composition of functions

$(gof)(n) = g(f(n)) = g(n+1) = (n+1)^2$ for all $n \in Z$ and

$(fog)(n) = f(g(n)) = f(n^2) = n^2+1$ for all $n \in Z$

b. Two functions from one set to another are equal if, and only if, they take the same values.

In this case,

$(gof)(1) = g(f(1)) = (1 + 1)^2 = 4$ where as

$(fog)(1) = f(g(1)) = 1^2 + 1 = 2$

Thus $fog \neq gof$

REMARK: The composition of functions is not a commutative operation.

COMPOSITION WITH THE IDENTITY FUNCTION:

Let X = {a,b,c,d} and Y={u,v,w} and suppose $f:X\to Y$ be defined by:

f(a) = u, f(b) = v, f(c) = v, f(d) = u

Find foi_x and $i_y of$, where i_x and i_y are identity functions on X and Y respectively.

SOLUTION:

The values of foi_x on X are obtained as:

$(foi_x)(a) = f(i_x(a)) = f(a) = u$

$(foi_x)(b) = f(i_x(b)) = f(b) = v$

$(foi_x)(c) = f(i_x(c)) = f(c) = v$

$(foi_x)(d) = f(i_x(d)) = f(d) = u$

For all elements x in X $(foi_x)(x) = f(x)$ so that $foi_x = f$

The values of $i_y of$ on X are obtained as:

$(i_y of)(a)=i_y(f(a)) = i_y(u) = u$

$(i_y of)(b)=i_y(f(b)) = i_y(v) = v$

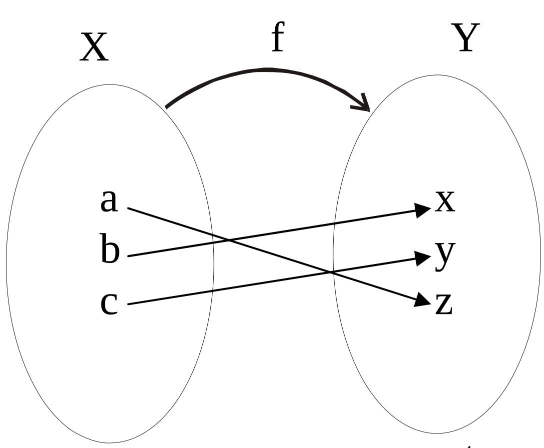
$(i_y of)(c)=i_y(f(c)) = i_y(v) = v$

$(i_y of)(d)=i_y(f(d)) = i_y(u) = u$

For all elements x in X $(i_y of)(x) = f(x)$ so that $i_y of = f$

COMPOSING A FUNCTION WITH ITS INVERSE:

Let X = {a,b,c} and Y= {x,y,z}. Define $f:X\to Y$ by the arrow diagram.

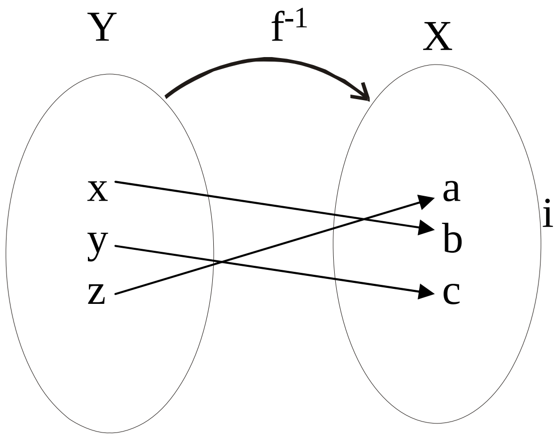


i.e. f(a) = z

f(b) = x

f(c) = y

Then f is one-to-one and onto. So f^{-1} exists and is represented by the arrow diagram Below.

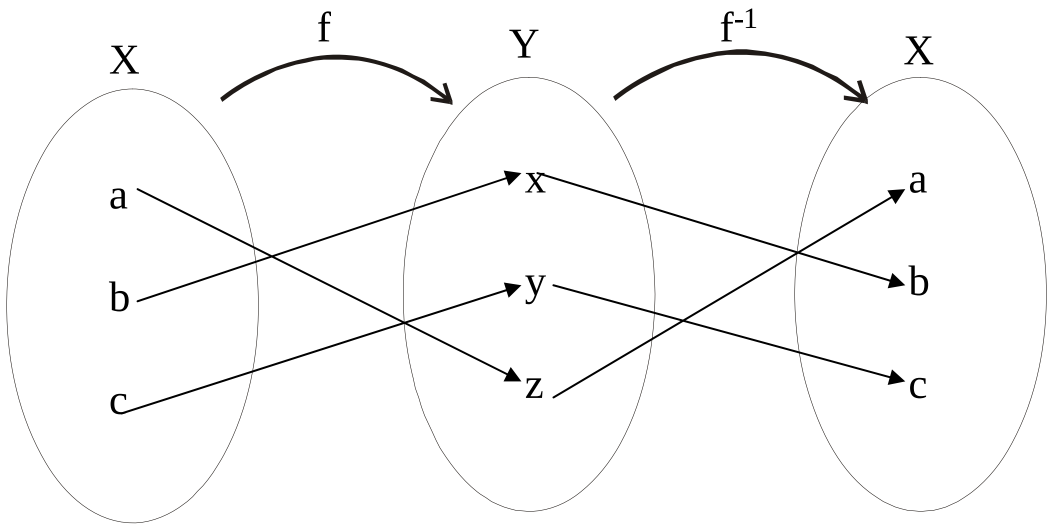


i.e. $f^{-1}(x) = b$

$f^{-1}(y) = c$

$f^{-1}(z) = a$

$f^{-1}of$ is found by following the arrows from X to Y by f and back to X by f^{-1}.



Thus, it is quite clear that
$(f^{-1}of)(a) = f^{-1}(f(a)) = f^{-1}(z) = a$
$(f^{-1}of)(b) = f^{-1}(f(b)) = f^{-1}(x) = b$ and $(f^{-1}of)(c) = f^{-1}(f(c)) = f^{-1}(y) = c$

REMARK 1:

$f^{-1}of : X \to X$ sends each element of X to itself. So by definition of identity function on X.
$f^{-1}of = i_x$
Similarly, the composition of f and f^{-1} sends each element of Y to itself. Accordingly
$$fof^{-1} = i_y$$

REMARK2:

The function $f:X \to Y$ and $g:Y \to X$ are inverses of each other iff
$$gof = i_x \text{ and } fog = i_y$$

EXERCISE:

Let $f: R \to R$ and $g:R \to R$ be defined by
$f(x) = 3x + 2$ for all $x \in R$

and $g(x) = \frac{x-2}{3}$ for all $x \in R$

Show that f and g are inverse of each other.

SOLUTION:

f and g are inverse of each other iff their composition gives the identity function. Now for all x $\in$R

$$
\begin{aligned}
(gof)(x) &= g(f(x)) \\
&= g(3x+2) \quad \text{(by definition of f)} \\
&= \frac{(3x+2)-2}{3} \quad \text{(by definition of g)} \\
&= \frac{3x}{3} = x
\end{aligned}
$$

$$
\begin{aligned}
(fog)(x) &= f(g(x)) \\
&= g(3x+2) \quad \text{(by definition of g)} \\
&= \frac{(3x+2)-2}{3} \quad \text{(by definition of f)} \\
&= (x-2)+2 \\
&= x
\end{aligned}
$$

Thus (gof)(x) = x = (fog)(x)

Hence gof and fog are identity functions. Accordingly f and g are inverse of each other.

