LECTURE 31

K-COMBINATIONS
K-SELECTIONS

K-COMBINATIONS:

With a *k*-combination the order in which the elements are selected does not matter and the elements cannot repeat.

DEFINITION:

A *k*-combination of a set of *n* elements is a choice of *k* elements taken from the set of *n* elements such that the order of the elements does not matter and elements can't be repeated.

The symbol *C*(*n*, *k*) denotes the number of *k*-combinations that can be chosen from a set of *n* elements.

NOTE:

k-combinations are also written ${}^{n}C_k$ as or $\binom{n}{k}$

REMARK:

With *k*-combinations of a set of *n* elements, repetition of elements is not allowed, therefore, *k* must be less than or equal to *n*, i.e., $k \le n$.

EXAMPLE:

Let *X* = {*a*, *b*, *c*}. Then 2-combinations of the 3 elements of the set *X* are:
{*a*, *b*}, {*a*, *c*}, and {*b*, *c*}. Hence *C*(3,2) = 3.

EXERCISE:

Let *X* = {*a*, *b*, *c*, *d*, *e*}.
List all 3-combinations of the 5 elements of the set *X*, and hence find the value of *C*(5,3).

SOLUTION:

Then 3-combinations of the 5 elements of the set *X* are:
{*a*, *b*, *c*}, {*a*, *b*, *d*}, {*a*, *b*, *e*}, {*a*, *c*, *d*}, {*a*, *c*, *e*},
{*a*, *d*, *e*}, {*b*, *c*, *d*}, {*b*, *c*, *e*}, {*b*, *d*, *e*}, {*c*, *d*, *e*}
Hence *C*(5, 3) = 10

PERMUTATIONS AND COMBINATIONS:

EXAMPLE:

Let *X* = {*A*, *B*, *C*, *D*}.
The 3-combinations of *X* are:
{*A*, *B*, *C*}, {*A*, *B*, *D*}, {*A*, *C*, *D*}, {*B*, *C*, *D*}
Hence *C*(4, 3) = 4

The 3-permutations of *X* can be obtained from 3-combinations of X as follows.
ABC, *ACB*, *BAC*, *BCA*, *CAB*, *CBA*
ABD, *ADB*, *BAD*, *BDA*, *DAB*, *DBA*
ACD, *ADC*, *CAD*, *CDA*, *DAC*, *DCA*
BCD, *BDC*, *CBD*, *CDB*, *DBC*, *DCB*
So that *P*(4, 3) = 24 = 4 · 6 = 4 · 3!
Clearly *P*(4, 3) = *C*(4, 3) · 3!
In general we have, *P*(*n*, *k*) = *C*(*n*, *k*) · *k*!
In general we have,

$$P(n, k) = C(n, k) \cdot k!$$

or

$$C(n,k) = \frac{P(n,k)}{k!}$$

But we know that $P(n,k) = \dfrac{n!}{(n-k)!}$

Hence, $C(n,k) = \dfrac{n!}{(n-k)!k!}$

COMPUTING *C*(*n*, *k*)

EXAMPLE:

Compute $C(9, 6)$.

SOLUTION:
$$C(9,6) = \frac{9!}{(9-6)!6!} = \frac{9\cdot 8\cdot 7\cdot 6!}{3!\cdot 6!} = \frac{9\cdot 8\cdot 7}{3\cdot 2} = 84$$

SOME IMPORTANT RESULTS

(a) $C(n, 0) = 1$
(b) $C(n, n) = 1$
(c) $C(n, 1) = n$
(d) $C(n, 2) = n(n-1)/2$
(e) $C(n, k) = C(n, n - k)$
(f) $C(n, k) + C(n, k + 1) = C(n + 1, k + 1)$

EXERCISE:

A student is to answer eight out of ten questions on an exam.
(a) Find the number m of ways that the student can choose the eight questions
(b) Find the number m of ways that the student can choose the eight questions, if the first three questions are compulsory.

SOLUTION:

(a) The eight questions can be answered in $m = C(10, 8) = 45$ ways.
(b) The eight questions can be answered in $m = C(7, 5) = 21$ ways.

EXERCISE:

An examination paper consists of 5 questions in section A and 5 questions in section B. A total of 8 questions must be answered. In how many ways can a student select the questions if he is to answer at least 4 questions from section A.

SOLUTION:

There are two possibilities:
(a) The student answers 4 questions from section A and 4 questions from section B.The number of ways 8 questions can be answered taking 4 questions from section A and 4 questions from section B are
$$C(5, 4) \cdot C(5, 4) = 5 \cdot 5 = 25.$$
(b) The student answers 5 questions from section A and 3 questions from section B.The number of ways 8 questions can be answered taking 5 questions from section A and 3 questions from section B are
$$C(5, 5) \cdot C(5, 3) = 1 \cdot 10 = 10.$$
Thus there will be a total of 25 + 10 = 35 choices.

EXERCISE:

A computer programming team has 14 members.
(a) How many ways can a group of seven be chosen to work on a project?

(b) Suppose eight team members are women and six are men
(i) How many groups of seven can be chosen that contain four women and three men
(ii) How many groups of seven can be chosen that contain at least one man?
(iii)How many groups of seven can be chosen that contain at most three women?
(c) Suppose two team members refuse to work together on projects. How many groups of seven can be chosen to work on a project?
(d) Suppose two team members insist on either working together or not at all on projects. How many groups of seven can be chosen to work on a project?
(a) How many ways a group of 7 be chosen to work on a project?

SOLUTION:

Number of committees of 7

$$
\begin{aligned}
C(14,7) &= \frac{14!}{(14-7)!\cdot 7!} \\
&= \frac{14\cdot 13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} \\
&= 3432
\end{aligned}
$$

(b) Suppose eight team members are women and six are men
(i) How many groups of seven can be chosen that contain four women and three men?

SOLUTION:

Number of groups of seven that contain four women and three men

$$
\begin{aligned}
C(8,4)\cdot C(6,3) &= \frac{8!}{(8-4)!\cdot 4!}\cdot\frac{6!}{(6-3)!\cdot 3!} \\
&= \frac{8\cdot 7\cdot 6\cdot 5}{4!}\cdot\frac{6\cdot 5\cdot 4}{3!} \\
&= \frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2}\cdot\frac{6\cdot 5\cdot 4}{3\cdot 2} \\
&= 70\cdot 20 = 1400
\end{aligned}
$$

(b) Suppose eight team members are women and six are men
(ii) How many groups of seven can be chosen that contain at least one man?

SOLUTION:

Total number of groups of seven

$$
\begin{aligned}
C(14,7) &= \frac{14!}{(14-7)!\cdot 7!} \\
&= \frac{14\cdot 13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} \\
&= 3432
\end{aligned}
$$

Number of groups of seven that contain no men

$$
\begin{aligned}
C(8,7) &= \frac{8!}{(8-7)!\cdot 7!} \\
&= 8
\end{aligned}
$$

Hence, the number of groups of seven that contain at least one man
$C(14,7) - C(8,7) = 3432 - 8 = 3424$

(b)Suppose eight team members are women and six are men

(iii) How many groups of seven can be chosen that contain at most three women?

SOLUTION:

Number of groups of seven that contain no women = 0

Number of groups of seven that contain one woman = $C(8,1) \cdot C(6,6)$
$= 8 \cdot 1 = 8$

Number of groups of seven that contain two women = $C(8,2) \cdot C(6,5)$
$= 28 \cdot 6 = 168$

Number of groups of seven that contain three women = $C(8,3) \cdot C(6,4)$
$= 56 \cdot 15 = 840$

Hence the number of groups of seven that contain at most three women
$= 0 + 8 + 168 + 840 = 1016$

(c) Suppose two team members refuse to work together on projects. How many groups of seven can be chosen to work on a project?

SOLUTION:

Call the members who refuse to work together *A* and *B*.

Number of groups of seven that contain neither *A* nor *B*

$$C(12,7) = \frac{12!}{(12-7)! \cdot 7!}$$
$$= 792$$

Number of groups of seven that contain *A* but not *B*

$C(12, 6) = 924$

Number of groups of seven that contain *B* but not *A*

$C(12,6) = 924$

Hence the required number of groups are

$C(12,7) + C(12,6) + C(12, 6)$
$= 792 + 924 + 924$
$= 2640$

(d)Suppose two team members insist on either working together or not at all on projects.

How many groups of seven can be chosen to work on a project?

SOLUTION:

Call the members who insist on working together *C* and *D*.

Number of groups of seven containing neither *C* nor *D*

$C(12, 7) = 792$

Number of groups of seven that contain both *C* and *D*

$C(12, 5) = 792$

Hence the required number
$= C(12, 7) + C(12, 5)$
$= 792 + 792 = 1584$

EXERCISE:

(a) How many 16-bit strings contain exactly 9 1's?

(b)How many 16-bit strings contain at least one 1?

SOLUTION:

(a) 16-bit strings that contain exactly 9 1's= $C(16,9) = \frac{16!}{(16-9)!9!} = 11440$

(b) Total no. of 16-bit strings = 2^{16}

Hence number of 16-bit strings that contain at least one 1

$2^{16} - 1 = 65536 - 1$

= 65535

K-SELECTIONS:

k-selections are similar to *k*-combinations in that the order in which the elements are selected does not matter, but in this case repetitions can occur.

DEFINITION:

A *k*-selection of a set of *n* elements is a choice of *k* elements taken from a set of *n* elements such that the order of elements does not matter and elements can be repeated.

REMARK:

1. *k*-selections are also called *k*-combinations with repetition allowed or multisets of size *k*.
2. With *k*-selections of a set of *n* elements repetition of elements is allowed. Therefore *k* need not to be less than or equal to *n*.

THEOREM:

The number of *k*-selections that can be selected from a set of *n* elements is

$$C(k+n-1, k) \text{ or } {}^{k+n-1}C_{k}$$

EXERCISE:

A camera shop stocks ten different types of batteries.
(a) How many ways can a total inventory of 30 batteries be distributed among the ten different types?
(b) Assuming that one of the types of batteries is A76, how many ways can a total inventory of 30 batteries be distributed among the 10 different types if the inventory must include at least four A76 batteries?

SOLUTION:

(a) $k = 30$
$n = 10$

The required number is

$$C(30 + 10 - 1, 30) = C(39, 30) = \frac{39!}{(39-30)!30!} = 211915132$$

(b) $k = 26$
$n = 10$

The required number is

$$C(26 + 10 - 1, 26) = C(35, 26) = \frac{35!}{(35-26)!26!} = 70607460$$

WHICH FORMULA TO USE?

	ORDER MATTERS	ORDER DOES NOT MATTER
REPETITION ALLOWED	k-sample n^k	k-selection C(n+k-1, k)
REPETITION NOT ALLOWED	k-permutation P(n, k)	k-combination C(n, k)

LECTURE 32

ORDERED AND UNORDERED PARTITIONS
PERMUTATIONS WITH REPETITIONS

K-SELECTIONS:

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REMARK:

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SOLUTION:

(a) $k = 30$
$n = 10$

The required number is

$$C(30 + 10 - 1, 30) = C(39, 30)$$

$$= \frac{39!}{(39-30)!30!}$$

$$= 211915132$$

(b) $k = 26$
$n = 10$

The required number is

$$C(26 + 10 - 1, 26) = C(35, 26)$$

$$= \frac{35!}{(35-26)!26!}$$

$$= 70607460$$

WHICH FORMULA TO USE?

	ORDER MATTERS	ORDER DOES NOT MATTER
REPETITION ALLOWED	k-sample n^k	k-selection C(n+k-1, k)
REPETITION NOT ALLOWED	k-permutation P(n, k)	k-combination C(n, k)

ORDERED AND UNORDERED PARTITIONS:

An unordered partition of a finite set S is a collection $[A_1, A_2, \ldots, A_k]$ of disjoint (nonempty) subsets of S (called cells) whose union is S.
The partition is ordered if the order of the cells in the list counts.

EXAMPLE:

Let S = {1, 2, 3, …, 7}

The collections

$P_1 = [\{1,2\}, \{3,4,5\}, \{6,7\}]$

And $P_2 = [\{6,7\}, \{3,4,5\}, \{1,2\}]$

determine the same partition of S but are distinct ordered partitions.

