LECTURE # 21

First of all instead of giving the definition of Recursion we give you an example, you already know the Set of Odd numbers Here we give the new definition of the same set that is the set of Odd numbers.

Definition for odd positive integers may be given as:

BASE:

1 is an odd positive integer.

RECURSION:

If k is an odd positive integer, then k + 2 is an odd positive integer.

Now, 1 is an odd positive integer by the definition base.
With k = 1, 1 + 2 = 3, so 3 is an odd positive integer.
With k = 3, 3 + 2 = 5, so 5 is an odd positive integer
and so, 7, 9, 11, … are odd positive integers.

REMARK: Recursive definitions can be used in a "generative" manner.

RECURSION:

The process of defining an object in terms of smaller versions of itself is called recursion.
A recursive definition has two parts:

1.**BASE:**

An initial simple definition which **cannot** be expressed in terms of smaller versions of itself.

2. **RECURSION:**

The part of definition which **can** be expressed in terms of smaller versions of itself.

RECURSIVELY DEFINED FUNCTIONS:

A function is said to be recursively defined if the function refers to itself such that

1. There are certain arguments, called base values, for which the function does not refer to itself.
2. Each time the function does refer to itself, the argument of the function must be closer to a base value.

EXAMPLE:

Suppose that f is defined recursively by

$$f(0) = 3$$
$$f(n + 1) = 2 f (n) + 3$$

Find f(1), f(2), f(3) and f(4)

SOLUTION:

From the recursive definition it follows that

$$f(1) = 2 f(0) + 3 = 2(3) + 3 = 6 + 3 = 9$$

In evaluating of f(1) we use the formula given in the example and we note that it involves f(0) and we are also given the value of that which we use to find out the functional value at 1.
Similarly we will use the preceding value
In evaluating the next values of the functions as we did below.

$$f(2) = 2 f(1) + 3 = 2(9) + 3 = 18 + 3 = 21$$
$$f(3) = 2 f(2) + 3 = 2(21) + 3 = 42 + 3 = 45$$
$$f(4) = 2 f(3) + 3 = 2(45) + 3 = 90 + 3 = 93$$

EXERCISE:

Find f(2), f(3), and f(4) if f is defined recursively by

$$f(0) = -1,\ f(1)=2 \text{ and for } n = 1, 2, 3, \ldots$$
$$f(n+1) = f(n) + 3 f(n - 1)$$

SOLUTION:

From the recursive definition it follows that

$$\begin{aligned} f(2) &= f(1) + 3 f(1-1) \\ &= f(1) + 3 f(0) \\ &= 2 + 3(-1) \\ &= -1 \end{aligned}$$

Now in order to find out the other values we will need the values of the preceding .So we write these values here again

$f(0) = -1,\ f(1)=2 \qquad f(n+1) = f(n) + 3 f(n - 1)$

$f(2) = -1$

By recursive formula we have

$$\begin{aligned} f(3) &= f(2) + 3 f(2-1) \\ &= f(2) + 3 f(1) \\ &= (-1) + 3(2) \\ &= 5 \end{aligned}$$

$$\begin{aligned} f(4) &= f(3) + 3 f(3-1) \\ &= f(2) + 3 f(2) \\ &= 5 + 3(-1) \\ &= 2 \end{aligned}$$

THE FACTORIAL OF A POSITIVE INTEGER:

For each positive integer n, the factorial of n denoted n! is defined to be the product of all the integers from 1 to n:

$$n! = n\cdot(n - 1)\cdot(n - 2) \cdots 3 \cdot 2 \cdot 1$$

Zero factorial is defined to be 1

$$0! = 1$$

EXAMPLE:

$0! = 1$	$1! = 1$
$2! = 2\cdot1 = 2$	$3! = 3\cdot2\cdot1 = 6$
$4! = 4\cdot3\cdot2\cdot1 = 24$	$5! = 5\cdot4\cdot3\cdot2\cdot1 = 120$
$6! = 6\cdot5\cdot4\cdot3\cdot2\cdot1= 720$	$7! = 7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1= 5040$

REMARK:

$$\begin{aligned} 5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ &= 5 \cdot(4 \cdot 3 \cdot 2 \cdot 1) \\ &= 5 \cdot 4! \end{aligned}$$

In general,

$n! = n(n-1)!$ for each positive integer n.

THE FACTORIAL FUNCTION DEFINED RECURSIVELY:

We can define the factorial function F(n) = n! recursively by specifying the initial value of this function, namely, F(0) = 1, and giving a rule for finding F(n) from F(n-1).{(n! = n(n-1)!}

Thus, the recursive definition of factorial function F(n) is:

1. F(0) = 1
2. F(n) = n F(n-1)

EXERCISE:

Let S be the function such that S(n) is the sum of the first n positive integers. Give a recursive definition of S(n).

SOLUTION:

The initial value of this function may be specified as S(0) = 0

Since

$$\begin{aligned} S(n) &= n + (n - 1) + (n - 2) + \ldots + 3 + 2 + 1 \\ &= n + [(n - 1) + (n - 2) + \ldots + 3 + 2 + 1] \\ &= n + S(n-1) \end{aligned}$$

which defines the recursive step.

Accordingly S may be defined as:

1. S(0)= 0
2. S(n) = n + S(n - 1) for $n \geq 1$

EXERCISE:

Let a and b denote positive integers. Suppose a function Q is defined recursively as follows:

(a) Find the value of Q(2,3) and Q(14,3)

(b) What does this function do? Find Q (3355, 7)

SOLUTION:

$$Q(a,b) = \begin{cases} 0 & \text{if } a \langle b \\ Q(a-b,b)+1 & \text{if } b \le a \end{cases}$$

(a) Q (2,3) = 0 since 2 < 3

Given Q(a,b) = Q(a-b,b) + 1 if b ≤a

Now

Q (14, 3) = Q (11,3) + 1
= [Q(8,3) + 1] + 1 = Q(8,3) + 2
= [Q(5,3) + 1] + 2 = Q(5,3) + 3
= [Q(2,3) + 1] + 3 = Q(2,3) + 4
= 0 + 4 (∴ Q(2,3) = 0)
= 4

(b)

$$Q(a,b) = \begin{cases} 0 & \text{if } a \langle b \\ Q(a-b,b)+1 & \text{if } b \le a \end{cases}$$

Each time b is subtracted from a, the value of Q is increased by 1. Hence Q(a,b) finds the integer quotient when a is divided by b.

Thus Q(3355, 7) = 479

THE FIBONACCI SEQUENCE:

The Fibonacci sequence is defined as follows.

$F_0 = 1$, $F_1 = 1$

$F_k = F_{k-1} + F_{k-2}$ for all integers $k \ge 2$

$F_2 = F_1 + F_0 = 1 + 1 = 2$

$F_3 = F_2 + F_1 = 2 + 1 = 3$

$F_4 = F_3 + F_2 = 3 + 2 = 5$

$F_5 = F_4 + F_3 = 5 + 3 = 8$

.
.
.

RECURRENCE RELATION:

A recurrence relation for a sequence $a_0, a_1, a_2, \ldots,$ is a formula that relates each term a_k to certain of its predecessors $a_{k-1}, a_{k-2}, \ldots, a_{k-i}$,

where i is a fixed integer and k is any integer greater than or equal to i. The initial conditions for such a recurrence relation specify the values of

$a_0, a_1, a_2, \ldots, a_{i-1}$.

EXERCISE:

Find the first four terms of the following recursively defined sequence.

$b_1 = 2$

$b_k = b_{k-1} + 2 \cdot k$, for all integers $k \ge 2$

\
SOLUTION:

$b_1 = 2$ (given in base step)

$b_2 = b_1 + 2 \cdot 2 = 2 + 4 = 6$

$b_3 = b_2 + 2 \cdot 3 = 6 + 6 = 12$

$b_4 = b_3 + 2 \cdot 4 = 12 + 8 = 20$

EXERCISE:

Find the first five terms of the following recursively defined sequence.

$t_0 = -1, \quad t_1 = 1$

$t_k = t_{k-1} + 2 \cdot t_{k-2},$ for all integers $k \geq 2$

SOLUTION:

$t_0 = -1,$ (given in base step)

$t_1 = 1$ (given in base step)

$t_2 = t_1 + 2 \cdot t_0 = 1 + 2 \cdot (-1) = 1 - 2 = -1$

$t_3 = t_2 + 2 \cdot t_1 = -1 + 2 \cdot 1 = -1 + 2 = 1$

$t_4 = t_3 + 2 \cdot t_2 = 1 + 2 \cdot (-1) = 1 - 2 = -1$

EXERCISE:

Define a sequence $b_0, b_1, b_2, \ldots$ by the formula

$b_n = 5^n$, for all integers $n \geq 0$.

Show that this sequence satisfies the recurrence relation $b_k = 5b_{k-1}$, for all integers $k \geq 1$.