LECTURE # 18

THEOREM:

If f and g are two one-to-one functions, then their composition that is gof is one-to-one.

PROOF:

We are taking functions f: X →Y and g: Y →Z are both one-to-one functions.

Suppose $x_1, x_2 \in X$ such that $(gof)(x_1) = (gof)(x_2)$

$\Rightarrow g(f(x_1)) = g(f(x_2))$ (definition of composition)

Since g is one-to-one, therefore

$$f(x_1) = f(x_2)$$

And since f is one-to-one, therefore

$$x_1 = x_2$$

Thus, we have shown that if

$$(gof)(x_1) = (gof)(x_2) \text{ then } x_1 = x_2$$

Hence, gof is one-to-one.

THEOREM:

If f: X→Y and g: Y → Z are both onto functions, then gof:X → Z is onto.

PROOF:

Suppose f: X →Y and g: Y →Z are both onto functions. We must show that gof: X →Z is onto.

Let z∈Z Since g:Y →Z is onto, so for z∈Z, there exists y∈Y such that g(y)=z. Further, since f: X →Y is onto, so for y∈Y, there exists x∈X such that f(x) = y.

Hence, there exists an element x in X such that (gof) (x) = g(f(x)) = g(y) = z

Thus, gof: X →Z is onto.

THEOREM:

If f: W →X, g:X →Y, and h:Y →Z are functions, then

$$(hog)of = ho(gof)$$

PROOF:

The two functions are equal if they assign the same image to each element in the domain, that is,

$$((hog)of)(x) = (ho(gof))(x) \quad \text{for every } x \in W$$

Computing

$$((hog)of)(x) = (hog)(f(x)) = h(g(f(x)))$$

and $(ho(gof))(x) = h((gof)(x)) = h(g(f(x)))$

Hence $(hog)of = ho(gof)$

REMARK: The composition of functions is associative.

EXERCISE:

Suppose f:X→Y and g:Y→Z and both of these are one-to-one and onto.

Provethat $(gof)^{-1}$ exists and that

$$(gof)^{-1} = f^{-1}og^{-1}$$

SOLUTION:

Suppose f: X →Y and g:Y →Z are bijective functions, then their composition gof: X →Z is also bijective. Hence $(gof)^{-1}: Z \to X$ exists.

Next, to establish $(gof)^{-1} = f^{-1}og^{-1}$, we show that

$(f^{-1}og^{-1})o(gof) = i_x$ and $(gof)o(f^{-1}og^{-1}) = i_z$

Now consider

$(f^{-1}og^{-1})o(gof) = f^{-1}o(g^{-1}o(gof))$ (associative law for o)

$= f^{-1}o((g^{-1}og)of)$ (associative law for o)

$= f^{-1}o(i_yof)$ ($g^{-1}og = iy$)

$= f^{-1}of$ ($i_yof = f$)

$= i_x \qquad (f:X\to Y)$

Also

$(gof)o(f^{-1}og^{-1}) = go(fo(f^{-1}og^{-1}))$ (associative law for o)

$= go((fof^{-1})og^{-1})$ (associative law for o)

$= go(i_y og^{-1}) \qquad (fof^{-1} = i_y)$

$= gog^{-1} \qquad (i_y og^{-1} = g^{-1})$

$= I_z \qquad (g:Y\to Z)$

Hence $f^{-1}og^{-1} = (gof)^{-1}$

REAL-VALUED FUNCTIONS:

Let X be any set and R be the set of real numbers.A function $f:X\to R$ that assigns to each $x\in X$ a real number $f(x) \in R$ is called a real-valued function.

If $f: R \to R$, then f is called a real-valued function of a real variable.

EXAMPLE:

1. $f: R+\to R$ defined by $f(x) = \log x$ is a real valued function.
2. $g:R \to R$ defined by $g(x) = e^x$ is a real valued function of a real variable.

OPERATIONS ON FUNCTIONS

SUM OF FUNCTIONS:

Let f and g be real valued functions with the same domain X. That is $f:X \to R$ and $g:X \to R$.

The sum of f and g denoted f+g is a real valued function with the same domain X i.e. $f+g: X \to R$ defined by

$$(f+g)(x) = f(x) + g(x) \quad \forall\ x\in X$$

EXAMPLE:

Let $f(x) = x^2 + 1$ and $g(x) = x + 2$ defines functions f and g from R to R.

Then $(f+g)(x) = f(x) + g(x)$

$= (x^2 + 1) + (x + 2)$

$= x^2 + x + 3 \quad \forall x\in R$

which defines the sum functions $f+g: X \to R$

DIFFERENCE OF FUNCTIONS:

Let $f: X \to R$ and $g:X \to R$ be real valued functions. The difference of f and g denoted by f-g which is a function from X to R defined by

$$(f-g)(x) = f(x) - g(x) \quad \forall\ x\in X$$

EXAMPLE:

Let $f(x) = x^2 + 1$ and $g(x)=x+2$ define functions f and g from R to R. Then

$(f-g)(x) = f(x) - g(x)$

$= (x^2 + 1) - (x + 2)$

$= x^2 - x - 1 \quad \forall x\in R$

which defines the difference function $f-g: X \to R$

PRODUCT OF FUNCTIONS:

Let $f: X \to R$ and $g:X \to R$ be real valued functions. The product of f and g denoted $f.g$ or simply fg is a function from X to R defined by

$$(f\,.\,g)(x) = f(x)\,.\,g(x) \quad \forall\ x\in X$$

EXAMPLE:

Let $f(x) = x^2 + 1$ and $g(x)=x+2$

define functions f and g from R to R.

Then $(f\,.\,g)(x) = f(x)\,.\,g(x)$

$= (x^2 + 1)\,.\,(x + 2)$

$= x^3 + 2x^2 + x + 2 \quad \forall x\in R$

which defines the product function f . g: X →R

QUOTIENT OF FUNCTIONS:

Let f:X→R and g: X →R be real valued functions. The quotient of f by g denoted $\frac{f}{g}$ is a function from X to R defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad g(x) \text{ is not equal to } 0$$

EXAMPLE:

Let f(x) = x^2 + 1 and g(x) = x + 2 defines functions f and g from R to R.
Then

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \forall x \in X \ \& \ g(x) \neq 0$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \forall x \in X \ \& \ g(x) \neq 0$$

which defines the quotient function : X →R.

SCALAR MULTIPLICATION:

Let f:X →R be a real valued function and c is a non-zero number. Then the scalar multiplication of f is a function c·f: R →R defined by (c·f)(x) = c·f(x) ∀x∈X

EXAMPLE:

Let f(x) = x^2 + 1 and g(x) = x+2 defines functions f and g from R to R.
Then

(3f - 2g)(x) =(3f)(x) - (2g)(x)
=3 · f(x) - 2 · g(x)
= 3(x^2+1) - 2 (x+2)
= 3x^2 - 2x-1 ∀x∈X

EXERCISE :

If f:R →R and g:R → R are both one-to-one, is f+g also one-to-one?