EXAMPLE:

Suppose a box **B** contains seven marbles numbered 1 through 7. Find the number m of ways of drawing from **B** firstly two marbles, then three marbles and lastly the remaining two marbles.

SOLUTION:

The number of ways of drawing 2 marbles from 7 = C(7, 2)
Following this, there are five marbles left in B.
The number of ways of drawing 3 marbles from 5 = C(5, 3)
Finally, there are two marbles left in B.
The number of way of drawing 2 marbles from 2 = C(2, 2)

Thus

$$\begin{aligned} m &= \binom{7}{2}\binom{5}{3}\binom{2}{2} \\ &= \frac{7!}{2!5!}\cdot\frac{5!}{2!3!}\cdot\frac{2!}{2!0!} \\ &= \frac{7!}{2!3!2!} = 210 \end{aligned}$$

THEOREM:

Let S contain n elements and let $n_1, n_2, \ldots, n_k$ be positive integers with

$$n_1+n_2+\ldots+n_k = n.$$

Then there exist $\dfrac{n!}{n_1!n_2!n_3!\cdots n_k!}$

different ordered partitions of S of the form $[A_1, A_2, \ldots, A_k]$, where

A_1 contains n_1 elements

A_2 contains n_2 elements

A_3 contains n_3 elements

………………………..

A_k contains n_k elements

REMARK:

To find the number of unordered partitions, we have to count the ordered partitions and then divide it by suitable number to erase the order in partitions.

EXERCISE:

Find the number m of ways that nine toys can be divided among four children if the youngest child is to receive three toys and each of the others two toys.

SOLUTION:

We find the number m of ordered partitions of the nine toys into four cells containing 3, 2, 2 and 2 toys respectively.

Hence

$$m = \frac{9!}{3!2!2!2!} = 2520$$

EXERCISE:

How many ways can 12 students be divided into 3 groups with 4 students in each group so that

(i) one group studies English, one History and one Mathematics.

(ii) all the groups study Mathematics.

SOLUTION:

(i) Since each group studies a different subject, so we seek the number of ordered partitions of the 12 students into cells containing 4 students each. Hence there are

$$\frac{12!}{4!4!4!} = 34,650$$ such partitions

(ii) When all the groups study the same subject, then order doesn't matter.

Now each partition $\{G_1, G_2, G_3\}$ of the students can be arranged in 3! ways as an ordered partition, hence there are

$$\frac{12!}{4!4!4!} \times \frac{1}{3!}$$

unordered partitions.

EXERCISE:

How many ways can 8 students be divided into two teams containing

(i) five and three students respectively.

(ii) four students each.

SOLUTION:

(i) The two teams (cells) contain different number of students; so the number of unordered partitions equals the number of ordered partitions, which is

$$\frac{8!}{5!3!} = 56$$

(ii) Since the teams are not labeled, so we have to find the number of unordered partitions of 8 students in groups of 4.

Firstly, note, there are $\frac{8!}{4!4!} = 70$ ordered partitions into two cells containing four students each.

Since each unordered partition determine 2! = 2 ordered partitions, there are

$$\frac{70}{2} = 35$$

unordered partitions

EXERCISE:

Find the number m of ways that a class X with ten students can be partitions into four teams A_1, A_2, B_1 and B_2 where A_1 and A_2 contain two students each and B_1 and B_2 contain three students each.

SOLUTION:

There are $\frac{10!}{2!2!3!3!} = 25,200$ ordered partitions of X into four cells containing 2, 2, 3 and 3 students respectively.

However, each unordered partition $[A_1, A_2, B_1, B_2]$ of X determines

$2!\cdot 2! = 4$ ordered partitions of X.

Thus,

$$m = \frac{25,200}{4} = 6300$$

EXERCISE:

Suppose 20 people are divided in 6 (numbered) committees so that 3 people each serve on committee C_1 and C_2, 4 people each on committees C_3 and C_4, 2 people on committee C_5 and 4 people on committee C_6. How many possible arrangements are there?

SOLUTION:

We are asked to count labeled group - the committee numbers labeled the group.So this is a problem of ordered partition. Now, the number of ordered partitions of 20 people into the specified committees is

$$\frac{20!}{3!3!4!4!2!4!} = 2444321880000$$

EXERCISE:

If 20 people are divided into teams of size 3, 3, 4, 4, 2, 4, find the number of possible arrangements.

SOLUTION:

Here, we are asked to count unlabeled groups. Accordingly, this is the case of ordered partitions.

Now number of ordered partitions $= \frac{20!}{3!3!4!4!2!4!} \times \frac{1}{3!2!}$

$$= 203693490000$$

GENERALIZED PERMUTATION or PERMUTATIONS WITH REPETITIONS:

The number of permutations of n elements of which n_1 are alike, n_2 are alike, …, n_k are alike is

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

REMARK:

The number $\frac{n!}{n_1!n_2!\cdots n_k!}$ is often called a multinomial coefficient, and is denoted by the symbol.

$$\binom{n}{n_1, n_2, \cdots, n_k}$$

EXERCISE:

Find the number of distinct permutations that can be formed using the letters of the word “BENZENE”.

SOLUTION:

The word "BENZENE" contains seven letters of which three are alike (the 3 E's) and two are alike (the 2 N's)

Hence, the number of distinct permutations are: $\dfrac{7!}{3!2!} = 420$

EXERCISE:

How many different signals each consisting of six flags hung in a vertical line, can be formed from four identical red flags and two identical blue flags?

SOLUTION:

We seek the number of permutations of 6 elements of which 4 are alike and 2 are alike.

There are $\dfrac{6!}{4!2!} = 15$ different signals.

EXERCISE:

(i)Find the number of "***words***" that can be formed of the letters of the word **ELEVEN.**

(ii)Find, if the *words* are to begin with L.

(iii)Find, if the *words* are to begin and end in E.

(iv)Find, if the *words* are to begin with E and end in N.

SOLUTION:

(i) There are six letters of which three are E; hence required number of "*words*" are

$$\frac{6!}{3!} = 120$$

(ii) If the first letter is L, then there are five positions left to fill where three are E; hence required number of *words* are $\dfrac{5!}{3!} = 20$

(iii) If the *words* are to begin and end in E, then there are only four positions to fill with four distinct letters.

Hence required number of words = 4! = 24

(iv) If the *words* are to begin with E and end in N, then there are four positions left to fill where two are E.

Hence required number of words = $\dfrac{4!}{2!} = 12$

EXERCISE:

(i)Find the number of permutations that can be formed from all the letters of the word **BASEBALL**

(ii)Find, if the two B's are to be next to each other.

(iii)Find, if the words are to begin and end in a vowel.

SOLUTION:

(i) There are eight letter of which two are B, two are A, and two are L. Thus,

Number of permutations $= \dfrac{8!}{2!2!2!}$

$= 5040$

(ii) Consider the two B's as one letter. Then there are seven letters of which two are A and two are L. Hence,

Number of permutations $= \dfrac{7!}{2!2!}$

$= 1260$

(iii) There are three possibilities, the words begin and end in A, the words begin in A and end in E, or the words begin in E and end in A.
In each case there are six positions left to fill where two are B and two are L. Hence,

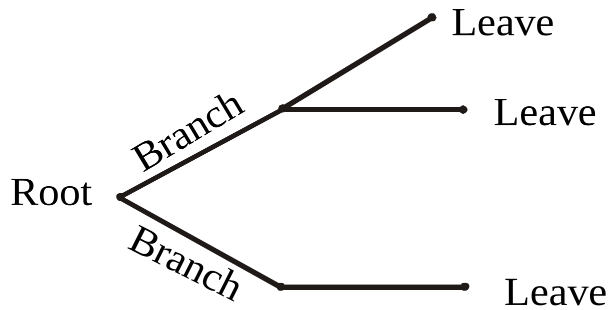
Number of permutations $= 3\dfrac{6!}{2!2!} = 540$

LECTURE 33

TREE DIAGRAM
INCLUSION - EXCLUSION PRINCIPLE

TREE DIAGRAM:

A tree diagram is a useful tool to list all the logical possibilities of a sequence of events where each event can occur in a finite number of ways.
A tree consists of a root, a number of branches leaving the root, and possible additional branches leaving the end points of other branches. To use trees in counting problems, we use a branch to represent each possible choice. The possible outcomes are represented by the leaves (end points of branches).
A tree is normally constructed from left to right.



A TREE STRUCTURE

EXAMPLE:

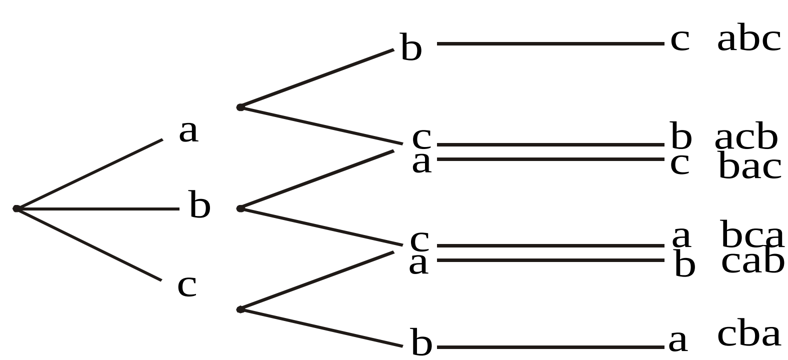
Find the permutations of {a, b, c}

SOLUTION:

The number of permutations of 3 elements is

$$P(3,3) = \frac{3!}{(3-3)!} = 3! = 6$$

We find the six permutations by constructing the appropriate tree diagram. The six permutations are listed on the right of the diagram.

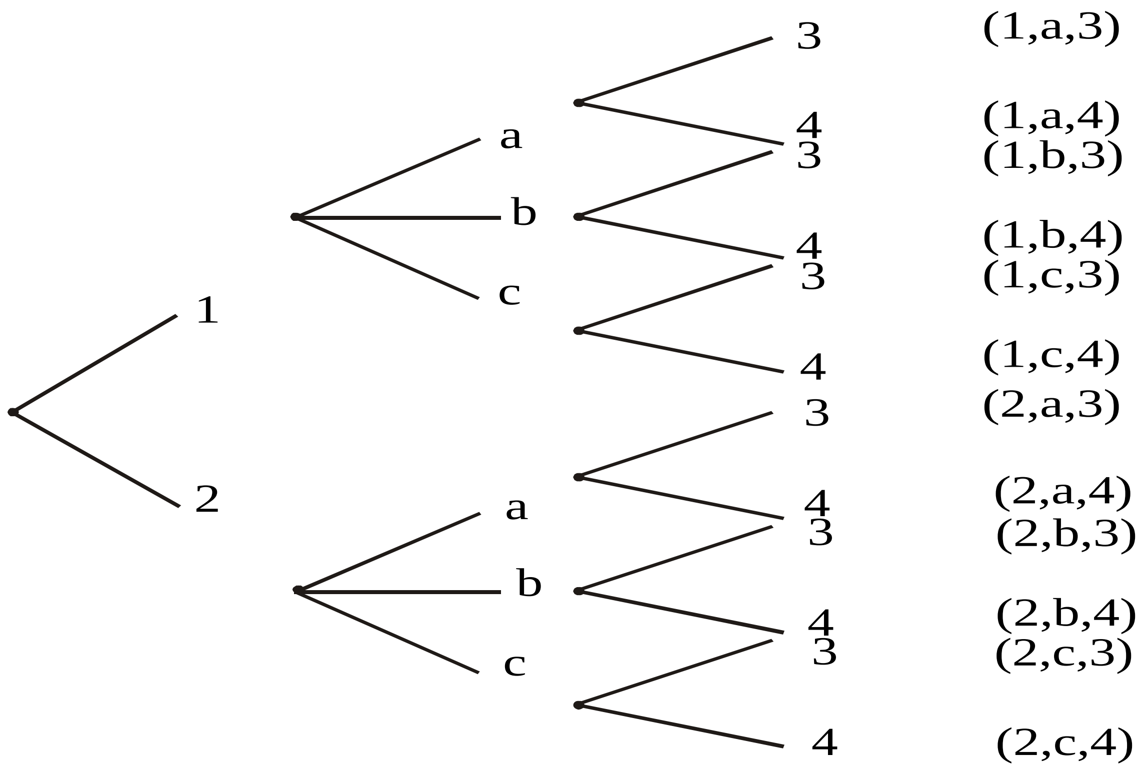


EXERCISE:

Find the product set A × B × C, where
A = {1,2}, B = {a,b,c}, and C = {3,4} by constructing the appropriate tree diagram.