SOLUTION:

The sequence is given by the formula

$b_n = 5^n$

Substituting k for n we get

$b_k = 5^k$ . . . . . (1)

Substituting k – 1 for n we get

$b_{k-1} = 5^{k-1}$ . . . . . (2)

Multiplying both sides of (2) by 5 we obtain

$5 \cdot b_{k-1} = 5 \cdot 5^{k-1}$

$= 5^k = b_k$ using (1)

Hence $b_k = 5b_{k-1}$ as required

EXERCISE:

Show that the sequence $0, 1, 3, 7, \ldots, 2^n - 1, \ldots$, for $n \geq 0$, satisfies the recurrence relation

$d_k = 3d_{k-1} - 2d_{k-2}$, for all integers $k \geq 2$

SOLUTION:

The sequence is given by the formula

$d_n = 2^n - 1$ for $n \geq 0$

Substituting k – 1 for n we get $d_{k-1} = 2^{k-1} - 1$

Substituting k – 2 for n we get $d_{k-2} = 2^{k-2} - 1$

We want to prove that

$d_k = 3d_{k-1} - 2d_{k-2}$

$$\begin{aligned}\text{R.H.S.} &= 3(2^{k-1}-1) - 2(2^{k-2}-1)\\ &= 3\cdot 2^{k-1} - 3 - 2\cdot 2^{k-2} + 2\\ &= 3\cdot 2^{k-1} - 2^{k-1} - 1\\ &= (3-1)\cdot 2^{k-1} - 1\\ &= 2\cdot 2^{k-1} - 1 = 2^{k} - 1 = d_k \quad = \text{L.H.S.}\end{aligned}$$

THE TOWER OF HANOI:

The puzzle was invented by a French Mathematician Adouard Lucas in 1883. It is well known to students of Computer Science since it appears in virtually any introductory text on data structures or algorithms.

There are three poles on first of which are stacked a number of disks that decrease in size as they rise from the base. The goal is to transfer all the disks one by one from the first pole to one of the others, but they must never place a larger disk on top of a smaller one.

Let m_n be the minimum number of moves needed to move a tower of n disks from one pole to another. Then m_n can be obtained recursively as follows.

- $m_1 = 1$
- $m_k = 2\, m_{k-1} + 1$

$m_2 = 2 \cdot m_1 + 1 = 2 \cdot 1 + 1 = 3$

$m_3 = 2 \cdot m_2 + 1 = 2 \cdot 3 + 1 = 7$

$m_4 = 2 \cdot m_3 + 1 = 2 \cdot 7 + 1 = 15$

$m_5 = 2 \cdot m_4 + 1 = 2 \cdot 15 + 1 = 31$

$m_6 = 2 \cdot m_5 + 1 = 2 \cdot 31 + 1 = 65$

Note that

$$m_n = 2^n - 1$$

$$m_{64} = 2^{64} - 1$$

$$\cong 584.5 \text{ billion years}$$

USE OF RECURSION:

At first recursion may seem hard or impossible, may be magical at best. However, recursion often provides elegant, short algorithmic solutions to many problems in computer science and mathematics.

Examples where recursion is often used

- math functions
- number sequences
- data structure definitions
- data structure manipulations
- language definitions

LECTURE # 22

RECURSIVELY DEFINED SETS

A recursive definition (i.e to build the new set elements from the previous one,s)for a set consists of the following three rules:

I. **BASE**: A statement that certain objects belong to the set.
II. **RECURSION**: A collection of rules indicating how to form new set objects from those already known to be in the set.
III. **RESTRICTION**: A statement that no objects belong to the set other than those coming from I and II.

EXERCISE:

Let S be a set defined recursively by

I. **BASE**: $5 \in S$.
II. **RECURSION**: If $x \in S$ and $y \in S$, then $x + y \in S$.
III. **RESTRICTION**: S contains no elements other than those obtained from rules I and II.

Show that S is the subset of all positive integers divisible by 5.

SOLUTION:

Let A be the set of all positive integers divisible by 5. Then

$A = \{5n \mid n \in N\}$.

We need to prove that $S \subseteq A$.

5 is divisible by 5 since $5 = 5 \times 1$

$\Rightarrow 5 \in A$

Now consider $x \in A$ and $y \in A$, we show that $x + y \in A$

$x \in A \Rightarrow 5 \mid x$ so that $x = 5 \cdot p$ for some $p \in N$

$y \in A \Rightarrow 5 \mid y$ so that $y = 5 \cdot q$ for some $q \in N$

Hence $x + y = 5 \cdot p + 5 \cdot q = 5 \cdot (p + q)$

$\Rightarrow 5 \mid (x + y)$ and so $(x + y) \in A$

Thus, S is a subset of A.

RECURSIVE DEFINITION OF BOOLEAN EXPRESSIONS:

I. **BASE**:

Each symbol of the alphabet is a Boolean expression.

II. **RECURSION**:

If P and Q are Boolean Expressions, then so are

(a) $(P \wedge Q)$
(b) $(P \vee Q)$ and
(c) $\sim P$.

III **RESTRICTION**:

There are no Boolean expressions over the alphabet other than those obtained from I and II.

EXERCISE:
RCISE

Show that the following is a Boolean expression over the English alphabet.

$$((p \vee q) \vee \sim ((p \wedge \sim s) \wedge r))$$

SOLUTION:

We will show that the given Boolean expression can be found out using the recursive definition of Boolean expressions. So first of all we will start with the symbols which are involved in the Boolean expressions.

(1) p, q, r, and s are Boolean expressions by I.

Now we start with the inner most expression which is $p \wedge \sim s$ before we check this one we will check ~s and we note that

(2) ~ s is a Boolean expressions by (1) and II(c).

Now from above we have p and ~s are Boolean expressions and we can say that

(3) $(p \wedge \sim s)$ is a Boolean expressions by (1), (2) and II(a).

Similarly we find that

(4) $(p \wedge \sim s) \wedge r)$ is a Boolean expressions by (1), (3) and II(a).

(5) $\sim (p \wedge \sim s) \wedge r)$ is a Boolean expressions by (4) and II(c).

(6) $(p \vee q)$ is a Boolean expressions by (1) and II(b).

(7) $((p \vee q) \vee \sim ((p \wedge \sim s) \wedge r))$ is a Boolean expressions by (5), (6) and II(b).

RECURSIVE DEFINITION OF THE SET OF STRINGS OVER AN ALPHABET:

Consider a finite alphabet $\Sigma = \{a, b\}$. The set of all finite strings over Σ, denoted Σ^*, is defined recursively as follows:

I. **BASE**: ε is in Σ^*, *where* ε is the null string.

II. **RECURSION**: If $s \in \Sigma^*$, then

(a) $sa \in \Sigma^*$ and

(b) $sb \in \Sigma^*$,

where sa and sb are concatenations of s with a and b respectively.

III. RESTRICTION: Nothing is in Σ^* other than objects defined in I and II above.

EXERCISE:

Give derivations showing that abb is in Σ^*.

SOLUTION

(1) $\varepsilon \in \Sigma^*$ by I.

(2) $a = \varepsilon a \in \Sigma^*$ by (1) and II(a).

(3) $ab \in \Sigma^*$ by (2) and II(b).

(4) $abb \in \Sigma^*$ by (3) and II(b).

EXERCISE:

Give a recursive definition of all strings of 0's and 1's for which all the 0's precede all the 1's.

SOLUTION:

Let S be the set of all strings of 0's and 1's for which all the 0's precede all the 1's. The following is a recursive definition of S.

I. **BASE**: The null string $\varepsilon \in S$.

II. **RECURSION**: If $s \in S$, then

(a) $0s \in S$ and (b) $s1 \in S$.

III. **RESTRICTION**: Nothing is in S other than objects defined in I and II above.

PARENTHESIS STRUCTURE:

Let P be the set of grammatical configurations of parentheses. The following is a recursive definition of P.

I. **BASE**: ( ) is in P.

II. **RECURSION**:

(a) If E is in P, so is (E).
(b) If E and F are in P, so is EF.

III. **RESTRICTION**: No configurations of parentheses are in P other than those derived from I and II above.

EXERCISE:

Derive the fact that ((( )) ( )) is in the set P of grammatical configuration of parentheses.

SOLUTION

Now we will show that the given structure of parenthesis can be obtained by using the recursive definition of Parenthesis Structure for this we will start with the inner most bracket and note that

1.( ) is in P, by I

Since in the recursive step (a) we say that parenthesis can be put into another parenthesis which shows that

2. (( )) is in P, by 1 and II(a)

Similarly you can see that

3. (( )) ( ) is in P, by 2, I and II(b)
4. ((( )) ( )) is in P, by 3, and II(a)

SET OF ARITHMETIC EXPRESSIONS:

The set of arithmetic expressions over the real numbers can be defined recursively as follows.

I. **BASE**: Each real number r is an arithmetic expression.

II. **RECURSION**: If u and v are arithmetic expressions, then the following are also arithmetic expressions.

a. (+ u) b. (– u) c. (u + v)

d. (u – v) e. (u · v) f. $\left(\frac{u}{v}\right)$

III. **RESTRICTION**: There are no arithmetic expressions other than those obtained from I and II above.

EXERCISE:

Give derivations showing that the following is an arithmetic expression.

$$\left(\frac{(9\cdot(6.1+2))}{((4-7)\cdot 6)}\right)$$

SOLUTION:

Here again our approach is same that we will trace the given expression and see that it can be obtained by using Recursive definition of Arithmetic Operations or not.