SOLUTION:

Here f+g is not one-to-one
As a counter example; define f:R →R and g: R →R by
f(x) = x and g(x) = -x ∀x∈R
Then obviously both f and g are one-to-one
Now
(f+g)(x) = f(x) + g(x) = x + (-x) = 0 ∀x∈R
Clearly f+g is not one-to-one because
(f+g)(1) = 0 and (f+g) (2) = 0 but 1≠2

EXERCISE:

If f:R→R and g:R →R are both onto, is f+g also onto? Prove or give a counter example.

SOLUTION:

f+g is not onto.
As a counter example, define f:R →R and g:R →R by
f(x) = x and g(x) = - x ∀x∈R
Then obviously both f and g are onto.
Now (f+g)(x)= f(x) + g(x)
= x + (-x)
= 0 ∀x∈R

Clearly f+g is not onto because only $0 \in R$ has its pre-image in R and no non-zero elementof co-domain R is the image of any element of R.

EXERCISE:

Let $f:R \to R$ be a function and $c(\neq 0) \in R$.

1. If f is one-to-one, is $c \cdot f$ also one-to-one?
2. If f is onto, is $c \cdot f$ also onto?

SOLUTION:

1. Suppose $f:R \to R$ is one-to-one and $c(\neq 0) \in R$

Let $(c \cdot f)(x_1) = (c \cdot f)(x_2)$ for x1, x2 $\in$ R

$\Rightarrow c \cdot f(x_1) = c \cdot f(x_2)$ (by definition of $c \cdot f$)

$\Rightarrow f(x_1) = f(x_2)$ (dividing by $c \neq 0$)

Since f is one-to-one, this implies

$x_1 = x_2$

Hence $c \cdot f$; $R \to R$ is also one-to-one.

2. Suppose $f:R \to R$ is onto and $(c \neq 0) \in R$.

Let $y \in R$. We search for an $x \in R$ such that

$(c \cdot f)(x) = y$ (1)

$\Rightarrow c \cdot f(x) = y$ (by definition of $c \cdot f$)

$\Rightarrow f(x) = \frac{y}{c}$ (dividing by $c \neq 0$)

Since $f: R \to R$ is onto, so for $\frac{y}{c} \in R$, there exists some $x \in R$

such that the above equation is true; and this leads back to equation (1).

Accordingly $c \cdot f: R \to R$ is also onto.

EXERCISE:

The real-valued function $0_X:X \to R$ which is defined by

$0_X(x) = 0$ for all $x \in X$

is called the zero function (on X).

Prove that for any function $f: X \to R$

1. $f + 0_X = f$ 2. $f \cdot 0_X = 0_X$

SOLUTION:

1. Since $(f + 0_X)(x) = f(x) + 0_X(x)$

$= f(x) + 0$

$= f(x) \quad \forall x \in X$

Hence $f+0_X= f$

2. Since $(f \cdot 0_X)(x) = f(x) \cdot 0_X(x)$

$= f(x) \cdot 0$

$= 0$

$= 0_X(x) \quad \forall x \in X$

Hence $f \cdot 0_X = 0_X$

EXERCISE:

Given a set S and a subset A, the characteristics function of A, denoted χ_A, is the function defined from S to the set {0,1} defined as

$$\chi_A(x) = \begin{cases} 1 \text{ if } & x \in A \\ 0 \text{ if } & x \notin A \end{cases}$$

Show that for all subsets A and B of S

1. $\chi_{A\cap B} = \chi_A \cdot \chi_B$
2. $\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$
3. $\chi_A(x) = 1 - \chi_A(x)$

SOLUTION:

1. Prove that $\chi_{A\cap B} = \chi_A \cdot \chi_B$

Let $x \in A\cap B$; therefore $x \in A$ and $x \in B$. Then

$\chi_{A\cap B}(x) = 1$; $\chi_A(x)=1$; $\chi_B(x)=1$

Hence $\chi_{A\cap B}(x) = 1 = (1)(1)=\chi_A(x)\,\chi_B(x)$

$=(\chi_A \cdot \chi_B)(x)$

SOLUTION:

Next, let $y \in (A\cap B)'$

$\Rightarrow y \in A' \cup B'$

$\Rightarrow y \in A'$ or $y \in B'$

Now $y \in (A\cap B)'$ i.e $y \notin (A\cap B)$

$\Rightarrow \chi_{(A\cap B)}(y) = 0$

and $y \in A'$ or $y \in B'$

$\Rightarrow \chi_A(y) = 0$ (as $y\notin A$) or $\chi_B(y) = 0$ (as $y\notin B$)

Thus $\chi_{A\cap B}(y) = 0 =(0)(0)= \chi_A(y)\,\chi_B(y)$

$= (\chi_A \cdot \chi_B)(y)$

Hence, $\chi_{A\cap B}$ and $\chi_A \cdot \chi_B$ assign the same number to each element x in S, so by definition

$\chi_{A\cap B} = \chi_A \cdot \chi_B$

SOLUTION:

2. Prove that $\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$

Let $x\in A\cup B$ then $x \in A$ or $x \in B$

Now $\chi_{A\cup B}(x) = 1$ and $\chi_A(x) = 1$ or $\chi_B(x) = 1$

Three cases arise depending upon which of $\chi_A(x)$ or $\chi_B(x)$ is 1.

<u>CASE-I</u> **(if $\chi_A(x)) = 1$ & $\chi_B(x) = 1)$**

Now $\chi_A(x) + \chi_B(x) - \chi_A(x) \cdot \chi_B(x)$

$= 1 + 1 - (1)(1)$

$= 1= \chi_{A\cup B}(x)$

<u>CASE-II</u> **(if $\chi_A(x) = 1$; $\chi_B(x) = 0$)**

Now $\chi_A(x) + \chi_B(x) - \chi_A(x)\cdot \chi_B(x)$

$= 1 + 0 - (1)(0)$

$= 1$

$= \chi_{A\cup B}(x)$

<u>CASE III</u> **(if $\chi_A(x) = 0$; $\chi_B(x) = 1$)**

Now $\chi_A(x) + \chi_B(x) - \chi_A(x)\cdot \chi_B(x)$

$= 0 + 1 + (0)(1)$

$= 1$

$= \chi_{A\cup B}(x)$

Thus in all cases

$\chi_{A\cup B}(x) = 1 = \chi_A(x) + \chi_B(x) - \chi_A(x)\cdot \chi_B(x) \quad \forall x\in A\cup B$

Next let $y\notin A\cup B$. Then $y\in (A\cup B)'$

$\Rightarrow$ $y \in A' \cap B'$ (DeMorgan's Law)
$\Rightarrow$ $y \in A'$ and $y \in B'$
$\Rightarrow$ $y \notin A$ and $y \notin B$
Thus $\chi_{A \cup B}(y) = 0$; $\chi_A(y) = 0$; $\chi_B(y) = 0$