SOLUTION:

The required diagram is shown next. Each path from the beginning of the tree to the end point designates an element of A × B × C which is listed to the right of the tree.

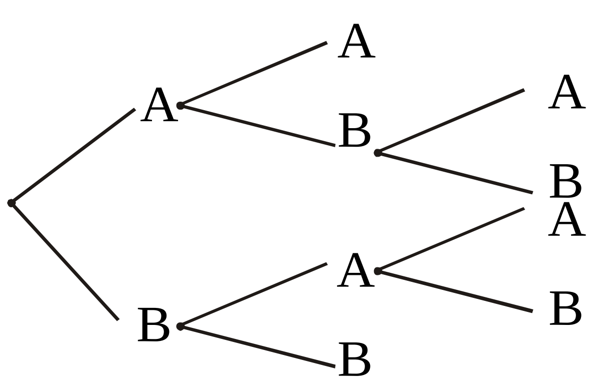


EXERCISE:

Teams A and B play in a tournament. The team that first wins two games wins the tournament. Find the number of possible ways in which the tournament can occur.

SOLUTION:

We construct the appropriate tree diagram.



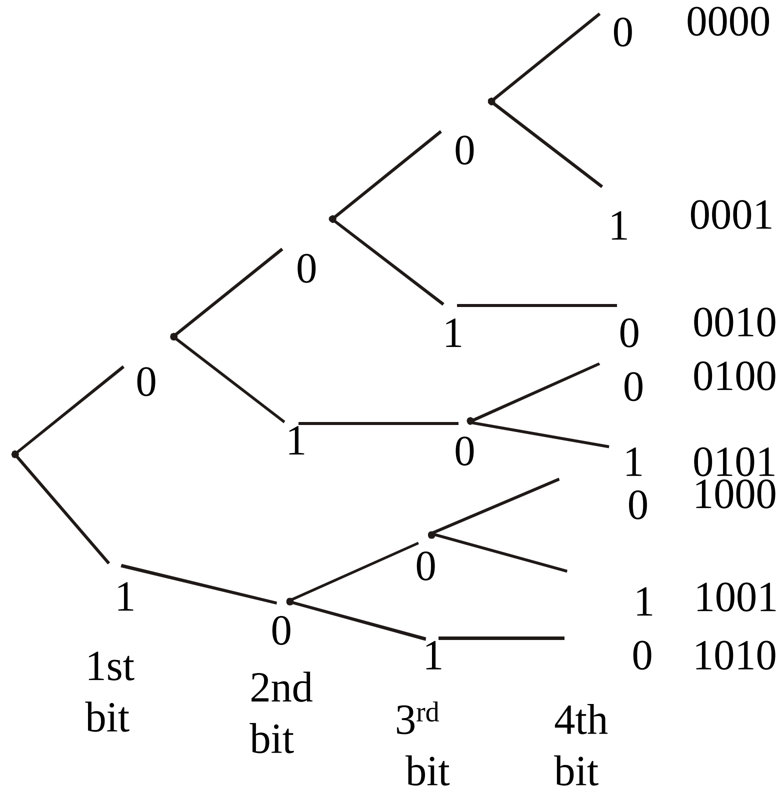
The tournament can occur in 6 ways: AA, ABA, ABB, BAA, BAB, BB

EXERCISE:

How many bit strings of length four do not have two consecutive 1's?

SOLUTION:

The following tree diagrams displays all bit strings of length four without two consecutive 1's. Clearly, there are 8 bit strings.



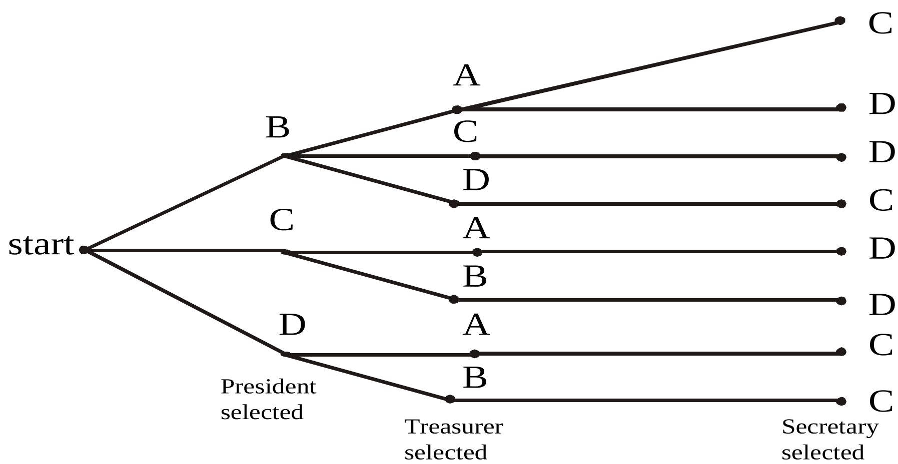
EXERCISE:

Three officers, a president, a treasurer, and a secretary are to be chosen from among four possible: A, B, C and D. Suppose that A cannot be president and either C or D must be secretary.

How many ways can the officers be chosen?

SOLUTION:

We construct the possibility tree to see all the possible choices.



From the tree, see that there are only eight ways possible to choose the offices under given conditions.

THE INCLUSION-EXCLUSION PRINCIPLE:

1 .If A and B are disjoint finite sets, then

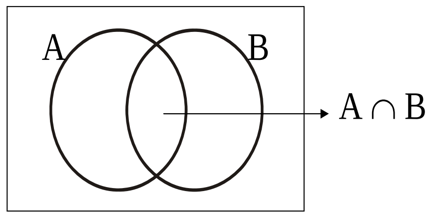
$n(A\cup B) = n(A) + n(B)$

2.If A and B are finite sets, then

$n(A\cup B) = n(A) + n(B) - n(A\cap B)$

REMARK:

Statement 1 follows from the sum rule
Statement 2 follows from the diagram



In counting the elements of $A \cup B$, we count the elements in A and count the elements in B.

There are n(A) in A and n(B) in B. However, the elements in $A \cap B$ were counted twice. Thus we subtract $n(A \cap B)$ from $n(A) + n(B)$ to get $n(A \cup B)$.
Hence,

$$n(A\cup B) = n(A) + n(B) - n(A\cap B)$$

EXAMPLE:

There are 15 girls students and 25 boys students in a class. How many students are there in total?

SOLUTION:

Let G be the set of girl students and B be the set of boy students.

Then $n(G) = 15$; $n(B) = 25$

and $n(G \cup B) = ?$

Since, the sets of boy and girl students are disjoint; here total number of students are

$$\begin{aligned} n(G \cup B) &= n(G) + n(B) \\ &= 15 + 25 \\ &= 40 \end{aligned}$$

EXERCISE:

Among 200 people, 150 either swim or jog or both. If 85 swim and 60 swim and jog, how many jog?

SOLUTION:

Let U be the set of people considered. Suppose S be the set of people who swim and J be the set of people who jog. Then given $n(U) = 200$; $n(S \cup J) = 150$

$n(S) = 85$; $n(S \cap J) = 60$ and $n(J) = ?$

By inclusion - exclusion principle,

$$n(S \cup J) = n(S) + n(J) - n(S \cap J)$$

$$150 = 85 + n(J) - 60$$

$$\Rightarrow \quad n(J) = 150 - 85 + 60$$

$$= 125$$

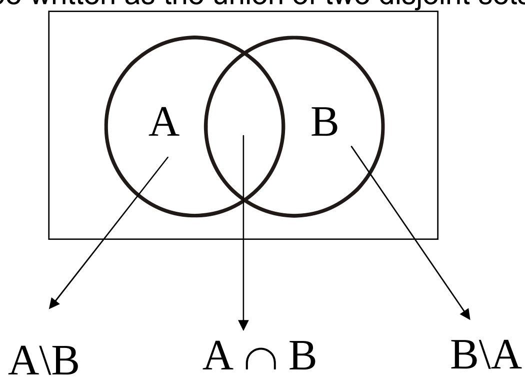
Hence 125 people jog.

EXERCISE:

Suppose A and B are finite sets. Show that
$n(A\backslash B) = n(A) - n(A \cap B)$

SOLUTION:

Set A may be written as the union of two disjoint sets $A\backslash B$ and $A\cap B$.



i.e., $A = (A\backslash B) \cup (A\cap B)$
Hence, by inclusion exclusion principle (for disjoint sets)

$$n(A) = n(A\backslash B) + n(A \cap B)$$
$$\Rightarrow \quad n(A\backslash B) = n(A) - n(A \cap B)$$

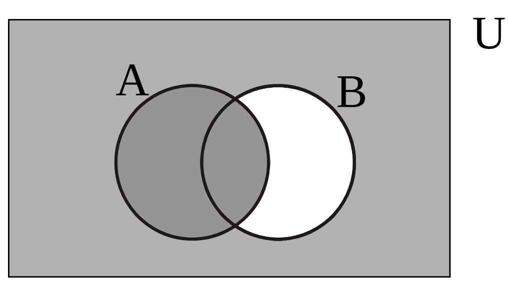
REMARK:

$$\begin{aligned} n(A') &= n(U\backslash A) \qquad \text{where U is the universal set} \\ &= n(U) - n(U \cap A) \\ &= n(U) - n(A) \end{aligned}$$

EXERCISE:

Let A and B be subsets of U with $n(A) = 10$, $n(B) = 15$, $n(A')=12$, and $n(A\cap B) = 8$. Find $n(A\cup B')$.

SOLUTION:



From the diagram $A \cup B' = U \backslash (B \backslash A)$
Hence

$$\begin{aligned} n(A \cup B') &= n(U \backslash (B\backslash A)) \\ &= n(U) - n(B\backslash A) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1) \end{aligned}$$

Now $U = A \cup A'$ where A & A′ are disjoint sets

$$\begin{aligned} \Rightarrow \quad n(U) &= n(A) + n(A') \\ &= 10 + 12 \\ &= 22 \end{aligned}$$

Also

$$n(B\backslash A) = n(B) - n(A\cap B)$$

$= 15 - 8$
$= 7$

Substituting values in (1) we get

$n(A \cup B') = n(U) - n(B \setminus A)$
$= 22 - 7$
$= 15$ Ans.

EXERCISE:

Let A and B are subset of U with n(U) = 100, n(A) = 50, n(B)= 60, and $n((A\cup B)') = 20$. Find $n(A\cap B)$

SOLUTION:

Since $(A \cup B)' = U \setminus (A \cup B)$

$\Rightarrow \quad n((A \cup B)') = n(U) - n(A \cup B)$

$\Rightarrow \quad 20 = 100 - n(A \cup B)$

$\Rightarrow \quad n(A \cup B) = 100 - 20 = 80$

Now, by inclusion - exclusion principle

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\Rightarrow \quad 80 = 50 + 60 - n(A \cap B)$

$\Rightarrow \quad n(A \cap B) = 50 + 60 - 80 = 30$

EXERCISE:

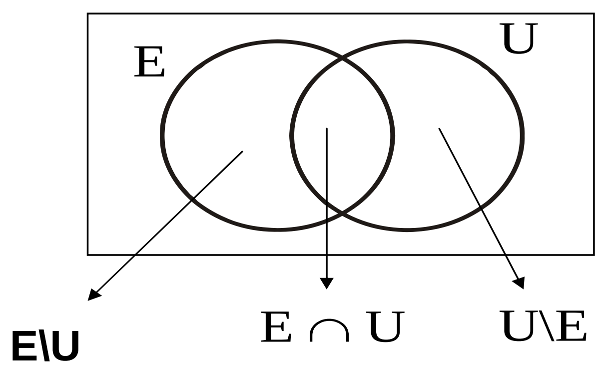
Suppose 18 people read English newspaper (E) or Urdu newspaper (U) or both. Given 5 people read only English newspaper and 7 read both, find the number "r" of people who read only Urdu newspaper.