(1) 9, 6.1, 2, 4, 7, and 6 are arithmetic expressions by I.
(2) (6.1 + 2) is an arithmetic expression by (1) and II(c).
(3) (9·(6.1 +2)) is an arithmetic expression by (1), (2) and II(e).
(4) (4 – 7) is an arithmetic expression by (1) and II(d).
(5) ((4 – 7)·6) is an arithmetic expression by (1), (4) and II(e).
(6)

$$\left(\frac{(9\cdot(6.1+2))}{((4-7)\cdot 6)}\right)$$ is an arithmetic expression by (3), (5) and II(f).

RECURSIVE DEFINITION OF SUM:

Given numbers $a_1, a_2, \ldots, a_n$, where n is a positive integer, the **summation from i = 1 to n of the a_i,** denoted $\sum_{i=1}^{n} a_i$, is defined as follows:

BASE:

$$\sum_{i=1}^{1} a_i = a_i$$

RECURSION:

$$\sum_{i=1}^{n} a_i = \left(\sum_{i=1}^{n-1} a_i\right) + a_n \quad \text{if } n \rangle 1$$

RECURSIVE DEFINITION OF UNION OF SETS:

Given sets $A_1, A_2, \ldots, A_n$, where n is a positive integer, the union of A_i from i = 1 to n, denoted $\bigcup_{i=1}^{n} A_i$ is defined by

BASE:

$$\bigcup_{i=1}^{1} A_i = A_1$$

RECURSION:

$$\bigcup_{i=1}^{n} A_i = \left(\bigcup_{i=1}^{n-1} A_i\right) \cup A_n.$$

RECURSIVE DEFINITION OF INTERSECTION OF SETS:

Given sets $A_1, A_2, \ldots, A_n$, where n is a positive integer, the intersection of A_i from i = 1 to n, denoted $\bigcap_{i=1}^{n} A_i,$ is defined by

BASE:

$$\bigcap_{i=1}^{1} A_i = A_1$$

RECURSION:

$$\bigcap_{i=1}^{n} A_i = \left(\bigcap_{i=1}^{n-1} A_i\right) \cap A_n.$$

LECTURE # 23

PRINCIPLE OF MATHEMATICAL INDUCTION:

Let P(n) be a propositional function defined for all positive integers n. P(n) is true for every positive integer n if

1. Basis Step:

The proposition P(1) is true.

2. Inductive Step:

If P(k) is true then P(k + 1) is true for all integers k ≥ 1.

i.e. $\forall$ k   p(k) $\rightarrow$ P(k + 1)

EXAMPLE:

Use Mathematical Induction to prove that

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$ for all integers n ≥1

SOLUTION:

Let $$P(n):1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

1.Basis Step:

P(1) is true.

For n = 1, left hand side of P(1) is the sum of all the successive integers starting at 1 and ending at 1, so LHS = 1 and RHS is

$$R.H.S=\frac{1(1+1)}{2}=\frac{2}{2}=1$$

so the proposition is true for n = 1.

2. Inductive Step: Suppose P(k) is true for, some integers k ≥ 1.

$$1+2+3+\cdots+k=\frac{k(k+1)}{2} \qquad (1)$$

To prove P(k + 1) is true. That is,

$$1+2+3+\cdots+(k+1)=\frac{(k+1)(k+2)}{2} \qquad (2)$$

Consider L.H.S. of (2)

$$\begin{aligned}1+2+3+\cdots+(k+1) &= 1+2+3+\cdots+k+(k+1)\\ &= \frac{k(k+1)}{2}+(k+1) \qquad \text{using (1)}\\ &= (k+1)\left[\frac{k}{2}+1\right]\\ &= (k+1)\left[\frac{k+2}{2}\right]\\ &= \frac{(k+1)(k+2)}{2} = \text{RHS of (2)}\end{aligned}$$

Hence by principle of Mathematical Induction the given result true for all integers greater or equal to 1.

EXERCISE:

Use mathematical induction to prove that
$1+3+5+\ldots+(2n-1) = n^2$ for all integers $n \geq 1$.

SOLUTION:

Let P(n) be the equation $1+3+5+\ldots+(2n-1) = n^2$

1. **Basis Step**:

P(1) is true
For n = 1, L.H.S of P(1) = 1and
R.H.S = 12 = 1
Hence the equation is true for n = 1

2. **Inductive Step**:

Suppose P(k) is true for some integer $k \geq 1$. That is,

$1 + 3 + 5 + \ldots + (2k - 1) = k^2$ …………………(1)

To prove P(k+1) is true; i.e.,

$1 + 3 + 5 + \ldots + [2(k+1)-1] = (k+1)^2$ ………………(2)

Consider L.H.S. of (2)

$$\begin{aligned}1+3+5+\cdots+[2(k+1)-1] &= 1+3+5+\cdots+(2k+1)\\ &= 1+3+5+\cdots+(2k-1)+(2k+1)\\ &= k^2+(2k+1) \qquad \text{using (1)}\\ &= (k+1)^2\\ &= \text{R.H.S. of (2)}\end{aligned}$$

Thus P(k+1) is also true. Hence by mathematical induction, the given equation is true for all integers $n \geq 1$.

EXERCISE:

Use mathematical induction to prove that
1+2+22 + … + 2n = 2n+1 - 1 for all integers $n \geq 0$

SOLUTION:

Let P(n): 1 + 2 + 22 + … + 2n = 2n+1 - 1

1. **Basis Step:**

P(0) is true.
For n = 0
L.H.S of P(0) = 1
R.H.S of P(0) = 20+1 - 1 = 2 - 1 = 1
Hence P(0) is true.

2. **Inductive Step:**

Suppose P(k) is true for some integer k ≥ 0; i.e.,
1+2+22+…+2k = 2k+1 – 1……………………(1)

To prove P(k+1) is true, i.e.,
1+2+22+…+2k+1 = 2k+1+1 – 1…………………(2)

Consider LHS of equation (2)
1+2+22+…+2k+1= (1+2+22+…+2k) + 2k+1
= (2k+1 - 1) +2k+1
= 2·2k+1 - 1
= 2k+1+1 - 1 = R.H.S of (2)

Hence P(k+1) is true and consequently by mathematical induction the given propositional function is true for all integers n≥0.

EXERCISE:

Prove by mathematical induction

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$ for all integers n ≥1.

SOLUTION:

Let P(n) denotes the given equation

1. **Basis step:**

P(1) is true
For n = 1
L.H.S of P(1) = 12 = 1

R.H.S of P(1) $= \frac{1(1+1)(2(1)+1)}{6}$

$$= \frac{(1)(2)(3)}{6} = \frac{6}{6} = 1$$

So L.H.S = R.H.S of P(1).Hence P(1) is true

2.Inductive Step:

Suppose P(k) is true for some integer k ≥1;

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6} \quad \text{………(1)}$$

To prove P(k+1) is true; i.e.;

$$1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6} \quad \text{…(2)}$$

Consider LHS of above equation (2)

$$
\begin{aligned}
1^2+2^2+3^2+\cdots+(k+1)^2 &= 1^2+2^2+3^2+\cdots+k^2+(k+1)^2\\
&= \frac{k(k+1)(2k+1)}{6}+(k+1)^2\\
&= (k+1)\left[\frac{k(2k+1)}{6}+(k+1)\right]\\
&= (k+1)\left[\frac{k(2k+1)+6(k+1)}{6}\right]\\
&= (k+1)\left[\frac{2k^2+k+6k+6}{6}\right]\\
&= \frac{(k+1)(2k^2+7k+6)}{6}\\
&= \frac{(k+1)(k+2)(2k+3)}{6}\\
&= \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}
\end{aligned}
$$

EXERCISE:

Prove by mathematical induction

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$ for all integers n≥1

SOLUTION:

Let P(n) be the given equation.

1.Basis Step:

P(1) is true

For n = 1

L.H.S of P(1) = $\frac{1}{1\cdot 2}=\frac{1}{1\times 2}=\frac{1}{2}$

R.H.S of P(1) = $\frac{1}{1+1}=\frac{1}{2}$

Hence P(1) is true

2.Inductive Step:

Suppose P(k) is true, for some integer k≥1. That is

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1} \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

To prove P(k+1) is true. That is

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)(k+1+1)}=\frac{k+1}{(k+1)+1} \qquad \ldots\ldots(2)$$

Now we will consider the L.H.S of the equation (2) and will try to get the R.H.S by using equation (1) and some simple computation.

Consider LHS of (2)

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)(k+2)}$$
$$=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$$
$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$
$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$
$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$
$$=\frac{(k+1)^2}{(k+1)(k+2)}$$
$$=\frac{k+1}{(k+2)}$$
$$=\text{RHS of (2)}$$

Hence P(k+1) is also true and so by Mathematical induction the given equation is true for all integers n ≥1.