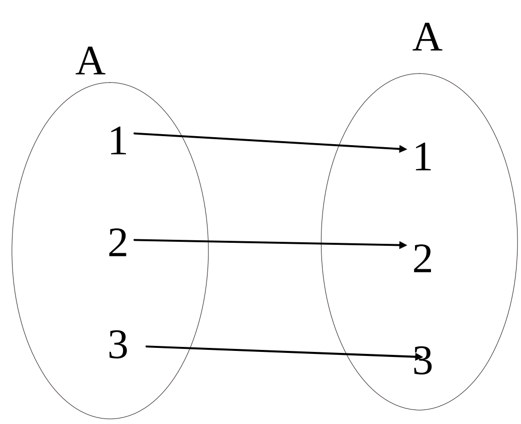
Consider $\chi_A(y) + \chi_B(y) - \chi_A(y) \cdot \chi_B(y)$
$= 0 + 0 - 0$
$= 0$
$= \chi_{A \cup B}(y)$
Hence for all elements of S
$\chi_{A \cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$

3. Prove that $\chi_{\bar{A}}(x) = 1 - \chi_A(x)$
Let $x \in \bar{A}$ .Then $x \notin A$ and so
$\chi_{\bar{A}}(x) = 1$ and $\chi_A(x) = 0$
$\therefore$ $\chi_{\bar{A}}(x) = 1 = 1 - 0 = 1 - \chi_A(x)$ (1)

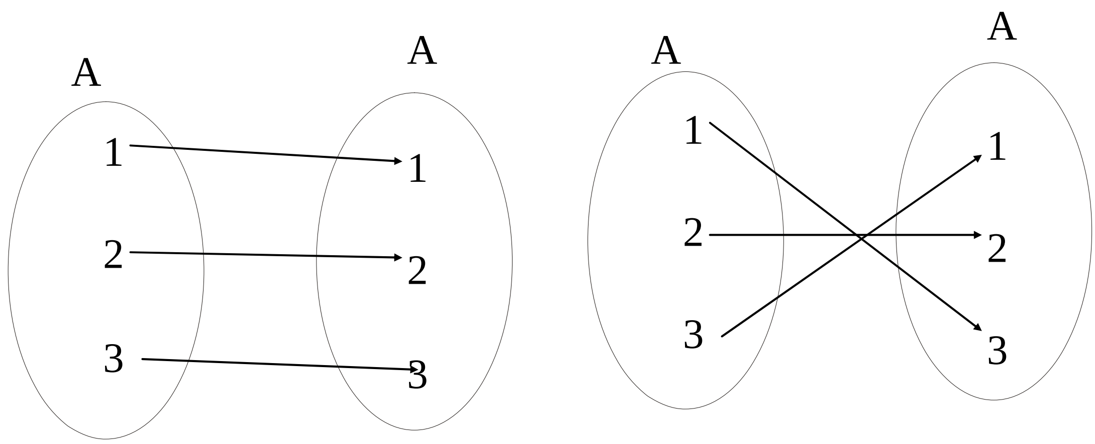
Also if $y \in A$, then $y \notin \bar{A}$ and so
$\chi_A(y) = 1$ and $\chi_{\bar{A}}(y) = 0$
$\therefore$ $\chi_{\bar{A}}(y) = 0 = 1 - 1 = 1 - \chi_A(y)$ (2)
By (1) and (2), for all elements of S
$\chi_A(x) = 1 - \chi_A(x)$

EXERCISE:
If F, G and H are functions from A = {1,2,3} to A what must be true if.
1. F is reflexive?
2. G is symmetric?
3. H is transitive, onto function?

SOLUTION:
1. F is reflexive iff every element of A is related to itself i.e.aFa $\forall a \in A$. Also F is a function from A to A, so each element of A is related to a unique (one and only one) element of A. Hence, F maps each element of A to itself so that F is an identity function.

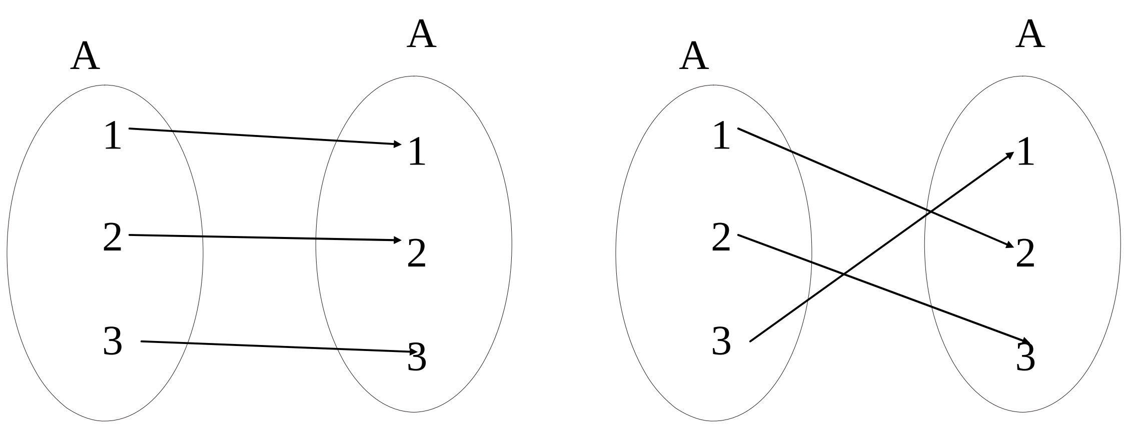


2. G is symmetric iff if aGb then bGa $\forall a,b \in A$ .Now, in the present case.



i.e. G is both one-to-one and onto (a bijective function)

3. H is transitive iff if aHb and bHc then aHc. $\forall a,b,c \in A$.
In our case



H is transitive, onto function if and only if it is an identity function.

FINITE AND INFINITE SETS

<u>**FINITE SET**</u>:

A set is called finite if, and only if, it is the empty set or there is one-to-one correspondence from {1,2,3,…,n} to it, where n is a positive integer.

<u>**INFINITE SET:**</u>

A non empty set that cannot be put into one-to-one correspondence with {1,2,3,…,n}, for any positive integer n, is called infinite set.

<u>**CARDINALITY:**</u>

Let A and B be any sets. A has the same cardinality as B if, and only if, there is a one-to-one correspondence from A to B(Cardinality means “the total number of elements in a set) .
Note:One-to-One correspondence means the condition of One-One and Onto.

<u>**COUNTABLE SET:**</u>

A set is **countably infinite** if, and only if, it has the same cardinality as the set of positive integers Z+.
A set is called **countable** if, and only if, it is finite or countably infinite.
A set that is not countable is called **uncountable**.