SOLUTION:

Given $n(E \cup U) = 18$ $n(E\setminus U) = 5$, $n(E \cap U) = 7$

$r = n(U \setminus E) = ?$

From thediagram



$E \cup U = (E\setminus U) \cup (E \cap U) \cup (U\setminus E)$

and the union is disjoint. Therefore,

$n(E \cup U) = n(E\setminus U) + n(E \cap U) + n(U\setminus E)$

$\Rightarrow \quad 18 = 5 + 7 + r$

$\Rightarrow \quad r = 18 - 5 – 7$

$\Rightarrow \quad r = 6$ Ans.

EXERCISE:

Fifty people are interviewed about their food preferences. 20 of them like Chinese food, 32 like fast food, and 12 like neither Chinese nor fast food.How many like Chinese but not fast food.

SOLUTION:

Let U denote the set of people interviewed and C and F denotes the sets of people who like Chinese food and fast food respectively.

Now given

$n(U) = 50$, $n(C) = 20$

$n(F) = 32$, $n((C \cup F')) = 12$

To find $n(C \cap F') = n(C \backslash F)$

Since $n((C \cup F)') = n(U) - n(C \cup F)$

$\Rightarrow$ $12 = 50 - n(C \cup F)$

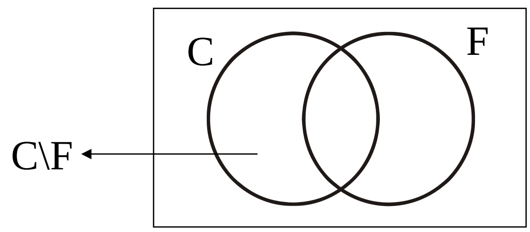
$\Rightarrow$ $n(C \cup F) = 50 - 12 = 38$

Next

$\Rightarrow$ $n(C \cup F) = n(C \backslash F) + n(F)$

$\Rightarrow$ $38 = n(C \backslash F) + 32$

$\Rightarrow$ $n(C \backslash F) = 38 - 32 = 6$



LECTURE 34

INCLUSION-EXCLUSION PRINCIPLE
PIGEONHOLE PRINCIPLE

EXERCISE:

a.How many integers from 1 through 1000 are multiples of 3 or multiples of 5?
b.How many integers from 1 through 1000 are neither multiples of 3 nor multiples of 5?

SOLUTION:

Let A and B denotes the set of integers from 1 through 1000 that are multiples of 3 and 5 respectively.
Then A $\cap$ B contains integers that are multiples of 3 and 5 both i.e., multiples of 15.
Now

$$n(A) = \left\lfloor \frac{1000}{3} \right\rfloor = 333 \quad \text{and} \quad n(B) = \left\lfloor \frac{1000}{5} \right\rfloor = 200$$

and

$$n(A \cap B) = \left\lfloor \frac{1000}{15} \right\rfloor = 66$$

Hence by the inclusion - exclusion principle

$$\begin{aligned} n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ &= 333 + 200 - 66 \\ &= 467 \end{aligned}$$

(b) The set (A $\cup$ B) contains those integers that are either multiples of 3 or multiples of 5.
Now

$$\begin{aligned} n((A \cup B)') &= n(U) - n(A \cup B) \\ &= 1000 - 467 \\ &= 533 \end{aligned}$$

where the universal set U contain integers 1 through 1000.

INCLUSION-EXCLUSION PRINCIPLE FOR 3 AND 4 SETS:

If A, B, C and D are finite sets, then

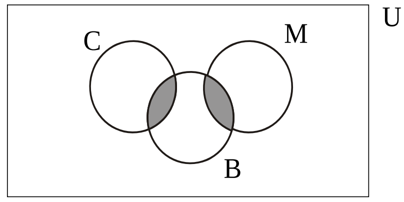
1. $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

2. $$\begin{aligned} n(A \cup B \cup C \cup D) &= n(A) + n(B) + n(C) + n(D) - n(A \cap B) - n(A \cap C) - n(A \cap D) \\ &\quad - n(B \cap C) - n(B \cap D) - n(C \cap D) + n(A \cap B \cap C) + n(A \cap B \cap D) \\ &\quad + n(A \cap C \cap D) + n(B \cap C \cap D) - n(A \cap B \cap C \cap D) \end{aligned}$$

EXERCISE:

A survey of 100 college students gave the following data:
8 owned a car (C)
20 owned a motorcycle (M)
48 owned a bicycle (B)
38 owned neither a car nor a motorcycle nor a bicycle
No student who owned a car, owned a motorcycle
How many students owned a bicycle and either a car or a motorcycle?

SOLUTION:



No. of elements in the shaded region to be determined
Let U represents the universal set of 100 college students. Now given that
$n(U) = 100;$ $n(C) = 8$
$n(M) = 20;$ $n(B) = 48$
$n((C \cup M \cup B)') = 38;$ $n(C \cap M) = 0$
and $n(B \cap C) + n(B \cap M) = ?$
Firstly note $n((C \cup M \cup B)') = n(U) - n(C \cup M \cup B)$
$\Rightarrow$ $38 = 100 - n(C \cup M \cup B)$
$\Rightarrow$ $n(C \cup M \cup B) = 100 - 38 = 62$
Now by inclusion - exclusion principle

$$n(C \cup M \cup B) = n(C) + n(M) + n(B) - n(C \cap M) - n(C \cap B) - n(M \cap B) + n(C \cap M \cap B)$$

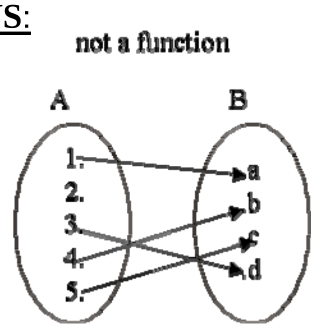
$$\Rightarrow \quad 62 = 8 + 20 + 48 - 0 - n(C \cap B) - n(M \cap B) - 0$$

$(\therefore n(C \cap B) = 0)$

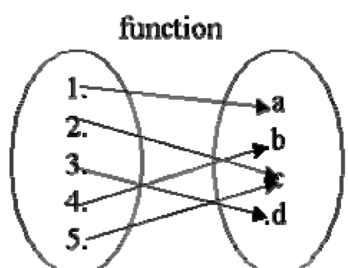
$$\Rightarrow n(C \cap B) + n(M \cap B) = 8 + 20 + 48 - 62 = 76 - 62 = 14$$

Hence, there are 14 students, who owned a bicycle and either a car or a motorcycle.

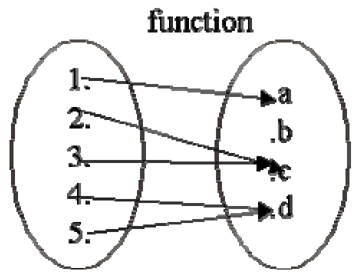
REVISION OF FUNCTIONS:



Clearly the above relation is not a function because 2 does not have any image under this relation. Note that if want to made it relation we have to must map the 2 into some element of B which is also the image of some element of A. Now



The above relation is a function because it satisfy the conditions of functions(as each element of 1st set have the images in 2nd set). The following is a function.



The above relation is a function because it satisfy the conditions of functions(as each element of 1st set have the images in 2nd set). Therefore the above is also a function.

PIGEONHOLE PRINCIPLE:

A function from a set of $k + 1$ or more elements to a set of k elements must have at least two elements in the domain that have the same image in the co-domain.

If $k + 1$ or more pigeons fly into k pigeonholes then at least one pigeonhole must contain two or more pigeons.

EXAMPLES:

1. Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

2.In any set of 27 English words, there must be at least two that begin with the same letter, since there are 26 letters in the English alphabet.

EXERCISE:

What is the minimum number of students in a class to be sure that two of them are born in the same month?

SOLUTION:

There are 12 (= n) months in a year. The pigeonhole principle shows that among any 13 (= $n + 1$) or more students there must be at least two students who are born in the same month.

EXERCISE:

Given any set of seven integers, must there be two that have the same remainder when divided by 6?

SOLUTION:

The set of possible remainders that can be obtained when an integer is divided by six is {0, 1, 2, 3, 4, 5}. This set has 6 elements. Thus by the pigeonhole principle if 7 = 6 + 1 integers are each divided by six, then at least two of them must have the same remainder.

EXERCISE:

How many integers from 1 through 100 must you pick in order to be sure of getting one that is divisible by 5?

SOLUTION:

There are 20 integers from 1 through 100 that are divisible by 5. Hence there are eighty integers from 1 through 100 that are not divisible by 5. Thus by the pigeonhole principle 81 = 80 + 1 integers from 1 though 100 must be picked in order to be sure of getting one that is divisible by 5.

EXERCISE:

Let A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. Suppose six integers are chosen from A. Must there be two integers whose sum is 11.

SOLUTION:

The set A can be partitioned into five subsets:
{1, 10}, {2, 9}, {3, 8}, {4, 7}, and {5, 6}

each consisting of two integers whose sum is 11.
These 5 subsets can be considered as 5 pigeonholes.
If 6 = (5 + 1) integers are selected from A, then by the pigeonhole principle at least two must be from one of the five subsets. But then the sum of these two integers is 11.

GENERALIZED PIGEONHOLE PRINCIPLE:

A function from a set of $n \cdot k + 1$ or more elements to a set of n elements must have at least $k + 1$ elements in the domain that have the same image in the co-domain.
If $n \cdot k + 1$ or more pigeons fly into n pigeonholes then at least one pigeonhole must contain $k + 1$ or more pigeons.

EXERCISE:

Suppose a laundry bag contains many red, white, and blue socks. Find the minimum number of socks that one needs to choose in order to get two pairs (four socks) of the same colour.

SOLUTION:

Here there are n = 3 colours (pigeonholes) and k + 1 = 4 or k = 3. Thus among any $n \cdot k + 1 = 3 \cdot 3 + 1 = 10$ socks (pigeons), at least four have the same colour.

DEFINITION:

1. Given any real number x, **the floor of *x***, denoted $\lfloor x \rfloor$, is the largest integer smaller than or equal to x.

2. Given any real number x, **the ceiling of *x***, denoted $\lceil x \rceil$, is the smallest integer greater than or equal to x.

EXAMPLE:

Compute $\lfloor x \rfloor$ and $\lceil x \rceil$ for each of the following values of x.

a. 25/4 **b.** 0.999 **c.** –2.01

SOLUTION:

a. $\lfloor 25/4 \rfloor = \lfloor 6 + 1/4 \rfloor = 6$
$\lceil 25/4 \rceil = \lceil 6 + 1/4 \rceil = 6 + 1 = 7$

b. $\lfloor 0.999 \rfloor = \lfloor 0 + 0.999 \rfloor = 0$
$\lceil 0.999 \rceil = \lceil 0 + 0.999 \rceil = 0 + 1 = 1$

c. $\lfloor -2.01 \rfloor = \lfloor -3 + 0.99 \rfloor = -3$
$\lceil -2.01 \rceil = \lceil -3 + 0.999 \rceil = -3 + 1 = -2$

EXERCISE:

What is the smallest integer N such that

a. $\lceil N/7 \rceil = 5$ b. $\lceil N/9 \rceil = 6$

SOLUTION:

a. $N = 7 \cdot (5 - 1) + 1 = 7 \cdot 4 + 1 = 29$
b. $N = 9 \cdot (6 - 1) + 1 = 9 \cdot 5 + 1 = 46$

PIGEONHOLE PRINCIPLE:

If N pigeons fly into k pigeonholes then at least one pigeonhole must contain $\lceil N/k \rceil$ or more pigeons.

EXAMPLE:

Among 100 people there are at least $\lceil 100/12 \rceil = \lceil 8 + 1/3 \rceil = 9$ who were born in the same month.

EXERCISE:

What is the minimum number of students required in a Discrete Mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F.

SOLUTION:

The minimum number of students needed to guarantee that at least six students receive the same grade is the smallest integer N such that $\lceil N/5 \rceil = 6$.

The smallest such integer is $N = 5(6-1)+1=5 \cdot 5 + 1 = 26$.