EXERCISE:

Use mathematical induction to prove that

$$\sum_{i=1}^{n+1} i2^i = n\cdot 2^{n+2}+2, \qquad \text{for all integers } n\geq 0$$

SOLUTION:

1.Basis Step:

To prove the formula for n = 0, we need to show that

$$\sum_{i=1}^{0+1} i.2^i = 0\cdot 2^{0+2}+2$$

Now, L.H.S = $\sum_{i=1}^{1} i\cdot 2^i = (1)2^1 = 2$

R.H.S = $0\cdot 2^2 + 2 = 0 + 2 = 2$

Hence the formula is true for n = 0

2.Inductive Step:

Suppose for some integer n=k ≥0

$$\sum_{i=1}^{k+1} i\cdot 2^i = k\cdot 2^{k+2}+2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

We must show that

$$\sum_{i=1}^{k+2} i\cdot 2^i = (k+1)\cdot 2^{k+1+2}+2 \qquad \ldots\ldots\ldots..(2)$$

Consider LHS of (2)

$$\begin{aligned}\sum_{i=1}^{k+2} i\cdot 2^{i} &= \sum_{i=1}^{k+1} i\cdot 2^{i} + (k+2)\cdot 2^{k+2}\\ &= (k\cdot 2^{k+2}+2)+(k+2)\cdot 2^{k+2}\\ &= (k+k+2)2^{k+2}+2\\ &= (2k+2)\cdot 2^{k+2}+2\\ &= (k+1)2\cdot 2^{k+2}+2\\ &= (k+1)\cdot 2^{k+1+2}+2\\ &= \text{RHS of equation (2)}\end{aligned}$$

Hence the inductive step is proved as well. Accordingly by mathematical induction the given formula is true for all integers n≥0.

EXERCISE:

Use mathematical induction to prove that

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n} \quad \text{for all integers } n \geq 2$$

SOLUTION:

1. **Basis Step**:

For n = 2

L.H.S $=1-\frac{1}{2^2}=1-\frac{1}{4}=\frac{3}{4}$

R.H.S $=\frac{2+1}{2(2)}=\frac{3}{4}$

Hence the given formula is true for n = 2

2. **Inductive Step**:

Suppose for some integer k ≥2

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

We must show that

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)=\frac{(k+1)+1}{2(k+1)} \quad \ldots\ldots\ldots\ldots(2)$$

Consider L.H.S of (2)

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)$$

$$=\left[\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\right]\left(1-\frac{1}{(k+1)^2}\right)$$

$$=\left(\frac{k+1}{2k}\right)\left(1-\frac{1}{(k+1)^2}\right)$$

$$=\left(\frac{k+1}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^2}\right)$$

$$=\left(\frac{1}{2k}\right)\left(\frac{k^2+2k+1-1}{(k+1)}\right)$$

$$=\frac{k^2+2k}{2k(k+1)}=\frac{k(k+2)}{2k(k+1)}$$

$$=\frac{k+1+1}{2(k+1)}= \quad \text{RHS of (2)}$$

Hence by mathematical induction the given equation is true

EXERCISE:

Prove by mathematical induction

$$\sum_{i=1}^{n} i(i!) = (n+1)!-1 \qquad \text{for all integers } n\geq 1$$

SOLUTION:

1.Basis step:

For n = 1

L.H.S $= \sum_{i=1}^{n} i(i!) = (1)(1!) = 1$

R.H.,S = (1+1)! - 1 = 2! - 1

= 2 -1 = 1

Hence

$$\sum_{i=1}^{1} i(i!) = (1+1)!-1$$

which proves the basis step.

2.Inductive Step:

Suppose for any integer k ≥1

$$\sum_{i=1}^{k} i(i!) = (k+1)!-1 \qquad \text{……………………….(1)}$$

We need to prove that

$$\sum_{i=1}^{k+1} i(i!) = (k+1+1)!-1 \qquad \text{……………………(2)}$$

Consider LHS of (2)

$$\sum_{i=1}^{k+1} i(i!) = \sum_{i=1}^{k} i(i!) + (k+1)(k+1)! \qquad \text{Using (1)}$$
$$= (k+1)! - 1 + (k+1)(k+1)!$$
$$= (k+1)! + (k+1)(k+1)! - 1$$
$$= [1 + (k+1)](k+1)! - 1$$
$$= (k+2)(k+1)! - 1$$
$$= (k+2)! - 1$$
$$= \text{RHS of (2)}$$

Hence the inductive step is also true.
Accordingly, by mathematical induction, the given formula is true for all integers n ≥1.

EXERCISE:

Use mathematical induction to prove the generalization of the following DeMorgan's Law:

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

where $A_1, A_2, \ldots, A_n$ are subsets of a universal set U and n≥2.

SOLUTION:

Let P(n) be the given propositional function

1.Basis Step:

P(2) is true.

L.H.S of P(2) = $\overline{\bigcap_{j=1}^{2} A_j} = \overline{A_1 \bigcap A_2}$ By DeMorgan's Law

$$= \overline{A_1} \bigcup \overline{A_2}$$
$$= \bigcup_{i=1}^{2} \overline{A_j} = \text{RHS of } P(2)$$

2.Inductive Step:

Assume that P(k) is true for some integer k ≥2; i.e.,

$$\overline{\bigcap_{j=1}^{k} A_j} = \bigcup_{j=1}^{k} \overline{A_j} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\text{(1)}$$

where $A_1, A_2, \ldots, A_k$ are subsets of the universal set U. If A_{k+1} is another set of U, then we need to show that

$$\overline{\bigcap_{j=1}^{k+1} A_j} = \bigcup_{j=1}^{k+1} \overline{A_j} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\text{(2)}$$

Consider L.H.S of (2)

$$\overline{\bigcap_{j=1}^{k+1} A_j} = \overline{\left(\bigcap_{j=1}^{k} A_j\right) \cap A_{k+1}}$$

$$= \left(\overline{\bigcap_{j=1}^{k} A_j}\right) \cup \overline{A_{k+1}} \qquad \text{By DeMorgan's Law}$$

$$= \left(\bigcup_{j=1}^{k} \overline{A_j}\right) \cup \overline{A_{k+1}}$$

$$= \bigcup_{j=1}^{k+1} \overline{A_j}$$

$$= \text{R.H.S of (2)}$$

Hence by mathematical induction, the given generalization of DeMorgan's Law holds.

LECTURE 24

MATHEMATICAL INDUCTION FOR
DIVISIBILITY PROBLEMS
INEQUALITY PROBLEMS

DIVISIBILITY:

Let n and d be integers and $d \neq 0$. Then n is divisible by d or d divides n written $d|n$. iff $n = d \cdot k$ for some integer k.
Alternatively, we say that
n is a multiple of d
d is a divisor of n
d is a factor of n
Thus $d|n \Leftrightarrow \exists$ an integer k such that $n = d \cdot k$

EXERCISE:

Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer.

SOLUTION:

1. Basis Step:

For n = 1

$$n^3 - n = 1^3 - 1 = 1 - 1 = 0$$

which is clearly divisible by 3, since $0 = 0 \cdot 3$
Therefore, the given statement is true for n = 1.

2.Inductive Step:

Suppose that the statement is true for n = k, i.e., k^3-k is divisible by 3 for all $n \in Z+$
Then

$$k^3-k = 3 \cdot q \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

for some $q \in Z$
We need to prove that $(k+1)^3 - (k+1)$ is divisible by 3.
Now

$$\begin{aligned}(k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - (k + 1)\\ &= k^3 + 3k^2 + 2k\\ &= (k^3-k) + 3k^2 + 2k + k\\ &= (k^3 - k) + 3k^2 + 3k\\ &= 3 \cdot q + 3 \cdot (k^2 + k) \qquad \text{using(1)}\\ &= 3[q+k^2 + k]\end{aligned}$$

$\Rightarrow$ $(k+1)^3 - (k+1)$ is divisible by 3.
Hence by mathematical induction $n^3 - n$ is divisible by 3, whenever n is a positive integer.

EXAMPLE:

Use mathematical induction to prove that for all integers $n \geq 1$, $2^{2n}-1$ is divisible by 3.

SOLUTION:

Let P(n): $2^{2n} -1$ is divisible by 3.

1.Basis Step:

P(1) is true
Now P(1): $2^{2(1)} - 1$ is divisible by 3.
Since $2^{2(1)} - 1 = 4 - 1 = 3$
which is divisible by 3.
Hence P(1) is true.

2.Inductive Step:
Suppose that P(k) is true. That is $2^{2k}-1$ is divisible by 3. Then, there exists an integer q such that
$2^{2k} - 1 = 3\cdot q$ ……………………………(1)
To prove P(k+1) is true, that is $2^{2(k+1)} - 1$ is divisible by 3.
Now consider

$$\begin{aligned} 2^{2(k+1)} - 1 &= 2^{2k+2} - 1 \\ &= 2^{2k}\, 2^2 - 1 \\ &= 2^{2k}4 - 1 \\ &= 2^{2k}(3+1) - 1 \\ &= 2^{2k}\cdot 3+(2^{2k} - 1) \\ &= 2^{2k}\cdot 3+3\cdot q \qquad \text{[by using (1)]} \\ &=3(2^{2k} + q) \end{aligned}$$

$\Rightarrow 2^{2(k+1)} - 1$ is divisible by 3.
Accordingly, by mathematical induction. $2^{2n} - 1$ is divisible by 3, for all integers $n \geq 1$.