EXAMPLE:
The set Z of all integers is countable.
SOLUTION:
We find a function from the set of positive integers Z+ to Z that is one-to-one and onto.
Define f: Z+ $\to$Z by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if n is an even positive integer} \\ -\frac{n-1}{2} & \text{if n is an odd positive integer} \end{cases}$$

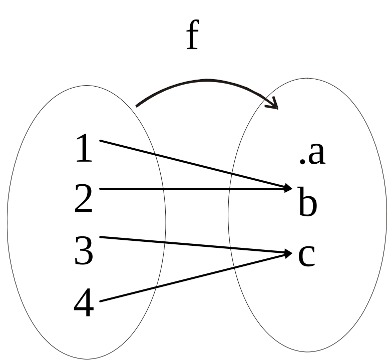
Then f clearly maps distinct elements of Z+ to distinct integers. Moreover, every integer m is the image of some positive integer under f. Thus f is bijective and so the set Z of all integers is countable (countably infinite).
EXERCISE:
Show that the set 2Z of all even integers is countable.
SOLUTION:
Consider the function h from Z to 2Z defined as follows

$$h(n) = 2n \text{ for all } n\in Z$$

Then clearly h is one-to-one. For if

$$h(n_1) = h(n_2) \text{ then}$$
$$2n_1 = 2n_2 \quad \text{(by definition of h)}$$
$$\Rightarrow \quad n_1 = n_2$$

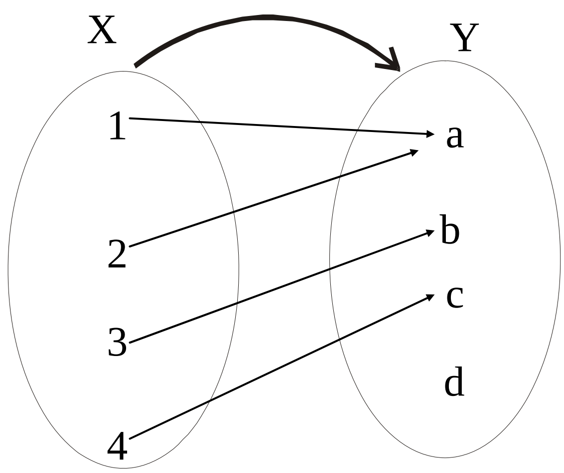
Also every even integer 2n is the image of integer n under h. Hence h is onto as well. Thus h:Z $\to$2Z is bijective. Since Z is countable, it follows that 2Z is countable.
IMAGE OF A SET:
Let f : X $\to$Y be a function and A $\subseteq$ X.
The image of A under f is denoted and defined as:
f(A) = {y$\in$Y | y = f(x), for some x in A}
EXAMPLE:
Let f: X $\to$Y be defined by the arrow diagram



Let A = {1,2}and B = {2,3} then
f(A)={b} and f(B) = {b,c}

INVERSE IMAGE OF A SET:
Let f: X $\to$Y be a function and C $\subseteq$ Y.
The inverse image of C under f is denoted and defined as:
f^{-1}(C)={x $\in$X | f(x) $\in$C}
EXAMPLE:

Let f: X →Y be defined by the arrow diagram.



Let C = {a},D = {b,c},E = {d} then f^{-1}(C)={1,2},
f^{-1}(D) = {3,4}, and f^{-1}(E) =∅

SOME RESULTS:

Let f: X →Y is a function. Let A and B be subsets of X and C and D be subsets of Y.

1. if A⊆ B then f(A) ⊆ f(B)
2. f(A∪B) = f(A) ∪f(B)
3. f(A∩B) ⊆ f(A) ∩ f(B)
4. f(A-B) ⊇ f(A) - f(B)
5. if C ⊆ D, then f^{-1}(C) ⊆ f^{-1}(D)
6. f^{-1}(C∪D) = f^{-1}(C) ∪ f^{-1}(D)
7. f^{-1}(C∩D) = f^{-1}(C) ∩f^{-1}(D)
8. f^{-1}(C-D) = f^{-1}(C) - f^{-1}(D)

LECTURE # 19

SEQUENCE:
A sequence is just a list of elements usually written in a row.

EXAMPLES:
1. 1, 2, 3, 4, 5, …
2. 4, 8, 12, 16, 20,…
3. 2, 4, 8, 16, 32, …
4. 1, 1/2, 1/3, 1/4, 1/5, …
5. 1, 4, 9, 16, 25, …
6. 1, -1, 1, -1, 1, -1, …

NOTE:
The symbol "…" is called ellipsis, and reads "and so forth"

FORMAL DEFINITION:
A sequence is a function whose domain is the set of integers greater than or equal to a particular integer n_0.
Usually this set is the set of Natural numbers {1, 2, 3, …} or the set of whole numbers {0, 1, 2, 3, …}.

NOTATION:
We use the notation a_n to denote the image of the integer n, and call it a term of the sequence. Thus

$$a_1, a_2, a_3, a_4 \ldots, a_n, \ldots$$

represent the terms of a sequence defined on the set of natural numbers N.
Note that a sequence is described by listing the terms of the sequence in order of increasing subscripts.

FINDING TERMS OF A SEQUENCE GIVEN BY AN EXPLICIT FORMULA:
An explicit formula or general formula for a sequence is a rule that shows how the values of a_k depends on k.

EXAMPLE:
Define a sequence $a_1, a_2, a_3, \ldots$ by the explicit formula

$$a_k = \frac{k}{k+1} \qquad \text{for all integers } k \geq 1$$

The first four terms of the sequence are:

$$a_1 = \frac{1}{1+1} = \frac{1}{2}, a_2 = \frac{2}{2+1} = \frac{2}{3}, a_3 = \frac{3}{3+1} = \frac{3}{4}$$

$$and \quad fourth\ term\ is\ a_4 = \frac{4}{4+1} = \frac{4}{5}$$

EXAMPLE:
Write the first four terms of the sequence defined by the formula

$bj = 1 + 2^j$, for all integers $j \geq 0$

SOLUTION:
$b_0 = 1 + 2^0 = 1 + 1 = 2$
$b_1 = 1 + 2^1 = 1 + 2 = 3$
$b_2 = 1 + 2^2 = 1 + 4 = 5$
$b_3 = 1 + 2^3 = 1 + 8 = 9$

REMARK:
The formula $bj = 1 + 2^j$, for all integers $j \geq 0$ defines an infinite sequence having infinite number of values.

EXERCISE:

Compute the first six terms of the sequence defined by the formula $C_n = 1 + (-1)^n$ for all integers $n \geq 0$

SOLUTION :

$C_0 = 1 + (-1)^0 = 1 + 1 = 2$ $\quad$ $C_1 = 1 + (-1)^1 = 1 + (-1) = 0$

$C_2 = 1 + (-1)^2 = 1 + 1 = 2$ $\quad$ $C_3 = 1 + (-1)^3 = 1 + (-1) = 0$

$C_4 = 1 + (-1)^4 = 1 + 1 = 2$ $\quad$ $C_5 = 1 + (-1)^5 = 1 + (-1) = 0$

REMARK:

(1)If n is even, then $C_n = 2$ and if n is odd, then $C_n = 0$

Hence, the sequence oscillates endlessly between 2 and 0.

(2)An infinite sequence may have only a finite number of values.

EXAMPLE:

Write the first four terms of the sequence defined by

$$C_n = \frac{(-1)^n n}{n+1} \quad \text{for all integers } n \geq 1$$

SOLUTION:

$$C_1 = \frac{(-1)^1(1)}{1+1} = \frac{-1}{2}, C_2 = \frac{(-1)^2(2)}{2+1} = \frac{2}{3}, C_3 = \frac{(-1)^3(3)}{3+1} = \frac{-3}{4}$$

$$And \quad fourth \ \ term \ isC_4 = \frac{(-1)^4(4)}{4+1} = \frac{4}{5}$$

REMARK:A sequence whose terms alternate in sign is called an alternating sequence.