Thus 26 is the minimum number of students needed to be sure that at least 6 students will receive the same grades.

LECTURE 35

INTRODUCTION TO PROBABILITY

INTRODUCTION:

Combinatorics and probability theory share common origins. The theory of probability was first developed in the seventeenth century when certain gambling games were analyzed by the French mathematician Blaise Pascal. It was in these studies that Pascal discovered various properties of the binomial coefficients. In the eighteenth century, the French mathematician Laplace, who also studied gambling, gave definition of the probability as the number of successful outcomes divided by the number of total outcomes.

DEFINITIONS:

An **experiment** is a procedure that yields a given set of possible outcomes.

The **sample space** of the experiment is the set of possible outcomes.

An **event** is a subset of the sample space.

EXAMPLE:

When a die is tossed the sample space S of the experiment have the following six outcomes. $S = \{1, 2, 3, 4, 5, 6\}$

Let E_1 be the event that a 6 occurs,
E_2 be the event that an even number occurs,
E_3 be the event that an odd number occurs,
E_4 be the event that a prime number occurs,
E_5 be the event that a number less than 5 occurs, and
E_6 be the event that a number greater than 6 occurs.

Then

$E_1 = \{6\}$ $\quad$ $E_2 = \{2, 4, 6\}$
$E_3 = \{1, 3, 5\}$ $\quad$ $E_4 = \{2, 3, 5\}$
$E_5 = \{1, 2, 3, 4\}$ $\quad$ $E_6 = \Phi$

EXAMPLE:

When a pair of dice is tossed, the sample space S of the experiment has the following thirty-six outcomes

$S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)$
$(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)$
$(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)$
$(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)$
$(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)$
$(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

or more compactly,

$\{11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26,$
$31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46,$
$51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66\}$

Let E be the event in which the sum of the numbers is ten.

Then

$E = \{(4, 6), (5, 5), (6, 4)\}$

DEFINITION:

Let *S* be a finite sample space such that all the outcomes are equally likely to occur.
The **probability** of an event *E*, which is a subset of sample space *S*, is

$$P(E) = \frac{\text{the number of outcomes in } E}{\text{the numbr of total outcomes in } S} = \frac{n(E)}{n(S)}$$

REMARK:
Since $\Phi \subseteq E \subseteq S$ therefore, $0 \le n(E) \le n(S)$. It follows that the probability of an event is always between 0 and 1.

EXAMPLE:
What is the probability of getting a number greater than 4 when a dice is tossed?

SOLUTION:
When a dice is rolled its sample space is S={1,2,3,4,5,6}
Let *E* be the event that a number greater than 4 occurs. Then *E* = {5, 6}
Hence,

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

EXAMPLE:
What is the probability of getting a total of eight or nine when a pair of dice is tossed?

SOLUTION:
When a pair of dice is tossed,its sample space S has the 36 outcomes which are as fellows:

S = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)
(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)
(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

Let *E* be the event that the sum of the numbers is eight or nine. Then
E = {(2, 6), (3, 5), (4, 4), (5, 3), (6, 2), (3, 6), (4, 5), (5, 4), (6, 3)}
Hence,

$$P(E) = \frac{n(E)}{n(S)} = \frac{9}{36} = \frac{1}{4}$$

EXAMPLE:
An urn contains four red and five blue balls. What is the probability that a ball chosen from the urn is blue?

SOLUTION:
Since there are four red balls and five blue balls so if we take out one ball from the urn then there is possibility that it may be one of from four red and one of from five blue balls hence there are total of nine possibilities. Thus we have
The total number of possible outcomes = 4 + 5 = 9
Now our favourable event is that we get the blue ball when we choose a ball from the urn. So we have
The total number of favorable outcomes = 5
Now we have Favorable outcomes 5 and our sample space has total outcomes 9 .Thus we have

The probability that a ball chosen = 5/9

EXERCISE:

Two cards are drawn at random from an ordinary pack of 52 cards. Find the probability **p** that (i) both are spades, (ii) one is a spade and one is a heart.

SOLUTION:

There are $\binom{52}{2} = 1326$ ways to draw 2 cards from 52 card

(i) There are $\binom{13}{2} = 78$ ways to draw 2 spades from 13 spades (as spades are 13 in 52 cards); hence

$$p = \frac{\text{number of ways 2 spades can be drawn}}{\text{number of ways 2 cards can be drawn}} = \frac{78}{1326} = \frac{1}{17}$$

ii) Since there are 13 spades and 13 hearts, there are $\binom{13}{1}\binom{13}{1} = 13.13 = 169$ ways to draw a spade and a heart; hence $p = \frac{169}{1326} = \frac{13}{102}$

EXAMPLE:

In a lottery, players win the first prize when they pick three digits that match, in the correct order, three digits kept secret. A second prize is won if only two digits match. What is the probability of winning (a) the first prize, (b) the second prize?

SOLUTION:

Using the product rule, there are 10^3 = 1000 ways to choose three digits.

(a) There is only one way to choose all three digits correctly. Hence the probability that a player wins the first prize is 1/1000 = 0.001.

(b) There are three possible cases:
(i)The first digit is incorrect and the other two digits are correct
(ii)The second digit is incorrect and the other two digits are correct
(iii)The thirds digit is incorrect and the other digits are correct

To count the number of successes with the first digit incorrect, note that there are nine choices for the first digit to be incorrect, and one each for the other two digits to be correct. Hence, there are nine ways to choose three digits where the first digit is incorrect, but the other two are correct. Similarly, there are nine ways for the other two cases. Hence, there are 9 + 9 + 9 = 27 ways to choose three digits with two of the three digits correct.
It follows that the probability that a player wins the second prize is 27/1000 = 0.027.

EXAMPLE:

What is the probability that a hand of five cards contains four cards of one kind?

SOLUTION:

(i) For determining the favorable outcomes we note that

The number of ways to pick one kind = $C(13, 1)$
The number of ways to pick the four of this kind out of the four of this kind in the deck = $C(4, 4)$
The number of ways to pick the fifth card from the remaining 48 cards = $C(48, 1)$
Hence, using the product rule the number of hands of five cards with four cards of one kind = $C(13, 1) \times C(4, 4) \times C(48, 1)$
(ii) The total number of different hands of five cards = $C(52, 5)$.

From (i) and (ii) it follows that the probability that a hand of five cards contains four cards of one kind is

$$\frac{C(13,1)\cdot C(4,4)\cdot C(48,1)}{C(52,5)} = \frac{13\cdot 1\cdot 48}{2,598,960} \approx 0.0024$$

EXAMPLE:
Find the probability that a hand of five cards contains three cards of one kind and two of another kind.

SOLUTION:
(i) For determining the favorable outcomes we note that
The number of ways to pick two kinds = $C(13, 2)$
The number of ways to pick three out four of the first kind = $C(4, 3)$
The number of ways to pick two out four of the second kind = $C(4, 2)$
Hence, using the product rule the number of hands of five cards with three cards of one kind and two of another kind = $C(13, 2) \times C(4, 3) \times C(4, 2)$
(ii) The total number of different hands of five cards = $C(52, 5)$.

From (i) and (ii) it follows that the probability that a hand of five cards contains three cards of one kind and two of another kind is

$$\frac{C(13,2)\cdot C(4,3)\cdot C(4,2)}{C(52,5)} = \frac{3744}{2,598,960} \approx 0.0014$$

EXAMPLE:
What is the probability that a randomly chosen positive two-digit number is a multiple of 6?

SOLUTION:
1.There are $\lfloor 99/6 \rfloor = \lfloor 16 + ½\rfloor = 16$ positive integers from 1 to 99 that are divisible by 6. Out of these 16 – 1 = 15 are two-digit numbers(as 6 is a multiple of 6 but not a two-digit number).
2.There are 99 – 9 = 90 positive two-digit numbers in all.
Hence, the probability that a randomly chosen positive two-digit number is a multiple of 6 = 15/90 = 1/6 ≈ 0.166667

DEFINITION:
Let E be an event in a sample space S, the complement of E is the event that occurs if E does not occur. It is denoted by E^c. Note that $E^c = S \backslash E$

EXAMPLE:
Let E be the event that an even number occurs when a die is tossed. Then E^c is the event that an odd number occurs.

THEOREM:
Let *E* be an event in a sample space *S*. The probability of the complementary event E^c of *E* is given by

$$P(E^c) = 1 - P(E).$$

EXAMPLE:
Let 2 items be chosen at random from a lot containing 12 items of which 4 are defective. What is the probability that (i) none of the items chosen are defective, (ii) at least one item is defective?

SOLUTION:
The number of ways 2 items can be chosen from 12 items = *C*(12, 2) = 66.
(i) Let *A* be the event that none of the items chosen are defective.
The number of favorable outcomes for *A* = The number of ways 2 items can be chosen from 8 non-defective items = *C*(8, 2) = 28.
Hence, P(A) = 28/66 = 14/33.

(ii)Let *B* be the event that at least one item chosen is defective.
Then clearly, $B = A^c$
It follows that
$P(B) = P(A^c)$
$=1 - P(A) = 1 - 14/33 = 19/33.$

EXERCISE:
Three light bulbs are chosen at random from 15 bulbs of which 5 are defective. Find the probability p that (i) none is defective, (ii) exactly one is defective, (iii) at least one is defective.

SOLUTION:
There are $\binom{15}{3} = 455$ ways to choose 3 bulbs from the 15 bulbs.

(i)Since there are 15 - 5 = 10 non-defective bulbs, there are $\binom{10}{3} = 120$ ways to choose 3 non-defective bulbs.

Thus

$$p = \frac{120}{455} = \frac{24}{91}$$

(ii) There are 5 defective bulbs and $\binom{10}{2} = 45$ different pairs of non-defective bulbs; hence there are $\binom{5}{1}\binom{10}{2} = 5.45 = 225$ ways to choose 3 bulbs of which one is defective.

Thus

$$P = \frac{225}{455} = \frac{45}{91}.$$

(iii)The event that at least one is defective is the complement of the event that none are defective which has by (i), probability $\frac{24}{91}$.

Hence $p(atleast\ one\ is\ defective) = 1 - p(none\ is\ defective) = 1 - \frac{24}{91} = \frac{67}{91}$

LECTURE 36
ADDITION LAW OF PROBABILITY

THEOREM:
If A and B are two disjoint (mutually exclusive) events of a sample space S, then

$$P(A \cup B) = P(A) + P(B)$$

In words, the probability of the happening of an event A or an event B or both is equal to the sum of the probabilities of event A and event B provided the events have nothing in common.

PROOF:
By inclusion - exclusion principle for mutually disjoint sets,

$$n(A \cup B) = n(A) + n(B)$$

Dividing both sides by n(S), we get

$$\frac{n(A \cup B)}{n(S)} = \frac{n(A) + n(B)}{n(S)}$$

$$= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)}$$

$$\Rightarrow \quad P(A \cup B) = p(A) + P(B)$$

EXAMPLE:
Suppose a die is rolled. Let A be the event that 1 appears & B be the event that some even number appears on the die. Then

S = {1, 2, 3, 4, 5, 6}, A = {1} & B = {2, 4, 6}

Clearly A & B are disjoint events and

$$P(A) = \frac{1}{6}, \qquad P(B) = \frac{3}{6}$$

Hence the probability that a 1 appears or some even number appears is given by

$$P(A \cup B) = P(A) + P(B)$$

$$= \frac{1}{6} + \frac{3}{6}$$

$$= \frac{4}{6} = \frac{2}{3} \qquad Ans.$$

EXERCISE:
A bag contains 6 white, 5 black and 4 red balls. Find the probability of getting a white or a black ball in a single draw.

SOLUTION:
Let A be the event of getting a white ball and B be the event of getting a black ball.

Total number of balls = 6 + 5 + 4 = 15

$$P(A) = \frac{6}{15}, \qquad P(B) = \frac{5}{15}$$

Since the two events are disjoint (mutually exclusive),therefore

$$P(A \cup B) = P(A) + P(B) = \frac{6}{15} + \frac{5}{15} = \frac{11}{15} \qquad Ans$$

EXERCISE:

A pair of dice is thrown. Find the probability of getting a total of 5 or 11.