EXERCISE:

Use mathematical induction to show that the product of any two consecutive positive integers is divisible by 2.

SOLUTION:

Let n and n + 1 be two consecutive integers. We need to prove that n(n+1) is divisible by 2.

1. Basis Step:
For n = 1
$n\,(n+1) = 1\cdot(1+1) = 1\cdot 2 = 2$
which is clearly divisible by 2.

2. Inductive Step:
Suppose the given statement is true for n = k. That is
k (k+1) is divisible by 2, for some $k \in Z+$
Then $k\,(k+1) = 2\cdot q$ ………………….(1) $q \in Z+$
We must show that
(k+1)(k+1+1) is divisible by 2.
Consider

$$\begin{aligned} (k+1)(k+1+1) &= (k+1)(k+2) \\ &= (k+1)k + (k+1)2 \\ &= 2q + 2\,(k+1) \quad \text{using (1)} \\ &= 2(q+k+1) \end{aligned}$$

Hence (k+1) (k+1+1) is also divisible by 2.
Accordingly, by mathematical induction, the product of any two consecutive positive integers is divisible by 2.

EXERCISE:

Prove by mathematical induction $n^3 - n$ is divisible by 6, for each integer $n \geq 2$.

SOLUTION:

1.Basis Step:

For n = 2

$n^3 - n = 2^3 - 2 = 8 - 2 = 6$

which is clearly divisible by 6, since $6 = 1 \cdot 6$

Therefore, the given statement is true for n = 2.

2.Inductive Step:

Suppose that the statement is true for n = k, i.e., $k^3 - k$ is divisible by 6, for all integers $k \geq 2$.

Then

$$k^3 - k = 6 \cdot q \ldots\ldots\ldots\ldots\ldots(1) \quad \text{for some } q \in Z.$$

We need to prove that

$(k+1)^3 - (k+1)$ is divisible by 6

Now $(k+1)^3 - (k+1) = (k^3 + 3k^3 + 3k + 1) - (k+1)$

$= k^3 + 3k^3 + 2k$

$= (k^3 - k) + (3k^3 + 2k + k)$

$= (k^3 - k) + 3k^3 + 3k$ Using (1)

$= 6 \cdot q + 3k(k+1) \ldots\ldots\ldots\ldots\ldots\ldots..(2)$

Since k is an integer, so k(k+1) being the product of two consecutive integers is an even number.

Let $k(k+1) = 2r \quad r \in Z$

Now equation (2) can be rewritten as:

$(k+1)^3 - (k+1) = 6 \cdot q + 3 \cdot 2\, r$

$= 6q + 6r$

$= 6(q+r) \quad q, r \in Z$

$\Rightarrow (k+1)^3 - (k+1)$ is divisible by 6.

Hence, by mathematical induction, $n^3 - n$ is divisible by 6, for each integer $n \geq 2$.

EXERCISE:

Prove by mathematical induction. For any integer $n \geq 1$, $x^n - y^n$ is divisible by $x - y$, where x and y are any two integers with $x \neq y$.

SOLUTION:

1.Basis Step:

For n = 1

$x^n - y^n = x^1 - y^1 = x - y$

which is clearly divisible by $x - y$. So, the statement is true for n = 1.

2.Inductive Step:

Suppose the statement is true for n = k, i.e.,

$x^k - y^k$ is divisible by $x - y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$

We need to prove that $x^{k+1} - y^{k+1}$ is divisible by $x - y$

Now

$x^{k+1} - y^{k+1} = x^k \cdot x - y^k \cdot y$

$= x^k \cdot x - x \cdot y^k + x \cdot y^k - y^k \cdot y$ (introducing $x.y^k$)
$= (x^k - y^k) \cdot x + y^k \cdot (x-y)$
The first term on R.H.S=$(x^k - y^k)$ is divisible by x - y by inductive hypothesis (1).
The second term contains a factor (x-y) so is also divisible by x - y.
Thus $x^{k+1} - y^{k+1}$ is divisible by x - y. Hence, by mathematical induction $x^n - y^n$ is divisible by x - y for any integer $n \geq 1$.

PROVING AN INEQUALITY:

Use mathematical induction to prove that for all integers $n \geq 3$.

$$2n + 1 < 2^n$$

SOLUTION:

1.Basis Step:

For n = 3
L.H.S= 2(3) + 1 = 6 + 1 = 7
R.H.S = $2^3 = 8$
Since 7 < 8, so the statement is true for n = 3.

2.Inductive Step:

Suppose the statement is true for n =k, i.e.,
$2k + 1 < 2^k$…………………(1) $k \geq 3$
We need to show that the statement is true for n = k+1,
i.e.;
$2(k+1) + 1 < 2^{k+1}$…………..(2)
Consider L.H.S of (2)

$= 2(k+1) + 1$
$= 2k + 2 + 1$
$= (2k + 1) + 2$
$< 2^k + 2$ using (1)
$< 2^k + 2^k$ (since $2 < 2^k$ for $k \geq 3$)
$< 2 \cdot 2^k = 2^{k+1}$

Thus 2(k+1)+1 < 2k+1 (proved)

EXERCISE:

Show by mathematical induction
$1 + n x \leq (1+x)^n$
for all real numbers x > - 1 and integers $n \geq 2$

SOLUTION:

1. Basis Step:

For n = 2
L.H.S = 1 + (2) x= 1 + 2x
RHS $= (1 + x)^2 = 1 + 2x + x^2 > 1 + 2x$ ($x^2 > 0$)
⇒ statement is true for n = 2.

2.Inductive Step:

Suppose the statement is true for n = k.
That is, for $k \geq 2$, $1 + k x \leq (1 + x)^k$………………..(1)
We want to show that the statement is also true for n = k + 1 i.e.,

$$1 + (k + 1)x \leq (1 + x)^{k+1}$$

Since x > - 1, therefore 1 + x > 0.
Multiplying both sides of (1) by (1+x) we get

$$\begin{aligned}(1+x)(1+x)^k &\geq (1 + x)(1 + kx) \\ &= 1 + kx + x + kx^2 \\ &= 1 + (k + 1)x + kx^2\end{aligned}$$

but

$$\left[\begin{array}{ll} x > -1, & \text{so } x^2 \geq 0 \\ \& k \geq 2, & \text{so } kx^2 \geq 0 \end{array}\right.$$

so

$$(1+x)(1+x)^k \geq 1 + (k + 1)x$$

Thus $1 + (k+1)x \leq (1+x)^{k+1}$. Hence by mathematical induction, the inequality is true.

PROVING A PROPERTY OF A SEQUENCE:

Define a sequence $a_1, a_2, a_3, \ldots$ as follows:

$a_1 = 2$

$a_k = 5a_{k-1}$ for all integers $k \geq 2$ ……………(1)

Use mathematical induction to show that the terms of the sequence satisfy the formula.

$a_n = 2\cdot5^{n-1}$ for all integers $n \geq 1$

SOLUTION:

1.Basis Step:

For n = 1, the formula gives

$a_1 = 2\cdot5^{1-1} = 2\cdot5^0 = 2\cdot1 = 2$

which confirms the definition of the sequence. Hence, the formula is true for n = 1.

2.Inductive Step:

Suppose, that the formula is true for n = k, i.e.,

$a_k = 2\cdot5^{k-1}$ for some integer $k \geq 1$

We show that the statement is also true for n = k + 1. i.e.,

$a_{k+1} = 2\cdot5^{k+1-1} = 2\cdot5^k$

Now

$$\begin{aligned} a_{k+1} &= 5\cdot a_{k+1-1} && \text{[by definition of } a_1, a_2, a_3 \ldots \text{ or by putting k+1 in (1)]} \\ &= 5\cdot a_k \\ &= 5\cdot(2\cdot5^{k-1}) && \text{by inductive hypothesis} \\ &= 2\cdot(5\cdot5^{k-1}) \\ &= 2\cdot5^{k+1-1} \\ &= 2\cdot5^k \end{aligned}$$

which was required.

EXERCISE:

A sequence $d_1, d_2, d_3, \ldots$ is defined by letting $d_1 = 2$ and $d_k = \frac{d_{k-1}}{k}$ for all integers $k \geq 2$. Show that $d_n = \frac{2}{n!}$ for all integers $n \geq 1$, using mathematical induction.

SOLUTION:

1.Basis Step:

For n = 1, the formula $d_n = \frac{2}{n!}$; n ≥1 gives

$$d_1 = \frac{2}{1!} = \frac{2}{1} = 2$$

which agrees with the definition of the sequence.

2.Inductive Step:

Suppose, the formula is true for n=k. i.e.,

$d_k = \frac{2}{k!}$ for some integer k ≥ 1…………..(1)

We must show that

$$d_{k+1} = \frac{2}{(k+1)!}$$

Now, by the definition of the sequence.

$$d_{k+1} = \frac{d_{(k+1)-1}}{(k+1)} = \frac{1}{(k+1)} d_k \qquad using\ d_k = \frac{d_{k-1}}{k}$$

$$= \frac{1}{(k+1)} \frac{2}{k!} \qquad \text{using (1)}$$

$$= \frac{2}{(k+1)!}$$

Hence the formula is also true for n = k + 1. Accordingly, the given formula defines all the terms of the sequence recursively.