EXERCISE:

Find explicit formulas for sequences with the initial terms given:

1. 0, 1, -2, 3, -4, 5, …

SOLUTION:

$a_n = (-1)^{n+1} n$ for all integers $n \geq 0$

2. $1 - \frac{1}{2}, \frac{1}{2} - \frac{1}{3}, \frac{1}{3} - \frac{1}{4}, \frac{1}{4} - \frac{1}{5}, \ldots$

SOLUTION:

$$b_k = \frac{1}{k} - \frac{1}{k+1} \quad \text{for all integers } n \geq 1$$

3. 2, 6, 12, 20, 30, 42, 56, …

SOLUTION:

$C_n = n\,(n + 1)$ for all integers $n \geq 1$

4. 1/4, 2/9, 3/16, 4/25, 5/36, 6/49, …

SOLUTION:

$$d_i = \frac{i}{(i+1)^2} \quad \text{for all integers} \quad i \geq 1$$

OR

$$d_j = \frac{j+1}{(j+2)^2} \quad \text{for all integers} \quad j \geq 0$$

ARITHMETIC SEQUENCE:

A sequence in which every term after the first is obtained from the preceding term by adding a constant number is called an arithmetic sequence or arithmetic progression (A.P.)
The constant number, being the difference of any two consecutive terms is called the common difference of A.P., commonly denoted by "d".

EXAMPLES:

1. 5, 9, 13, 17, … (common difference = 4)
2. 0, -5, -10, -15, … (common difference = -5)
3. x + a, x + 3a, x + 5a, … (common difference = 2a)

GENERAL TERM OF AN ARITHMETIC SEQUENCE:

Let **a** be the first term and **d** be the common difference of an arithmetic sequence. Then the sequence is a, a+d, a+2d, a+3d, …

If a_i, for $i \geq 1$, represents the terms of the sequence then

a_1 = first term = a = a + (1-1) d

a_2 = second term = a + d = a + (2-1) d

a_3 = third term = a + 2d = a + (3 -1) d

By symmetry

a_n = nth term = a + (n - 1)d for all integers $n \geq 1$.

EXAMPLE:

Find the 20th term of the arithmetic sequence

3, 9, 15, 21, …

SOLUTION:

Here a = first term = 3

d = common difference = 9 - 3 = 6

n = term number = 20

a_{20} = value of 20th term = ?

Since $a_n = a + (n - 1) d; \quad n \geq 1$

$\therefore \quad a_{20} = 3 + (20 - 1) 6$

$= 3 + 114$

$= 117$

EXAMPLE:

Which term of the arithmetic sequence

4, 1, -2, …, is -77

SOLUTION:

Here a = first term = 4

d = common difference = 1 - 4 = -3

a_n = value of nth term = - 77

n = term number = ?

Since

$an = a + (n - 1) d \quad n \geq 1$

$\Rightarrow \quad - 77 = 4 + (n - 1) (-3)$

$\Rightarrow \quad - 77 - 4 = (n - 1) (-3)$

OR

$$\frac{-81}{-3} = n - 1$$

OR

27 = n – 1

n = 28

Hence –77 is the 28th term of the given sequence.

EXERCISE:

Find the 36th term of the arithmetic sequence whose 3rd term is 7 and 8th term is 17.

SOLUTION:

Let **a** be the first term and **d** be the common difference of the arithmetic sequence.

Then

$a_n = a + (n - 1)d \qquad n \geq 1$

$\Rightarrow \quad a_3 = a + (3 - 1) d$

and $\quad a_8 = a + (8 - 1) d$

Given that $a_3 = 7$ and $a_8 = 17$. Therefore

$7 = a + 2d$……………………(1)

and $\quad 17 = a + 7d$……………………..(2)

Subtracting (1) from (2), we get,

$10 = 5d$

$\Rightarrow \quad d = 2$

Substituting d = 2 in (1) we have

$7 = a + 2(2)$

which gives a = 3

Thus, $\quad a_n = a + (n - 1) d$

$a_n = 3 + (n - 1) 2$ (using values of a and d)

Hence the value of 36th term is

$a_{36} = 3 + (36 - 1) 2$

$= 3 + 70$

$= 73$

GEOMETRIC SEQUENCE:

A sequence in which every term after the first is obtained from the preceding term by multiplying it with a constant number is called a geometric sequence or geometric progression (G.P.)

The constant number, being the ratio of any two consecutive terms is called the common ratio of the G.P. commonly denoted by "r".

EXAMPLE:

1. 1, 2, 4, 8, 16, … (common ratio = 2)
2. 3, - 3/2, 3/4, - 3/8, … (common ratio = - 1/2)
3. 0.1, 0.01, 0.001, 0.0001, … (common ratio = 0.1 = 1/10)

GENERAL TERM OF A GEOMETRIC SEQUENCE:

Let **a** be the first tem and **r** be the common ratio of a geometric sequence. Then the sequence is $a, ar, ar^2, ar^3, \ldots$

If a_i, for $i \geq 1$ represent the terms of the sequence, then

a_1 = first term = $a = ar^{1-1}$

a_2 = second term = $ar = ar^{2-1}$

a_3 = third term = $ar^2 = ar^{3-1}$

………………

………………

a_n = nth term = ar^{n-1}; for all integers $n \geq 1$

EXAMPLE:

Find the 8th term of the following geometric sequence

4, 12, 36, 108, …

SOLUTION:

Here a = first term = 4

r = common ratio = $\frac{12}{4} = 3$

n = term number = 8

a_8 = value of 8th term = ?

Since $\quad a_n = ar^{n-1}; \qquad n \geq 1$

$\Rightarrow \quad a_8 = (4)(3)^{8-1}$

= 4 (2187)
= 8748

EXAMPLE:

Which term of the geometric sequence is 1/8 if the first term is 4 and common ratio ½

SOLUTION:

Given a = first term = 4
r = common ratio = 1/2
a_n = value of the nth term = 1/8
n = term number = ?

Since $a_n = ar^{n-1}$ $n \geq 1$

$$\Rightarrow \quad \frac{1}{8} = 4\left(\frac{1}{2}\right)^{n-1}$$

$$\Rightarrow \quad \frac{1}{32} = \left(\frac{1}{2}\right)^{n-1}$$

$$\Rightarrow \quad \left(\frac{1}{2}\right)^{5} = \left(\frac{1}{2}\right)^{n-1}$$

$$\Rightarrow \quad n-1 = 5 \quad \Rightarrow n = 6$$

Hence 1/8 is the 6th term of the given G.P.

EXERCISE:

Write the geometric sequence with positive terms whose second term is 9 and fourth term is 1.

SOLUTION:

Let **a** be the first term and **r** be the common ratio of the geometric sequence. Then

$a_n = ar^{n-1}$ $n \geq 1$

Now $a_2 = ar^{2-1}$

$\Rightarrow$ 9 = ar…………………(1)

Also $a_4 = ar^{4-1}$

1 = ar^3 ………………(2)

Dividing (2) by (1), we get,

$$\frac{1}{9} = \frac{ar^3}{ar}$$

$$\Rightarrow \quad \frac{1}{9} = r^2$$

$$\Rightarrow \quad r = \frac{1}{3} \qquad \left(\text{rejecting } r = -\frac{1}{3}\right)$$

Substituting r = 1/3 in (1), we get

$$9 = a\left(\frac{1}{3}\right)$$

$$\Rightarrow \quad a = 9 \times 3 = 27$$

Hence the geometric sequence is

27, 9, 3, 1, 1/3, 1/9, …

SEQUENCES IN COMPUTER PROGRAMMING:

An important data type in computer programming consists of finite sequences known as one-dimensional arrays; a single variable in which a sequence of variables may be stored.