SOLUTION:

When two dice are thrown, the sample space has 6 * 6 = 36 outcomes. Let A be the event that a total of 5 occurs and B be the event that a total of 11 occurs.
Then
A = {(1,4), (2,3), (3,2), (4,1)} and B = {(5,6), (6,5)}
Clearly, the events A and B are disjoint (mutually exclusive) with probabilities given by

Now by using the sum Rule for Mutually Exclusive events we get

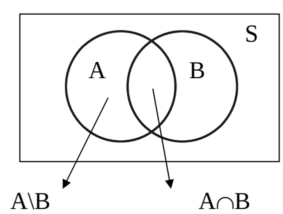
$$P(A \cup B) = P(A) + P(B) = \frac{4}{36} + \frac{2}{36} = \frac{6}{36} = \frac{1}{6} \qquad Ans$$

EXERCISE:

For any two event A and B of a sample space S. Prove that
$P(A\backslash B) = P(A \cap B') = P(A) - P(A \cap B)$

SOLUTION:

The event A can be written as the union of two disjoint events A\B and $A \cap B$. i.e. $A = (A\backslash B) \cup (A \cap B)$
Hence, by addition law of probability
$P(A) = P(A\backslash B) + P(A \cap B)$
$\Rightarrow P(A\backslash B) = P(A) - P(A \cap B)$



GENERAL ADDITION LAW OF PROBABILITY

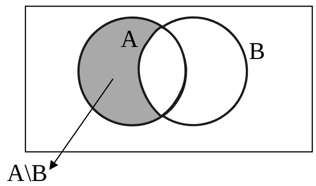
THEOREM

If A and B are any two events of a sample space S, then
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

PROOF:

The event $A \cup B$ may be written as the union of two disjoint events A\B and B.
i.e., $A \cup B = (A\backslash B) \cup B$
Hence, by addition law of probability (for disjoint events)

$$\begin{aligned} P(A \cup B) &= P(A\backslash B) + P(B) \\ &= [P(A) - P(A \cap B)] + P(B) \\ &= P(A) + P(B) - P(A \cap B) \qquad \text{(proved)} \end{aligned}$$



REMARK:

By inclusion - exclusion principle

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$ (where A and B are finite)

Dividing both sides by n(S) and denoting the ratios as respective probabilities we get the Generalized Addition Law of probability.

i.e $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

EXERCISE:

Let A and B be events in a sample space S, and let

$P(A) = 0.65$, $P(B) = 0.30$ and $P(A \cap B) = 0.15$

Determine the probability of the events

(a)$A \cap B'$ (b) $A \cup B$ (c)$A' \cap B'$

SOLUTION:

(a)As we know that

$$P(A \cap B') = P(A) - P(A \cap B) \quad (\text{as } A-B=A \cap B')$$
$$= 0.65 - 0.15$$
$$= 0.50$$

(b)By addition Law of probability

$$P(A \cup B)= P(A) + P(B) - P(A \cap B) \quad (\text{as } A \cap B \neq \phi)$$
$$= 0.65 + 0.30 - 0.15$$
$$= 0.80$$

(c) By DeMorgan's Law

$$A' \cap B'= (A \cup B)'$$
$$\therefore P(A' \cap B')= P(A \cup B)'$$
$$= 1 - P(A \cup B)$$
$$= 1 - 0.80$$
$$= 0.20 \quad \text{Ans.}$$

EXERCISE:

Let A, B, C and D be events which form a partition of a sample space S. If P(A) = P(B), P(C) = 2 P(A) and P(D) = 2 P(C). Determine each of the following probabilities.

(a)P(A) (b)$P(A \cup B)$ (c)$P(A \cup C \cup D)$

SOLUTION:

(a) Since A, B, C and D form a partition of S, therefore

$S = A \cup B \cup C \cup D$ and A, B, C, D are pair wise disjoint. Hence, by addition law of probability.

$$P(S) = P(A) + P(B) + P(C) + P(D)$$
$$\Rightarrow 1 = P(A) + P(A) + 2P(A) +2 P(C)$$
$$\Rightarrow 1 = 4P(A) + 2 (2P(A))$$
$$\Rightarrow 1 = 8 P(A)$$
$$\Rightarrow P(A) = \frac{1}{8}$$

(b) $P(A \cup B) = P(A) + P(B)$

= P(A) + P(A) [$\therefore$ P(B) = P(A)]
= 2 P(A)

= $2\left(\frac{1}{8}\right)=\frac{1}{4}$

(c) P(A $\cup$C $\cup$D) = P(A) + P(C) + P(D)
= P(A) + 2 P(A) + 2 (2P(A)) [$\therefore$ P(C) = 2 P(A) & P(D) = 2 P(C)]
= 7 P(A)

=$7\left(\frac{1}{8}\right)=\frac{7}{8}$ Ans.

<u>EXERCISE:</u>

A card is drawn from a well-shuffled pack of playing card. What is the probability that it is either a spade or an ace?

<u>SOLUTION:</u>

Let A be the event of drawing a spade and B be the event of drawing an ace. Now A and B are not disjoint events. A $\cap$ B represents the event of drawing an ace of spades.

Now

$$P(A) = \frac{13}{52}; \qquad P(B)=\frac{4}{52}$$

$$P(A\cap B) = \frac{1}{52}$$

$$\therefore \quad P(A\cup B) = P(A)+P(B)-P(A\cap B)$$

$$=\frac{13}{52}+\frac{4}{52}-\frac{1}{52}$$

$$=\frac{16}{52}=\frac{4}{13}$$

<u>EXERCISE:</u>

A class contains 10 boys and 20 girls of which half the boys and half the girls have brown eyes. Find the probability that a student chosen at random is a boy or has brown eyes.

<u>SOLUTION:</u>

Let A be the event that a boy is chosen and B be the event that a student with brown eyes is chosen.Then A and B are not disjoint events. Infact, A$\cap$B represents the event that a boy with brown eyes is chosen.

$$P(A) = \frac{10}{10+20}=\frac{10}{30}$$

$$P(B) = \frac{5+10}{10+20}=\frac{15}{30}$$

and

$$P(A\cap B)=\frac{5}{10+20}=\frac{5}{30} \quad (as\,some\,boys\,also\,have\,brown\,eyes)$$

$$\therefore \quad P(A\cup B) = P(A)+P(B)-P(A\cap B)$$

$$=\frac{10}{30}+\frac{15}{30}-\frac{5}{30}$$

$$=\frac{20}{30}=\frac{2}{3} \qquad Ans.$$

EXERCISE:

An integer is chosen at random from the first 100 positive integers. What is the probability that the integer chosen is divisible by 6 or by 8?

SOLUTION:

Let A be the event that the integer chosen is divisible by 6, and B be the event that the integer chosen is divisible by 8.

$A \cap B$ is the event that the integer is divisible by both 6 and 8 (i.e. as their L.C.M. is 24)

Now

$$n(A) = \left\lfloor \frac{100}{6} \right\rfloor = 16; \qquad n(B) = \left\lfloor \frac{100}{8} \right\rfloor = 12$$

and

$$n(A \cap B) = \left\lfloor \frac{100}{24} \right\rfloor = 4$$

$$\begin{aligned} \text{Hence} \quad P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{16}{100} + \frac{12}{100} - \frac{4}{100} \\ &= \frac{24}{100} = \frac{6}{25} \qquad \textit{Ans} \end{aligned}$$

OR

Let A denote the event that the integer chosen is divisible by 6, and B denote the event that the integer chosen is divisible by 8 i.e

A={6,12,18,24,…,90,96} $\Rightarrow$ n(A)=16 $\quad \Rightarrow P(A) = \frac{16}{100}$

B={8,16,24,40,…,88,96} $\Rightarrow$ n(B)=12 $\quad \Rightarrow P(B) = \frac{12}{100}$

$A \cap B$={24,48,72,96} $\quad \Rightarrow$ n($A \cap B$)=12 $\quad \Rightarrow P(A \cap B) = \frac{4}{100}$

EXERCISE:

A student attends mathematics class with probability 0.7 skips accounting class with probability 0.4, and attends both with probability 0.5. Find the probability that

(1)he attends at least one class

(2)he attends exactly one class

SOLUTION:

(1) Let A be the event that the student attends mathematics class and B be the event that the student attends accounting class.

Then given

P(A) = 0.7; P(B) = 1 - 0.4 = 0.6

And P($A \cap B$) = 0.5, P($A \cup B$) = ?

By addition law of probability

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.7 + 0.6 - 0.5 \\ &= 0.8 \end{aligned}$$

(2) Students can attend exactly one class in two ways
(a)He attends mathematics class but not accounting i.e., event $A \cap B^c$ or
(b)He does not attend mathematics class and attends accounting class i.e., event $A^c \cap B$

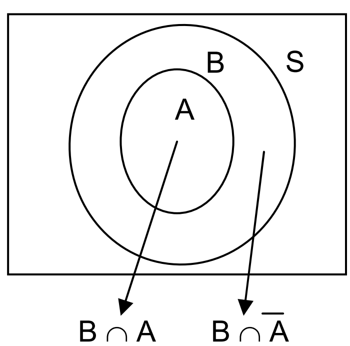
Since the two event $A \cap B^c$ and $A^c \cap B$ are disjoint, hence required probability is
$P(A \cap B^c) + P(A^c \cap B)$
Now
$P(A \cap B^c) = P(A \backslash B)$
$= P(A) - P(A \cap B)$
$= 0.7 - 0.5$
$= 0.2$
and
$P(A^c \cap B) = P(B \backslash A)$
$= P(B) - P(A \cap B)$
$= 0.6 - 0.5$
$= 0.1$
Hence required probability is
$P(A \cap B^c) + P(A^c \cap B) = 0.2 + 0.1 = 0.3$

PROBABILITY OF SUB EVENT

THEOREM:

If A and B are two events such that $A \subseteq B$, then $P(A) \le P(B)$

PROOF:
Suppose $A \subseteq B$. The event B may be written as the union of disjoint events $B \cap A$ and $B \cap \overline{A}$
i.e., $B = (B \cap A) \cup (B \cap \overline{A})$
But $B \cap A = A$ (as $A \subseteq B$)
So $B = A \cup (B \cap \overline{A})$
$\therefore P(B) = P(A) + P(B \cap \overline{A})$
But $P(B \cap \overline{A}) \ge 0$
Hence $P(B) \ge P(A)$
Or $P(A) \le P(B)$



EXERCISE:

Let A and B be subsets of a sample space S with $P(A) = 0.7$ and $P(B) = 0.5$. What are the maximum and minimum possible values of $P(A \cup B)$.

SOLUTION:
By addition law of probabilities
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= 0.7 + 0.5 - P(A \cap B)$
$= 1.2 - P(A \cap B)$
Since probability of any event is always less than or equal to 1, therefore
max $P(A \cup B) = 1$, for which $P(A \cap B) = 0.2$
Next to find the minimum value, we note
$A \cap B \subseteq B$
$\Rightarrow P(A \cap B) \le P(B) = 0.5$
Thus for min $P(A \cup B)$ we take maximum possible value of $P(A \cap B)$ which is 0.5. Hence
min $P(A \cup B) = 1.2 -$ max $P(A \cap B)$
$= 1.2 - 0.5$
$= 0.7$ is the required minimum value.

ADDITION LAW OF PROBABILITY FOR THREE EVENTS:
If A, B and C are any three events, then
$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

REMARK:
If A, B, C are mutually disjoint events, then $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

EXERCISE:
Three newspapers A, B, C are published in a city and a survey of readers indicates the following:
20% read A, 16% read B, 14% read C
8% read both A and B, 5% read both A and C
4% read both B and C, 2% read all the three
For a person chosen at random, find the probability that he reads none of the papers.