EXERCISE:

Prove by mathematical induction that

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$$

Whenever n is a positive integer greater than 1.

SOLUTION:

1. Basis Step: for n = 2

L.H.S $= 1 + \frac{1}{4} = \frac{5}{4} = 1.25$

R.H.S $= 2 - \frac{1}{2} = \frac{3}{2} = 1.5$

Clearly LHS < RHS

Hence the statement is true for n = 2.

2.Inductive Step:

Suppose that the statement is true for some integers k > 1, i.e.;

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2}<2-\frac{1}{k} \qquad (1)$$

We need to show that the statement is true for n = k + 1. That is

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(k+1)^2}<2-\frac{1}{k+1} \qquad (2)$$

Consider the LHS of (2)

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(k+1)^2}=1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2}+\frac{1}{(k+1)^2}$$

$$<\left(2-\frac{1}{k}\right)+\frac{1}{(k+1)^2}$$

$$=2-\left(\frac{1}{k}-\frac{1}{(k+1)^2}\right)$$

We need to prove that

$$2-\left(\frac{1}{k}-\frac{1}{(k+1)^2}\right)\leq 2-\frac{1}{k+1}$$

or $$-\left(\frac{1}{k}-\frac{1}{(k+1)^2}\right)\leq -\frac{1}{k+1}$$

or $$\frac{1}{k}-\frac{1}{(k+1)^2}\geq\frac{1}{k+1}$$

or $$\frac{1}{k}-\frac{1}{k+1}\geq\frac{1}{(k+1)^2}$$

Now $$\frac{1}{k}-\frac{1}{k+1}=\frac{k+1-k}{k(k+1)}$$

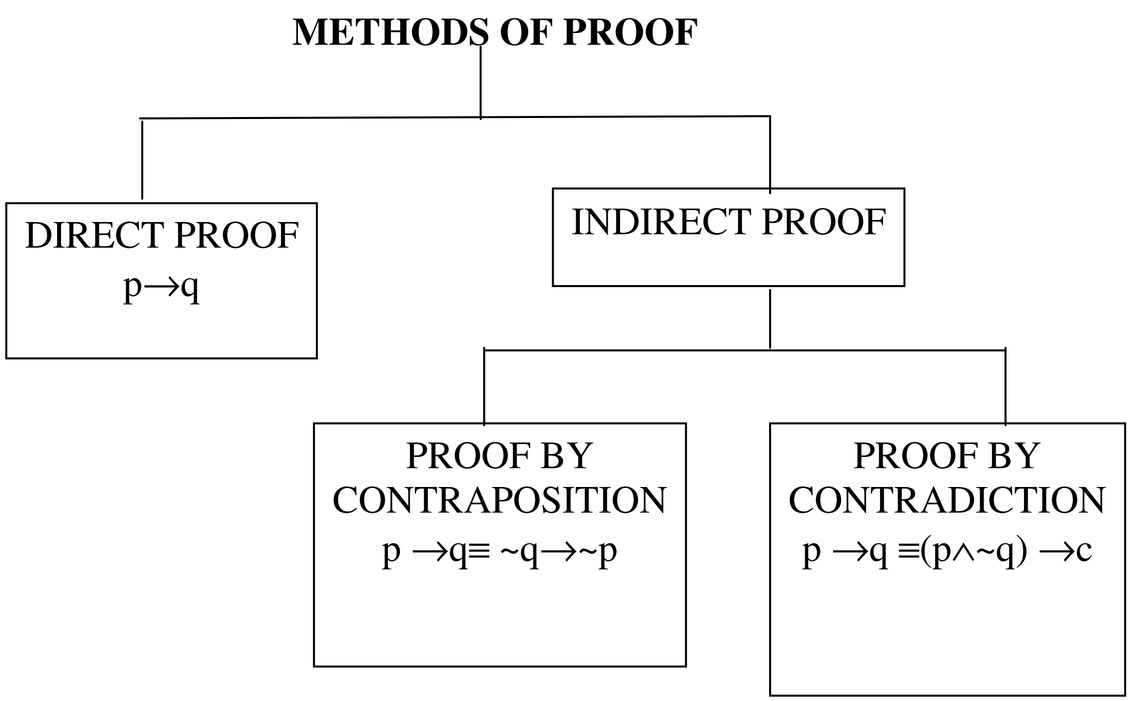
$$=\frac{1}{k(k+1)}>\frac{1}{(k+1)^2}$$

LECTURE 25

METHODS OF PROOF
DIRECT PROOF
DISPROOF BY
COUNTER EXAMPLE

INTRODUCTION:

To understand written mathematics, one must understand what makes up a correct mathematical argument, that is, a proof. This requires an under standing of the techniques used to build proofs. The methods we will study for building proofs are also used throughout computer science, such as the rules computers used to reason, the techniques used to verify that programs are correct, etc.
Many theorems in mathematics are implications, $p \to q$. The techniques of proving implications give rise to different methods of proofs.



DIRECT PROOF:

The implication $p \to q$ can be proved by showing that if p is true, the q must also be true. This shows that the combination p true and q false never occurs. A proof of this kind is called a direct proof.

p	q	p→q
T	T	T
T	F	F
F	T	T
F	F	T

SOME BASICS:

1. An integer n is even if, and only if, n = 2k for some integer k.
2. An integer n is odd if, and only if, n = 2k + 1 for some integer k.
3. An integer n is prime if, and only if, n > 1 and for all positive integers r and s, if n = r·s, then r = 1 or s = 1.
4. An integer n > 1 is composite if, and only if, n = r·s for some positive integers r and s with r ≠ 1 and s ≠ 1.
5. A real number r is rational if, and only if, for some integers a and b with b≠0.
6. If n and d are integers and d ≠0, then d divides n, written d|n if, and only if, n = d.k for some integers k.
7. An integer n is called a perfect square if, and only if, $n = k^2$ for some integer k.

EXERCISE:

Prove that the sum of two odd integers is even.

SOLUTION:

Let m and n be two odd integers. Then by definition of odd numbers

$m = 2k + 1$ for some $k \in Z$

$n = 2l + 1$ for some $l \in Z$

Now $m + n = (2k + 1) + (2l + 1)$

$= 2k + 2l + 2$

$= 2 (k + l + 1)$

$= 2r$ where $r = (k + l + 1) \in Z$

Hence m + n is even.

EXERCISE:

Prove that if n is any even integer, then $(-1)^n = 1$

SOLUTION:

Suppose n is an even integer. Then n = 2k for some integer k.

Now

$(-1)^n = (-1)^{2k}$

$= [(-1)^2]^k$

$= (1)^k$

$= 1$ (proved)

EXERCISE:

Prove that the product of an even integer and an odd integer is even.

SOLUTION:

Suppose m is an even integer and n is an odd integer. Then

$m = 2k$ for some integer k

and $n = 2l + 1$ for some integer l

Now

$m \cdot n = 2k \cdot (2l + 1)$

$= 2 \cdot k\,(2l + 1)$
$= 2 \cdot r$ where $r = k(2l + 1)$ is an integer

Hence $m \cdot n$ is even. (Proved)

EXERCISE:

Prove that the square of an even integer is even.

SOLUTION:

Suppose n is an even integer. Then $n = 2k$

Now

$$\text{square of } n = n^2 = (2 \cdot k)^2 = 4k^2 = 2 \cdot (2k^2) = 2 \cdot p \quad \text{where } p = 2k^2 \in Z$$

Hence, n^2 is even. (proved)

EXERCISE:

Prove that if n is an odd integer, then $n^3 + n$ is even.

SOLUTION:

Let n be an odd integer, then $n = 2k + 1$ for some $k \in Z$

Now $n^3 + n = n\,(n^2 + 1)$

$= (2k + 1)\,((2k+1)^2 + 1)$
$= (2k + 1)\,(4k^2 + 4k + 1 + 1)$
$= (2k + 1)\,(4k^2 + 4k + 2)$
$= (2k + 1)\, 2.\,(2k^2 + 2k + 1)$
$= 2 \cdot (2k + 1)\,(2k^2 + 2k + 1)$ $\quad k \in Z$
= an even integer

EXERCISE:

Prove that, if the sum of any two integers is even, then so is their difference.

SOLUTION:

Suppose m and n are integers so that $m + n$ is even. Then by definition of even numbers

$m + n = 2k$ for some integer k

$\Rightarrow m = 2k - n$ ……………….(1)

Now $m - n = (2k - n) - n$ using (1)

$= 2k - 2n$
$= 2\,(k - n) = 2r$ where $r = k - n$ is an integer

Hence $m - n$ is even.

EXERCISE:

Prove that the sum of any two rational numbers is rational.

SOLUTION:

Suppose r and s are rational numbers.

Then by definition of rational

$$r = \frac{a}{b} \quad \text{and} \quad s = \frac{c}{d}$$

for some integers a, b, c, d with $b \neq 0$ and $d \neq 0$

Now

$$r + s = \frac{a}{b} + \frac{c}{d}$$

$$= \frac{ad + bc}{bd}$$

$$= \frac{p}{q}$$ where p = ad + bc $\in$ Z and q =bd $\in$ Z and q $\neq$0

Hence r + s is rational.