EXAMPLE:

The names of k students in a class may be represented by an array of k elements “name” as:

name [0], name[1], name[2], …, name[k-1]

LECTURE # 20

SERIES:

The sum of the terms of a sequence forms a series. If $a_1, a_2, a_3, \ldots$ represent a sequence of numbers, then the corresponding series is

$$a_1 + a_2 + a_3 + \ldots = \sum_{k=1}^{\infty} a_k$$

SUMMATION NOTATION

The capital Greek letter sigma $\sum$ is used to write a sum in a short hand notation.
where k varies from 1 to n represents the sum given in expanded form by
$= a_1 + a_2 + a_3 + \ldots + a_n$
More generally if m and n are integers and $m \le n$, then the summation from k equal m to n of a_k is

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

Here **k** is called the index of the summation; **m** the lower limit of the summation and **n** the upper limit of the summation.

COMPUTING SUMMATIONS:

Let $a_0 = 2$, $a_1 = 3$, $a_2 = -2$, $a_3 = 1$ and $a_4 = 0$.Compute each of the summations:

1. $\sum_{i=0}^{4} a_i$ 2. $\sum_{j=0}^{2} a_{2j}$ $\sum_{k=1}^{1} a_k$

SOLUTION:

1. $\sum_{i=0}^{4} a_i = a_0 + a_1 + a_2 + a_3 + a_4$
$= 2 + 3 + (-2) + 1 + 0 = 4$

2 $\sum_{j=0}^{2} a_{2j} = a_0 + a_2 + a_4$
$= 2 + (-2) + 0 = 0$

3. $\sum_{k=1}^{1} a_k = a_1$
$= 3$

EXERCISE:

Compute the summations

1. $\sum_{i=1}^{3} (2i-1) = [2(1)-1]+[2(2)-1]+[2(3)-1]$
$= 1+3+5$
$= 9$

2. $\sum_{k=-1}^{1} (k^3+2) = [(-1)^3+2]+[(0)^3+2]+[(1)^3+2]$
$= [-1+2]+[0+2]+[1+2]$
$= 1+2+3$
$= 6$

SUMMATION NOTATION TO EXPANDED FORM:

Write the summation $\sum_{i=0}^{n} \frac{(-1)^i}{i+1}$ to expanded form:

SOLUTION:

$$\sum_{i=0}^{n} \frac{(-1)^i}{i+1} = \frac{(-1)^0}{0+1} + \frac{(-1)^1}{1+1} + \frac{(-1)^2}{2+1} + \frac{(-1)^3}{3+1} + \cdots + \frac{(-1)^n}{n+1}$$

$$= \frac{1}{1} + \frac{(-1)}{2} + \frac{1}{3} + \frac{(-1)}{4} + \cdots + \frac{(-1)^n}{n+1}$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1}$$

EXPANDED FORM TO SUMMATION NOTATION:

Write the following using summation notation:

1. $$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n}$$

SOLUTION:

We find the kth term of the series.

The numerators forms an arithmetic sequence 1, 2, 3,…, n+1, in which

a = first term = 1

& d = common difference = 1

a_k = a + (k - 1)d

= 1 + (k - 1) (1) = 1 + k - 1 = k

Similarly, the denominators forms an arithmetic sequence
n, n+1, n+2, …, 2n, in which

a = first term = n

d = common difference = 1

∴ a_k = a + (k - 1) d

= n + (k - 1) (1)

= k + n - 1

Hence the kth term of the series is

$$\frac{k}{(n-1)+k}$$

And the expression for the series is given by

$$\therefore \quad \frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n} = \sum_{k=1}^{n+1} \frac{k}{(n-1)+k}$$

$$= \sum_{k=0}^{n} \frac{k+1}{n+k}$$

TRANSFORMING A SUM BY A CHANGE OF VARIABLE:

Consider $$\sum_{k=1}^{3} k^2 = 1^2 + 2^2 + 3^2$$

and $$\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2$$

Hence $$\sum_{k=1}^{3} k^2 = \sum_{i=1}^{3} i^2$$

The index of a summation can be replaced by any other symbol. The index of a summation is therefore called a dummy variable.

EXERCISE:

Consider $$\sum_{k=1}^{n+1} \frac{k}{(n-1)+k}$$

Substituting k = j + 1 so that j = k – 1
When k = 1, j = k - 1 = 1 - 1 = 0
When k = n + 1, j = k - 1 = (n + 1) - 1 = n
Hence

$$\sum_{k=1}^{n+1} \frac{k}{(n-1)+k} = \sum_{j=0}^{n} \frac{j+1}{(n-1)+(j+1)}$$

$$= \sum_{j=0}^{n} \frac{j+1}{n+j} = \sum_{k=0}^{n} \frac{k+1}{n+k} \quad \text{(changing variable)}$$

Transform by making the change of variable j = i - 1, in the summation

$$\sum_{i=1}^{n-1} \frac{i}{(n-i)^2} \qquad **$$

PROPERTIES OF SUMMATIONS:

1. $\sum_{k=m}^{n} (a_k + b_k) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k; \quad a_k, b_k \in R$
2. $\sum_{k=m}^{n} ca_k = c\sum_{k=m}^{n} a_k \quad c \in R$
3. $\sum_{k=a-i}^{b-i} (k+i) = \sum_{k=a}^{b} k \quad i \in N$
4. $\sum_{k=a+i}^{b+i} (k-i) = \sum_{k=a}^{b} k \quad i \in N$
5. $\sum_{k=1}^{n} c = c + c + \cdots + c = nc$

EXERCISE:

Express the following summation more simply:

$$3\sum_{k=1}^{n} (2k-3) + \sum_{k=1}^{n} (4-5k)$$

SOLUTION:

$$3\sum_{k=1}^{n}(2k-3)+\sum_{k=1}^{n}(4-5k)$$

$$=3\sum_{k=1}^{n}3(2k-3)+\sum_{k=1}^{n}(4-5k)$$

$$=\sum_{k=1}^{n}[3(2k-3)+(4-5k)]$$

$$=\sum_{k=1}^{n}(k-5)$$

$$=\sum_{k=1}^{n}k-\sum_{k=1}^{n}5$$

$$=\sum_{k=1}^{n}k-5n$$

ARITHMETIC SERIES:

The sum of the terms of an arithmetic sequence forms an arithmetic series (A.S). For example

$$1 + 3 + 5 + 7 + \ldots$$

is an arithmetic series of positive odd integers.