SOLUTION:
Given
$$P(A) = 20\% = \frac{20}{100} = 0.2; \qquad P(B) = 16\% = \frac{16}{100} = 0.16$$
$$P(C) = 14\% = \frac{14}{100} = 0.14; \qquad P(A \cap B) = 8\% = \frac{8}{100} = 0.08$$
$$P(A \cap C) = 5\% = \frac{5}{100} = 0.05; P(B \cap C) = 4\% = \frac{4}{100} = 0.04$$
and
$$P(A \cap B \cap C) = 2\% = \frac{2}{100} = 0.02$$
Now the probability that person reads A or B or C = $P(A \cup B \cup C)$
$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$
$$= \frac{20}{100} + \frac{16}{100} + \frac{14}{100} - \frac{8}{100} - \frac{5}{100} - \frac{4}{100} + \frac{2}{100}$$
$$= \frac{35}{100}$$

Hence, the probability that he reads none of the papers

$$
\begin{aligned}
&= P((A \cup B \cup C)^c) \\
&= 1 - P(A \cup B \cup C) \\
&= 1 - \frac{35}{100} \\
&= \frac{65}{100} \\
&= 65\%
\end{aligned}
$$

EXERCISE:

Let A, B and C be events in a sample space S, with $A \cup B \cup C = S$, $A \cap (B \cup C) = \phi$, P(A) = 0.2, P(B) = 0.5 and P(C) = 0.7. Find $P(A^c)$, $P(B \cup C)$, $P(B \cap C)$.

SOLUTION:

$P(A^c) = 1 - P(A)$
$= 1 - 0.2$
$P(A^c) = 0.8$

Next, given that the events A and $B \cup C$ are disjoint, since $A \cap (B \cup C) = \phi$, therefore

$P(A \cup (B \cup C)) = P(A) + P(B \cup C)$
$\Rightarrow \quad P(S) = 0.2 + P(B \cup C)$
$\Rightarrow \quad 1 = 0.2 + P(B \cup C)$
$\Rightarrow \quad P(B \cup C) = 1 - 0.2 = 0.8$

Finally, by addition law of probability
$P(B \cup C) = P(B) + P(C) - P(B \cap C)$
$\Rightarrow \quad 0.8 = 0.5 + 0.7 - P(B \cap C)$
$\Rightarrow P(B \cap C) = 0.4$ is the required probability.

LECTURE 37

CONDITIONAL PROBABILITY
MULTIPLICATION THEOREM
INDEPENDENT EVENTS

EXAMPLE:

a. What is the probability of getting a 2 when a dice is tossed?
b. An even number appears on tossing a die.
(i) What is the probability that the number is 2?
(ii) What is the probability that the number is 3?

SOLUTION:

When a dice is tossed,the sample space is S={1,2,3,4,5,6}
$\Rightarrow$ n(S)=6
a. Let "A" denote the event of getting a 2 i.e A={ 2 } $\Rightarrow$ n(A)=1

P(2 appears when the die is tossed) = $\frac{n(A)}{n(S)} = \frac{1}{6}$

b. (i) Let "S_1" denote the total number of even numbers from a sample space S,when a dice is tossed (i.e $S_1 \subseteq S$)
S_1={ 2,4,6 } $\Rightarrow$ n(S_1)=3
Let "B" denote the event of getting a 2 from total number of even number i.e B={ 2 }
$\Rightarrow$ n(B) = 1

P(2 appears; given that the number is even)=P(B)= $\frac{n(B)}{n(S_1)} = \frac{1}{3}$

(ii) Let "C" denote the event of getting a 3 in S_1(among the even numbers)i.e C={ }
$\Rightarrow$ n(C)=0

P(3 appears; given that the number is even) =P(C) = $\frac{n(C)}{n(S_1)} = \frac{0}{3} = 0$

EXAMPLE:

Suppose that an urn contains 3 red balls, 2 blue balls, and 4 white balls, and that a ball is selected at random.
Let E be the event that the ball selected is red.
Then P(E) = 3/9 (as there are 3 red balls out of total 9 balls)
Let F be the event that the ball selected is not white.
Then the probability of E if it is already known that the selected ball is not white would be P(red ball selected; given that the selected ball is not white) = 3/5 (as we count no white ball so there are total 9 balls(i.e 2 blue and 3 red balls))
This is called the conditional probability of E given F and is denoted by P(E|F).

DEFINITION:

Let E and F be two events in the sample space of an experiment with P(F) $\neq$ 0. The **conditional probability** of E given F, denoted by P(E|F), is defined as

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

EXAMPLE:

Let A and B be events of an experiment such that
P(B) = 1/4 and P(A$\cap$B) = 1/6.

What is the conditional probability P(A|B)?

SOLUTION:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{1/4} = \frac{4}{6} = \frac{2}{3}$$

EXERCISE:

Let A and B be events with $P(A) = \frac{1}{2}, P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{4}$

Find

(i) $P(A|B)$ (ii) $P(B|A)$

(iii) $P(A \cup B)$ (iv) $P(A^c | B^c)$

SOLUTION:

Using the formula of the conditional Probability we can write

(i) $P(A|B)$
$$= \frac{P(A \cap B)}{P(B)}$$
$$= \frac{1/4}{1/3} = \frac{3}{4}$$

(ii) $P(B|A)$
$$= \frac{P(B \cap A)}{P(A)}$$
$$= \frac{1/4}{1/2} = \frac{2}{4} = \frac{1}{2}$$ (As $P(B \cap A) = P(A \cap B) = 1/4$)

(iii) $P(A \cup B)$
$$= P(A) + P(B) - P(A \cap B)$$
$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$$
$$= \frac{7}{12}$$

(iv) $P(A^c | B^c)$
$$= \frac{P(A^c \cap B^c)}{P(B^c)}$$
$$= \frac{P((A \cup B)^c)}{P(B^c)} \quad \text{(By using DeMorgan's Law)}$$
$$= \frac{1 - P(A \cup B)}{1 - P(B)} \quad [P(E^c) = 1 - P(E)]$$
$$= \frac{1 - 7/12}{1 - 1/3} = \frac{5/12}{2/3} = \frac{5}{12} \times \frac{3}{2} = \frac{5}{8}$$

EXERCISE:

Find P(B|A) if

(i) A is a subset of B

(ii) A and B are mutually exclusive

SOLUTION:

(i) When A $\subseteq$ B, then B $\cap$ A = A (As A$\cap$A$\subseteq$B$\cap$A $\Rightarrow$ A $\subseteq$ B$\cap$A …………….(i) also we know that $B \cap A \subseteq A$ ……………..(ii) , From (i) and (ii) clearly $B \cap A = A$)

$$\therefore \quad P(B|A) = \frac{P(B \cap A)}{P(A)} \quad (as\, B \cap A = A \Rightarrow P(B \cap A) = P(A))$$

$$= \frac{P(A)}{P(A)} = 1$$

(ii) When A and B are mutually exclusive, then $B \cap A = \varnothing$

$$\therefore \quad P(B|A) = \frac{P(B \cap A)}{P(A)}$$

$$= \frac{P(\phi)}{P(A)} \qquad (Since\ B \cap A = \phi \Rightarrow P(B \cap A) = 0)$$

$$= \frac{0}{P(A)} = 0 \qquad (\ as\ P(\phi) = 0\ )$$

EXAMPLE:

Suppose that an urn contains **three** red balls marked 1, 2, 3, one blue ball marked 4, and **four** white balls marked 5, 6, 7, 8.

A ball is selected at random and its color and number noted.

(i) What is the probability that it is red?

(ii) What is the probability that it has an even number marked on it?

(iii) What is the probability that it is red, if it is known that the ball selected has an even number marked on it?

(iv) What is the probability that it has an even number marked on it, if it is known that the ball selected is red?

SOLUTION:

Let E be the event that the ball selected is red .

(i) P(E) = 3/8

let F be the event that the ball selected has an even number marked on it.

(ii) P(F) = 4/8 (as there are four even numbers 2,4,6 & 8 out of total eight numbers).

(iii) $E \cap F$ is the event that the ball selected is red and has an even number marked on it.

Clearly $P(E \cap F)$ =1/8 (as there is only one ball which is red and marked an even number ”2” out of total eight balls).

Hence,

P(Selected ball is red, given that the ball selected has an even number marked on it.) = P(E|F)

$$= \frac{P(E \cap F)}{P(F)} = \frac{1/8}{4/8} = 1/4$$

(iv) P(Selected ball has an even number marked on it, given that the ball selected is red) = P(F|E)

$$= \frac{P(E \cap F)}{P(E)} = \frac{1/8}{3/8} = \frac{1}{3}$$

EXAMPLE:

Let a pair of dice be tossed. If the sum is 7, find the probability that one of the dice is 2.

SOLUTION:

Let E be the event that a 2 appears on at least one of the two dice, and F be the event that the sum is 7.

Then

E = {(1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6) ,(3, 2) (4, 2), (5, 2), ,(6, 2)}

F = {(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}

$E\cap F$ = {(2, 5), (5, 2)}

P(F) = 6/36 and $P(E\cap F) = 2/36$.

Hence,

P(Probability that one of the dice is 2, given that the sum is 7)

= P(E|F)

$$= \frac{P(E\cap F)}{P(E)} = \frac{2/36}{6/36} = \frac{1}{3}$$

EXAMPLE:

A man visits a family who has two children. One of the children, a boy, comes into the room.

Find the probability that the other child is also a boy if

(i) The other child is known to be elder,

(ii) Nothing is known about the other child.

SOLUTION:

The sample space of the experiment is S = {*bb*, *bg*, *gb*, *gg*}

(The outcome *bg* specifies that younger is a boy and elder is a girl, etc.)

Let A be the event that both the children are boys.

Then, A = {*bb*}.

(i) Let B be the event that the younger is a boy. Then, B = {*bb*, *bg*},and $A\cap B$ = {*bb*}. Hence, the required probability is

P(Probability that the other child is also a boy, given that the other child is elder) =

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{1/4}{2/4} = \frac{1}{2}$$

(ii) Let C be the event that one of the children is a boy.

Then C = {*bb*, *bg*, *gb*}, and $A\cap C$ = {*bb*}.

Hence, the required probability is

P(Probability that both the children are boys, given that one of the children is a boy)

= P(A|C)

$$= \frac{P(A\cap C)}{P(C)} = \frac{1/4}{3/4} = \frac{1}{3}$$

MULTIPLICATION THEOREM

Let E and F be two events in the sample space of an experiment, then

$P(E\cap F) = P(F)P(E \mid F)$

Or $P(E\cap F) = P(E)P(F \mid E)$

Let $E_1, E_2, \ldots, E_n$ be events in the sample space of an experiment, then

$$P(E_1\cap E_2 \cap \ldots \cap E_n) = P(E_1)P(E_2 \mid E_1)\, P(E_3 \mid E_1\cap E_2) \ldots P(E_n \mid E_1\cap E_2 \cap \ldots \cap E_n)$$

EXAMPLE:

A lot contains 12 items of which 4 are defective. Three items are drawn at random from the lot one after the other. What is the probability that all three are non-defective?

SOLUTION:

Let A_1 be the event that the first item is not defective.
Let A_2 be the event that the second item is not defective.
Let A_3 be the event that the third item is not defective.
Then $P(A_1) = 8/12$, $P(A_2|A_1) = 7/11$, and $P(A_3|A_1 \cap A_2) = 6/10$
Hence, by multiplication theorem, the probability that all three are non-defective is
$P(A_1 \cap A_2 \cap A_3) = P(A_1)\, P(A_2|A_1)\, P(A_3|A_1 \cap A_2)$

$$= \frac{8}{12} \cdot \frac{7}{11} \cdot \frac{6}{10} = \frac{14}{55}$$

INDEPENDENCE:

An event A is said to be independent of an event B if the probability that A occurs is not influenced by whether B has or has not occurred. That is, $P(A|B) = P(A)$.
It follows then from the Multiplication Theorem that,
$P(A \cap B) = P(B)P(A|B) = P(B)P(A)$
We also know that,

$$\begin{aligned} P(B|A) &= \frac{P(A \cap B)}{P(A)} \\ &= \frac{P(A)P(B)}{P(A)} \qquad \textit{Because } P(A \cap B) = P(A)P(B) \text{ ,due to independence} \\ &= P(B) \end{aligned}$$

EXAMPLE:

Let A be the event that a randomly generated bit string of length four begins with a 1 and let B be the event that a randomly generated bit string of length four contains an even number of 0s.
Are A and B independent events?