EXERCISE:

Given any two distinct rational numbers r and s with r < s. Prove that there is a rational number x such that r < x < s.

SOLUTION:

Given two distinct rational numbers r and s such that

r < s ………………….(1)

Adding r to both sides of (1), we get

r + r < r + s

2r < r + s

$\Rightarrow$ $r < \frac{r+s}{2}$ ………………….(2)

Next adding s to both sides of (1), we get

r + s < s + s

$\Rightarrow$ r + s < 2s

$\Rightarrow$ $\frac{r+s}{2} < s$ ………………………(3)

Combining (2) and (3), we may write

$$r < \frac{r+s}{2} < s \quad \text{……………………..(4)}$$

Since the sum of two rationals is rational, therefore r + s is rational. Also the quotient of a rational by a non-zero rational, is rational, therefore $\frac{r+s}{2}$ is rational and by (4) it lies between r & s.

Hence, we have found a rational number such that r < x < s. (proved)

EXERCISE:

Prove that for all integers a, b and c, if a|b and b|c then a|c.

PROOF:

Suppose a|b and b|c where a, b, c $\in$ Z. Then by definition of divisibility b=a·r and c=b·s for some integers r and s.

Now c = b·s

= (a·r)·s (substituting value of b)

= a·(r·s) (associative law)

$= a \cdot k$ where $k = r \cdot s \in Z$

$\Rightarrow$ $a|c$ by definition of divisibility

EXERCISE:

Prove that for all integers a, b and c if a|b and a|c then a|(b+c)

PROOF:

Suppose a|b and a|c where $a, b, c \in Z$

By definition of divides

$b = a \cdot r$ and $c = a \cdot s$ for some $r, s \in Z$

Now

$b + c = a \cdot r + a \cdot s$ (substituting values)

$= a \cdot (r+s)$ (by distributive law)

$= a \cdot k$ where $k = (r + s) \in Z$

Hence $a|(b + c)$ by definition of divides.

EXERCISE:

Prove that the sum of any three consecutive integers is divisible by 3.

PROOF:

Let n, n + 1 and n + 2 be three consecutive integers.

Now

$n + (n + 1) + (n + 2) = 3n + 3$

$= 3(n + 1)$

$= 3 \cdot k$ where $k=(n+1) \in Z$

Hence, the sum of three consecutive integers is divisible by 3.

EXERCISE:

Prove the statement:

There is an integer n > 5 such that $2^n - 1$ is prime.

PROOF:

Here we are asked to show a single integer for which 2^n -1is prime. First of all we will check the integers from 1 and check whether the answer is prime or not by putting these values in 2^n-1.when we got the answer is prime then we will stop our process of checking the integers and we note that,

Let n = 7, then

$2^n - 1 = 2^7 - 1 = 128 - 1 = 127$

and we know that 127 is prime.

EXERCISE:

Prove the statement: There are real numbers a and b such that

$$\sqrt{a+b} = \sqrt{a} + \sqrt{b}$$

PROOF:

Let $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$

Squaring,we get $a + b = a + b + 2\sqrt{a}\sqrt{b}$

$\Rightarrow$ $0 = 2\sqrt{a}\sqrt{b}$ canceling a+b

$\Rightarrow \quad 0 = \sqrt{ab}$
$\Rightarrow \quad 0 = ab$ squaring
$\Rightarrow$ either a = 0 or b = 0
It means that if we want to find out the integers which satisfy the given condition then one of them must be zero.
Hence if we let a = 0 and b = 3 then

$R.H.S = \sqrt{a+b} = \sqrt{0+3}$

$R.H.S = \sqrt{3}$

$Now\ L.H.S = \sqrt{a} + \sqrt{b}$ *by putting the values of a and b we get*

$= \sqrt{0} + \sqrt{3}$

$L.H.S = \sqrt{3}$

From above it quite clear that the given condition is satisfied if we take a=0 and b=3.

PROOF BY COUNTER EXAMPLE:

Disprove the statement by giving a counter example.
For all real numbers a and b, if a < b then $a^2 < b^2$.

SOLUTION:

Suppose a = -5 and b = -2
then clearly - 5 < - 2
But $a^2 = (-5)^2 = 25$ and $b^2 = (-2)^2 = 4$
But 25 > 4
This disproves the given statement.

EXERCISE:

Prove or give counter example to disprove the statement.
For all integers n, $n^2 - n + 11$ is a prime number.

SOLUTION:

The statement is not true
For n = 11
we have , $n^2 - n + 11 = (11)^2 - 11 + 11$
$= (11)^2$
$= (11)(11)$
$= 121$
which is obviously not a prime number.

EXERCISE:

Prove or disprove that the product of any two irrational numbers is an irrational number.

SOLUTION:

We know that $\sqrt{2}$ is an irrational number. Now

$$(\sqrt{2})(\sqrt{2}) = (\sqrt{2})^2 = 2 = \frac{2}{1}$$

which is a rational number. Hence the statement is disproved.

EXERCISE:

Find a counter example to the proposition:
For every prime number n, n + 2 is prime.

SOLUTION:

Let the prime number n be 7 then
$n + 2 = 7 + 2 = 9$
which is not prime.

LECTURE # 26

PROOF BY CONTRADICTION:

A proof by contradiction is based on the fact that either a statement is true or it is false but not both. Hence the supposition, that the statement to be proved is false, leads logically to a contradiction, impossibility or absurdity, then the supposition must be false. Accordingly, the given statement must be true.

This method of proof is also known as reductio ad absurdum because it relies on reducing a given assumption to an absurdity.

Many theorems in mathematics are conditional statements ($p \to q$). Now the negation of he implication $p \to q$ is

$$\begin{aligned} \sim(p \to q) &\equiv \sim(\sim p \vee q) \\ &\equiv \sim(\sim p) \wedge (\sim q) \qquad \text{DeMorgan's Law} \\ &\equiv p \wedge \sim q \end{aligned}$$

Clearly if the implication is true, then its negation must be false, i.e., leads to a contradiction.

Hence $p \to q \equiv (p \wedge \sim q) \to c$

where c is a contradiction.

Thus to prove an implication $p \to q$ by contradiction method we suppose that the condition p and the negation of the conclusion q, i.e., $(p \wedge \sim q)$ is true and ultimately arrive at a contradiction.

The method of proof by contradiction, may be summarized as follows:

1. Suppose the statement to be proved is false.
2. Show that this supposition leads logically to a contradiction.
3. Conclude that the statement to be proved is true.

THEOREM:

There is no greatest integer.

PROOF:

Suppose there is a greatest integer N. Then $n \leq N$ for every integer n.

Let $M = N + 1$

Now M is an integer since it is a sum of integers.

Also $M > N$ since $M = N + 1$

Thus M is an integer that is greater than the greatest integer, which is a contradiction.

Hence our supposition is not true and so there is no greatest integer.

EXERCISE:

Give a proof by contradiction for the statement:

"If n^2 is an even integer then n is an even integer."

PROOF:

Suppose n^2 is an even integer and n is not even, so that n is odd.

Hence $n = 2k + 1$ for some integer k.

Now

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2\cdot(2k^2 + 2k) + 1 \end{aligned}$$

$= 2r + 1$ where $r = (2k^2 + 2k) \in Z$

This shows that n^2 is odd, which is a contradiction to our supposition that n^2 is even. Hence the given statement is true.

EXERCISE:

Prove that if n is an integer and $n^3 + 5$ is odd, then n is even using contradiction method.

SOLUTION:

Suppose that $n^3 + 5$ is odd and n is not even (odd). Since n is odd and the product of two odd numbers is odd, it follows that n^2 is odd and $n^3 = n^2 . n$ is odd. Further, since the difference of two odd number is even, it follows that

$$5 = (n^3 + 5) - n^3$$

is even. But this is a contradiction. Therefore, the supposition that $n^3 + 5$ and n are both odd is wrong and so the given statement is true.

EXERCISE:

Prove by contradiction method, the statement: If n and m are odd integers, then n + m is an even integer.

SOLUTION:

Suppose n and m are odd and n + m is not even (odd i.e by taking contradiction).

Now $n = 2p + 1$ for some integer p

and $m = 2q + 1$ for some integer q

Hence $n + m = (2p + 1) + (2q + 1)$

$= 2p + 2q + 2 = 2\cdot (p + q + 1)$

which is even, contradicting the assumption that n + m is odd.

THEOREM:

The sum of any rational number and any irrational number is irrational.

PROOF:

We suppose that the negation of the statement is true. That is, we suppose that there is a rational number r and an irrational number s such that r + s is rational. By definition of ration

$$r = \frac{a}{b} \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

and

$$r + s = \frac{c}{d} \qquad \ldots\ldots\ldots\ldots\ldots(2)$$

for some integers a, b, c and d with $b \neq 0$ and $d \neq 0$.

Using (1) in (2), we get

$$\frac{a}{b} + s = \frac{c}{d}$$

$$\Rightarrow \quad s = \frac{c}{d} - \frac{a}{b}$$

$$s = \frac{bc - ad}{bd} \qquad (bd \neq 0)$$

Now bc - ad and bd are both integers, since products and difference of integers are integers. Hence s is a quotient of two integers bc-ad and bd with bd $\neq$ 0. So by definition of rational, s is rational.