In general, if a is the first term and d the common difference of an arithmetic series, then the series is given as: a + (a+d) + (a+2d) +…

SUM OF n TERMS OF AN ARITHMETIC SERIES:

Let a be the first term and d be the common difference of an arithmetic series. Then its nth term is:

$$a_n = a + (n - 1)d; \quad n \geq 1$$

If S_n denotes the sum of first n terms of the A.S, then

$$S_n = a + (a + d) + (a + 2d) + \ldots + [a + (n-1) d]$$
$$= a + (a+d) + (a + 2d) + \ldots + a_n$$
$$= a + (a+d) + (a + 2d) + \ldots + (a_n - d) + a_n \ldots\ldots\ldots(1)$$

where $a_n = a + (n - 1) d$

Rewriting the terms in the series in reverse order,

$$Sn = a_n + (a_n - d) + (a_n - 2d) + \ldots + (a + d) + a \ldots\ldots\ldots.(2)$$

Adding (1) and (2) term by term, gives

$$2 S_n = (a + a_n) + (a + a_n) + (a + a_n) + \ldots + (a + a_n) \quad \text{(n terms)}$$
$$2 S_n = n (a + a_n)$$
$$\Rightarrow \quad S_n = n(a + a_n)/2$$
$$S_n = n(a + l)/2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3)$$

Where $l = an = a + (n - 1)d$

Therefore

$$S_n= n/2 [a + a + (n - 1) d]$$
$$S_n \; n/2 [2 a + (n - 1) d]\ldots\ldots\ldots.(4)$$

EXERCISE:

Find the sum of first n natural numbers.

SOLUTION:

Let $S_n = 1 + 2 + 3 + \ldots + n$

Clearly the right hand side forms an arithmetic series with

a = 1, d = 2 - 1 = 1 and n = n

$$\therefore \quad S_n = \frac{n}{2}\left[2a+(n-1)d\right]$$
$$= \frac{n}{2}\left[2(1)+(n-1)(1)\right]$$
$$= \frac{n}{2}\left[2+n-1\right]$$
$$= \frac{n(n+1)}{2}$$

EXERCISE:

Find the sum of all two digit positive integers which are neither divisible by 5 nor by 2.

SOLUTION:

The series to be summed is:

11 + 13 + 17 + 19 + 21 + 23 + 27 + 29 + … + 91 + 93 + 97 + 99

which is not an arithmetic series.

If we make group of four terms we get

(11 + 13 + 17 + 19) + (21 + 23 + 27 + 29) + (31 + 33 + 37 + 39) + … + (91 + 93 + 97 + 99) = 60 + 100 + 140 + … + 380

which now forms an arithmetic series in which

$a = 60$; $d = 100 - 60 = 40$ and $l = a_n = 380$

To find n, we use the formula

$$a_n = a + (n - 1)d$$
$$\Rightarrow 380 = 60 + (n - 1)(40)$$
$$\Rightarrow 380 - 60 = (n - 1)(40)$$
$$\Rightarrow 320 = (n - 1)(40)$$
$$\frac{320}{40} = n-1$$
$$8 = n - 1$$
$$\Rightarrow n = 9$$

Now

$$S_n = \frac{n}{2}(a+l)$$
$$\therefore \quad S_9 = \frac{9}{2}(60+380) = 1980$$

GEOMETRIC SERIES:

The sum of the terms of a geometric sequence forms a geometric series (G.S.). For example

1 + 2 + 4 + 8 + 16 + …

is geometric series.

In general, if **a** is the first term and **r** the common ratio of a geometric series, then the series is given as: $a + ar + ar^2 + ar^3 + \ldots$

SUM OF n TERMS OF A GEOMETRIC SERIES:

Let **a** be the first term and **r** be the common ratio of a geometric series. Then its nth term is:

$$a_n = ar^{n-1}; \quad n \geq 1$$

If Sn denotes the sum of first n terms of the G.S. then

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1} \ldots\ldots\ldots\ldots\ldots(1)$$

Multiplying both sides by r we get.

$$r S_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Subtracting (2) from (1) we get

$$S_n - rS_n = a - ar^n$$

$$\Rightarrow \quad (1 - r)\, S_n = a\,(1 - r^n)$$

$$\Rightarrow \quad S_n = \frac{a(1-r^n)}{1-r} \qquad (r \neq 1)$$

EXERCISE:

Find the sum of the geometric series

$$6-2+\frac{2}{3}-\frac{2}{9}+\cdots+ \text{ to 10 terms}$$

SOLUTION:

In the given geometric series

$$a=6,\quad r=\frac{-2}{6}=-\frac{1}{3} \quad \text{and } n=10$$

$$\therefore \quad S_n = \frac{a(1-r^n)}{1-r}$$

$$S_{10} = \frac{6\left(1-\left(-\frac{1}{3}\right)^{10}\right)}{1-\left(-\frac{1}{3}\right)} = \frac{6\left(1+\frac{1}{3^{10}}\right)}{\left(\frac{4}{3}\right)}$$

$$= \frac{9\left(1+\frac{1}{3^{10}}\right)}{2}$$

INFINITE GEOMETRIC SERIES:

Consider the infinite geometric series

$$a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$$

then

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r} \qquad (r \neq 1)$$

If $S_n \to S$ as $n \to \infty$, then the series is convergent and S is its sum.

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$

$$\therefore \quad S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$$

$$= \frac{a}{1-r}$$

If S_n increases indefinitely as n becomes very large then the series is said to be divergent.

EXERCISE:

Find the sum of the infinite geometric series:

$$\frac{9}{4}+\frac{3}{2}+1+\frac{2}{3}+\cdots$$

SOLUTION:

Here we have

$$a = \frac{9}{4}, \quad r = \frac{3/2}{9/4} = \frac{2}{3}$$

Note that |r| < 1 So we can use the above formula.

$$\therefore \quad S = \frac{a}{1-r}$$
$$= \frac{9/4}{1-2/3}$$
$$= \frac{9/4}{1/3} = \frac{9}{4} \times \frac{3}{1} = \frac{27}{4}$$

EXERCISE:

Find a common fraction for the recurring decimal 0.81

SOLUTION:

0.81= 0.8181818181 …
= 0.81 + 0.0081 + 0.000081 + …

which is an infinite geometric series with

$$a = 0.81, \quad r = \frac{0.0081}{0.81} = 0.01$$

$$\therefore \quad \text{Sum} = \frac{a}{1-r}$$
$$= \frac{0.81}{1-0.01} = \frac{0.81}{0.99}$$
$$= \frac{81}{99} = \frac{9}{11}$$

IMPORTANT SUMS:

1. $$1 + 2 + 3 + \cdots + n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$
2. $$1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$
3. $$1^3 + 2^3 + 3^3 + \cdots + n^3 = \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)}{4} = \left[\frac{n(n+1)}{2}\right]^2$$

EXERCISE:

Sum to n terms the series 1·5+5 ·11+9 ·17+…

SOLUTION:

Let T_k denote the kth term of the given series.

Then T_k= [1+(k-1)4] [5+(k-1)6]
= (4k-3)(6k-1)
= $24k^2$ - 22k + 3

Now $S_k = T_1 + T_2 + T_3 + \ldots + T_n$

$$= \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n} (24k^2 - 22k + 3)$$

$$= 24\sum_{k=1}^{n} k^2 - 22\sum_{k=1}^{n} k + \sum_{k=1}^{n} 3$$

$$= 24\left(\frac{n(n+1)(2n+1)}{6}\right) - 22\left(\frac{n(n+1)}{2}\right) + 3n$$

$$= n[(8n^2 + 12n + 4) - (11n + 11) + 3]$$