This contradicts the supposition that s is irrational. Hence the supposition is false and the theorem is true.

EXERCISE:

Prove that $\sqrt{2}$ is irrational.

PROOF:

Suppose $\sqrt{2}$ is rational. Then there are integers m and n with no common factors so that

$$\sqrt{2} = \frac{m}{n}$$

Squaring both sides gives

$$2 = \frac{m^2}{n^2}$$

Or $m^2 = 2n^2$ ……………………….(1)

This implies that m^2 is even (by definition of even). It follows that m is even. Hence

m = 2 k for some integer k (2)

Substituting (2) in (1), we get

$$(2k)^2 = 2n^2$$
$$\Rightarrow \quad 4k^2 = 2n^2$$
$$\Rightarrow \quad n^2 = 2k^2$$

This implies that n^2 is even, and so n is even. But we also know that m is even. Hence both m and n have a common factor 2. But this contradicts the supposition that m and n have no common factors. Hence our supposition is false and so the theorem is true.

Substituting (2) in (1), we get

$$(2k)^2 = 2n^2$$
$$\Rightarrow \quad 4k^2 = 2n^2$$
$$\Rightarrow \quad n^2 = 2k^2$$

This implies that n^2 is even, and so n is even. But we also know that m is even. Hence both m and n have a common factor 2. But this contradicts the supposition that m and n have no common factors. Hence our supposition is false and so the theorem is true.

EXERCISE:

Prove by contradiction that $6-7\sqrt{2}$ is irrational.

PROOF:

Suppose $6-7\sqrt{2}$ is rational.
Then by definition of rational,

$$6-7\sqrt{2}=\frac{a}{b}$$

for some integers a and b with $b\neq0$.
Now consider,

$$7\sqrt{2}=6-\frac{a}{b}$$

$$\Rightarrow \quad 7\sqrt{2}=\frac{6b-a}{b}$$

$$\Rightarrow \quad \sqrt{2}=\frac{6b-a}{7b}$$

Since a and b are integers, so are 6b-a and 7b and $7b\neq0$;
hence $\sqrt{2}$ is a quotient of the two integers 6b-a and 7b with $7b\neq0$.
Accordingly, $\sqrt{2}$ is rational (by definition of rational).
This contradicts the fact because $\sqrt{2}$ is irrational.
Hence our supposition is false and so $6-7\sqrt{2}$ is irrational.

EXERCISE:

Prove that $\sqrt{2}+\sqrt{3}$ is irrational.

SOLUTION:

Suppose $\sqrt{2}+\sqrt{3}$ is rational. Then, by definition of rational, there exists integers a and b with $b\neq0$ such that

$$\sqrt{2}+\sqrt{3}=\frac{a}{b}$$

Squaring both sides, we get

$$2+3+2\sqrt{2}\sqrt{3}=\frac{a^2}{b^2}$$

$$\Rightarrow \qquad 2\sqrt{2\times 3}=\frac{a^2}{b^2}-5$$

$$\Rightarrow \qquad 2\sqrt{6}=\frac{a^2-5b^2}{b^2}$$

$$\Rightarrow \qquad \sqrt{6}=\frac{a^2-5b^2}{2b^2}$$

Since a and b are integers, so are therefore $a^2 - 5b^2$ and $2b^2$ with $2b^2 \neq 0$. Hence $\sqrt{6}$ is the quotient of two integers $a^2 - 2b^2$ and $2b^2$ with $2^2 \neq 0$. Accordingly, $\sqrt{6}$ is rational. But this is a contradiction, since $\sqrt{6}$ is not rational. Hence our supposition is false and so $\sqrt{2}+\sqrt{3}$ is irrational.

REMARK:

The sum of two irrational numbers need not be irrational in general for

$(6-7\sqrt{2})+(6+7\sqrt{2})=6+6=12$

which is rational.

EXERCISE:

Prove that for any integer a and any prime number p, if p|a, then $P \nmid (a + 1)$.

PROOF:

Suppose there exists an integer **a** and a prime number **p** such that p|a and p|(a+1). Then by definition of divisibility there exist integer r and s so that

$a = p \cdot r$ and $a + 1 = p \cdot s$

It follows that

$1 = (a + 1) - a$

$= p \cdot s - p \cdot r$

$= p \cdot (s-r)$ where $s - r \in Z$

This implies p|1.

But the only integer divisors of 1 are 1 and -1 and since p is prime p>1. This is a contradiction.

Hence the supposition is false, and the given statement is true.

THEOREM:

The set of prime numbers is infinite.

PROOF:

Suppose the set of prime numbers is finite.

Then, all the prime numbers can be listed, say, in ascending order:

$p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ p_4 = 7,\ \ldots,\ p_n$

Consider the integer

$N = p_1.p_2.p_3. \ldots . p_n + 1$

Then N > 1. Since any integer greater than 1 is divisible by some prime number p, therefore $p \mid N$.

Also since p is prime, p must equal one of the prime numbers

$p_1, p_2, p_3, \ldots , p_n$.

Thus

$P \mid (p_1, p_2, p_3, \ldots , p_n)$

But then

$P \nmid (p_1, p_2, p_3, \ldots , p_n + 1)$

So $P \nmid N$

Thus $p \mid N$ and $p \nmid N$, which is a contradiction.

Hence the supposition is false and the theorem is true.

PROOF BY CONTRAPOSITION:

A proof by contraposition is based on the logical equivalence between a statement and its contrapositive. Therefore, the implication $p \to q$ can be proved by showing that its contrapositive $\sim q \to \sim p$ is true. The contrapositive is usually proved directly.

The method of proof by contrapositive may be summarized as:

1. Express the statement in the form if p then q.
2. Rewrite this statement in the contrapositive form if not q then not p.
3. Prove the contrapositive by a direct proof.

EXERCISE:

Prove that for all integers n, if n^2 is even then n is even.

PROOF:

The contrapositive of the given statement is:

"if n is not even (odd) then n^2 is not even (odd)"

We prove this contrapositive statement directly.

Suppose n is odd. Then $n = 2k + 1$ for some $k \in Z$

Now $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$

$= 2\cdot(2k^2 + 2k) + 1$

$= 2\cdot r + 1$ where $r = 2k^2 + 2k \in Z$

Hence n^2 is odd. Thus the contrapositive statement is true and so the given statement is true.

EXERCISE:

Prove that if $3n + 2$ is odd, then n is odd.

PROOF:

The contrapositive of the given conditional statement is
" if n is even then $3n + 2$ is even"
Suppose n is even, then

$n = 2k$ for some k €Z

Now $3n + 2 = 3 (2k) + 2$
$= 2. (3k + 1)$
$= 2.r$ where $r = (3k + 1)$ € Z

Hence $3n + 2$ is even. We conclude that the given statement is true since its contrapositive is true.

EXERCISE:

Prove that if n is an integer and $n^3 + 5$ is odd, then n is even.

PROOF:

Suppose n is an odd integer. Since, a product of two odd integers is odd, therefore $n^2 = n.n$ is odd; and $n^3 = n^2.n$ is odd.
Since a sum of two odd integers is even therefore $n^2 + 5$ is even.
Thus we have prove that if n is odd then $n^3 + 5$ is even.
Since this is the contrapositive of the given conditional statement, so the given statement is true.

EXERCISE:

Prove that if n^2 is not divisible by 25, then n is not divisible by 5.

SOLUTION:

The contra positive statement is:
"if n is divisible by 5, then n^2 is divisible by 25"
Suppose n is divisible by 5. Then by definition of divisibility

$n = 5\cdot k$ for some integer k

Squaring both sides

$n^2 = 25\cdot k^2$ where k^2 € Z

n^2 is divisible by 25

EXERCISE:

Prove that if $|x| > 1$ then $x > 1$ or $x < -1$ for all x € R.

PROOF:

The contrapositive statement is:
if $x \leq 1$ and $x \geq -1$ then $|x| \leq 1$ for x € R.
Suppose that $x \leq 1$ and $x \geq -1$

$\Rightarrow$ $x \leq 1$ and $x \geq -1$

$\Rightarrow$ $-1 \leq x \leq 1$

and so

$|x| \leq 1$

Equivalently $|x| > 1$

EXERCISE:

Prove the statement by contraposition:

For all integers m and n, if m + n is even then m and n are both even or m and n are both odd.

PROOF:

The contrapositive statement is:

"For all integers m and n, if m and n are not both even and m and n are not both odd, then m + n is not even.

Or more simply,

"For all integers m and n, if one of m and n is even and the other is odd, then m + n is odd"

Suppose m is even and n is odd. Then

$m = 2p$ for some integer p

and $n = 2q + 1$ for some integer q

Now $m + n = (2p) + (2q + 1)$

$= 2\cdot(p+q) + 1$

$= 2\cdot r + 1$ where $r = p+q$ is an integer

Hence m + n is odd.

Similarly, taking m as odd and n even, we again arrive at the result that m + n is odd. Thus, the contrapositive statement is true. Since an implication is logically equivalent to its contrapositive so the given implication is true.