SOLUTION:

A = {1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111}
B = {0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111}

Since there are 16 bit strings of length four, we have
$P(A) = 8/16 = 1/2$, $P(B) = 8/16 = 1/2$
Also,
$A \cap B$ = {1001, 1010, 1100, 1111} so that $P(A \cap B) = 4/16 = 1/4$
We note that,

$$P(A \cap B) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A)P(B)$$

Hence A and B are independent events.

EXAMPLE:

Let a fair coin be tossed three times. Let A be the event that first toss is heads, B be the even that the second toss is a heads, and C be the event that exactly two heads are tossed in a row. Examine pair wise independence of the three events.

SOLUTION:

The sample space of the experiment is
S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT} and the events are
A = {HHH, HHT, HTH, HTT}

B = {HHH, HHT, THH, THT}
C = {HHT, THH}
$A\cap B$ = {HHH, HHT}, $A\cap C$ = {HHT}, $B\cap C$ = {HHT, THH}

It follows that
P(A) = 4/8=1/2
P(B) = 4/8= 1/2
P(C) = 2/8=1/4
and
$P(A\cap B)$=2/8=1/4
$P(A\cap C)$= 1/8
$P(B\cap C)$ =2/8= 1/4

Note that,

$$P(A)P(B) \ = \ \frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}=P(A\cap B), \text{ so that A and B are independent.}$$

$$P(A)P(C) \ = \ \frac{1}{2}\cdot\frac{1}{4}=\frac{1}{8}=P(A\cap C), \text{ so that A and C are independent.}$$

$$P(B)P(C) \ = \ \frac{1}{2}\cdot\frac{1}{4}=\frac{1}{8}\neq P(B\cap C), \text{ so that B and C are dependent.}$$

EXAMPLE:

The probability that A hits a target is 1/3 and the probability that B hits the target is 2/5. What is the probability that target will be hit if A and B each shoot at the target?

SOLUTION:

It is clear from the nature of the experiment that the two events are independent.

Hence,
$P(A\cap B) = P(A)P(B)$

It follows that,

$$\begin{aligned} P(A\cup B) &= P(A) + P(B) - P(A\cap B) \\ &= P(A) + P(B) - P(A)P(B) \quad \text{(due to independence)} \\ &= \frac{1}{3}+\frac{2}{5}-\frac{1}{3}\cdot\frac{2}{5} \\ &= \frac{3}{5} \end{aligned}$$

LECTURE 38

RANDOM VARIABLE
PROBABILITY DISTRIBUTION
EXPECTATION AND VARIANCE

INTRODUCTION:

Suppose S is the sample space of some experiment. The outcomes of the experiment, or the points in S, need not be numbers. For example in tossing a coin,the outcomes are H (heads) or T(tails), and in tossing a pair of dice the outcomes are pairs of integers. However, we frequently wish to assign a specific number to each outcome of the experiment. For example, in coin tossing, it may be convenient to assign 1 to H and 0 to T; or in the tossing of a pair of dice, we may want to assign the sum of the two integers to the outcome. Such an assignment of numerical values is called a random variable.

RANDOM VARIABLE:

A random variable X is a rule that assigns a numerical value to each outcome in a sample Space S.

OR

It is a function which maps each outcome of the sample space into the set of real numbers. We shall let X(S) denote the set of numbers assigned by a random variable X, and refer to X(S) as the range space.

In formal terminology, X is a function from S(sample space) to the set of real numbers R, and X(S) is the range of X.

REMARK:

1. A random variable is also called a chance variable, or a stochastic variable(not called simply a variable, because it is a function).
2. Random variables are usually denoted by capital letters such as X, Y, Z; and the values taken by them are represented by the corresponding small letters.

EXAMPLE:

A pair of fair dice is tossed. The sample space S consists of the 36 ordered pairs i.e

S = {(1,1), (1,2), (1,3), …, (6,6)}

Let X assign to each point in S the sum of the numbers; then X is a random variable with range space i.e

X(S) = {2,3,4,5,6,7,8,9,10,11,12}

Let Y assign to each point in S the maximum of the two numbers in the outcomes; then Y is a random variable with range space.

Y(S) = {1,2,3,4,5,6}

PROBABILITY DISTRIBUTION OF A RANDOM VARIABLE:

Let $X(S) = \{x_1, x_2, \ldots, x_n\}$ be the range space of a random variable X defined on a finite sample space S.

Define a function f on X(S) as follows:

$f(\mathbf{x_i}) = P(X = \mathbf{x_i})$

$=$ sum of probabilities of points in S whose image is $\mathbf{x_i}$.

This function f is called the probability distribution or the probability function of X.

The probability distribution f of X is usually given in the form of a table.

x_1	x_2	…	x_n
$f(x_1)$	$f(x_2)$	…	$f(x_n)$

The distribution f satisfies the conditions.

$(i) \quad f(x_i) \ge 0$ and $(ii) \quad \sum_{i=1}^{n} f(x_i) = 1$

EXAMPLE:

A pair of fair dice is tossed. Let X assign to each point (a, b) in S = {(1,1), (1,2), …, (6,6)}, the sum of its number, i.e., X (a,b) = a+b. Compute the distribution f of X.

SOLUTION:

X is clearly a random variable with range space X(S) = {2,3,4,5,6,7,8,9,10, 11, 12}
(because X (a,b) = a+b ⇒ X (1,1) = 1+1=2, X (1,2) = 1+2=3, X (1,3) = 1+3=4 etc).

The distribution f of X may be computed as:

$$f(2) = P(X=2)=P(\{(1,1)\}) = \frac{1}{36}$$

$$f(3) = P(X=3) = P(\{(1,2), (2,1)\}) = \frac{2}{36}$$

$$f(4) = P(X=4) = P(\{(1,3), (2,2),(3,1)\}) = \frac{3}{36}$$

$$f(5) = P(X=5) = P(\{(1,4), (2,3),(3,2),(4,1)\}) = \frac{4}{36}$$

$$f(6) = P(X=6) = P(\{(1,5),(2,4),(3,3),(4,2),(5,1)\}) =$$

$$f(7) = P(X=7) = P(\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,2$$

$$f(8) = P(X=8) = P(\{(2,6),(3,5),(4,4),(5,3),(6,2)\}) = \frac{5}{36}$$

$$f(9) = P(X=9) = P(\{(3,6),(4,5),(5,4),(6,3)\}) = \frac{4}{36}$$

$$f(10) = P(X=10) = P(\{(4,6),(5,5),(6,4)\}) = \frac{3}{36}$$

$$f(11) = P(X=11) = P(\{(5,6),(6,5)\}) = \frac{2}{36}$$

$$f(12) = P(X=12) = P(\{(6,6)\}) = \frac{1}{36}$$

The distribution of X consists of the points in X(S) with their respective probabilities.

x_i	2	3	4	5	6	7	8	9	10	11	12
$f(x_i)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

EXAMPLE:

A box contains 12 items of which three are defective. A sample of three items is selected from the box.

If X denotes the number of defective items in the sample; find the distribution of X.

SOLUTION:

The sample space S consists of $\binom{12}{3} = 220$ that is 220 different samples of size 3.

The random variable X, denoting the number of defective items has the range space X(S) = {0,1,2,3}

There are $\binom{3}{0}\binom{9}{3} = 84$ samples of size 3 with no defective items;

hence

$$p_0 = P(X = 0) = \frac{84}{220}$$

There are $\binom{3}{1}\binom{9}{2} = 108$ samples of size 3 containing one defective item; hence .

$$p_1 = P(X = 1) = \frac{108}{220}$$

There are $\binom{3}{2}\binom{9}{1} = 27$ samples of size 3 containing two defective items;

hence

$$p_2 = P(X = 2) = \frac{27}{220}$$

Finally, there is $\binom{3}{3}\binom{9}{0} = 27$, only one sample of size 3 containing three defective items;

hence

$$p_3 = P(X = 3) = \frac{1}{220}$$

The distribution of X follows:

x_i	0	1	2	3
p_i	84/220	108/220	27/220	1/220

EXPECTATION OF A RANDOM VARIABLE

Let X be a random variable with probability distribution

x_i	x_1	x_2	x_3	…	x_n
$f(x_i)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	…	$f(x_n)$

The mean (denoted μ) or the expectation of X (written E(X)) is defined by

$\mu = E(X) = x_1 f(x_1) + x_2 f(x_2) + \ldots + x_n f(x_n)$

$$= \sum_{i=1}^{n} x_i f(x_i)$$

EXAMPLE:

What is the expectation of the number of heads when three fair coins are tossed?

SOLUTION:

The sample space of the experiment is:

S = {TTT, TTH, THT, HTT, THH, HTH, HHT, HHH}

Let the random variable X represents the number of heads (i.e 0,1,2,3) when three fair coins are tossed. Then X has the probability distribution.

x_i	$x_0=0$	$x_1=1$	$x_2=2$	$x_3=3$
$f(x_i)$	1/8	3/8	3/8	1/8

Hence, expectation of X is

$$E(X) = x_0\ f(x_0) + x_1 f(x_1) + x_2 f(x_2) + x_3 f(x_3)$$
$$= 0\cdot\frac{1}{8} + 1\cdot\frac{3}{8} + 2\cdot\frac{3}{8} + 3\cdot\frac{1}{8} = \frac{12}{8} = 1\cdot 5$$

EXERCISE:

A player tosses two fair coins. He wins Rs. 1 if one head appears, Rs.2 if two heads appear. On the other hand, he loses Rs.5 if no heads appear. Determine the expected value E of the game and if it is favourable to be player.

SOLUTION:

The sample space of the experiment is S = {HH, HT, TH, TT}

Now

$$P(\text{Two heads}) = P(HH) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$
$$P(\text{One head}) = P(HT, TH) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}$$
$$P(\text{No heads}) = P(TT) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

Thus, the probability of winning Rs.2 is $\frac{1}{4}$, of winning Rs 1 is $\frac{1}{2}$ and of losing Rs 5 is $\frac{1}{4}$.

Hence,

$$E = 2(\frac{1}{4}) + 1(\frac{1}{2}) - 5(\frac{1}{4})$$
$$= -\frac{1}{4} = -0.25$$

Since, the expected value of the game is negative, so it is unfavorable to the player.

EXAMPLE:

A coin is weighted so that and $P(H) = \frac{3}{4}$ and $P(T) = \frac{1}{4}$

The coin is tossed three times.

Let X denotes the number of heads that appear.

(a) Find the distribution of X

(b) Find the expectation of E(X)

SOLUTION:

(a) The sample space is S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

The probabilities of the points in sample space are

$$p(HHH) = \frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4} = \frac{27}{64} \qquad p(HHT) = \frac{3}{4}\cdot\frac{3}{4}\cdot\frac{1}{4} = \frac{9}{64}$$

$$p(HTH) = \frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4} = \frac{9}{64} \qquad p(THH) = \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{3}{4} = \frac{9}{64}$$

$$p(HTT) = \frac{3}{4}\cdot\frac{1}{4}\cdot\frac{1}{4} = \frac{3}{64} \qquad p(THT) = \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{4} = \frac{3}{64}$$

$$p(TTH) = \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{3}{4} = \frac{3}{64} \qquad p(TTT) = \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4} = \frac{1}{64}$$

The random variable X denoting the number of heads assumes the values 0,1,2,3 with the probabilities:

$$P(0) = P(TTT) = \frac{1}{64}$$

$$P(1) = P(HTT, THT, TTH) = \frac{3}{64} + \frac{3}{64} + \frac{3}{64} = \frac{9}{64}$$

$$P(2) = P(HHT, HTH, THH) = \frac{9}{64} + \frac{9}{64} + \frac{9}{64} = \frac{27}{64}$$

$$P(3) = P(HHH) = \frac{27}{64}$$

Hence, the distribution of X is

x_i	0	1	2	3
$P(x_i)$	1/64	9/64	27/64	27/64

(b) The expected value E(X) is obtained by multiplying each value of X by its probability and taking the sum.
The distribution of X is

x_i	0	1	2	3
$P(x_i)$	1/64	9/64	27/64	27/64

Hence

$$\begin{aligned} E(X) &= 0\left(\frac{1}{64}\right) + 1\left(\frac{9}{64}\right) + 2\left(\frac{27}{64}\right) + 3\left(\frac{27}{64}\right) \\ &= \frac{144}{64} \\ &= 2\cdot 25 \end{aligned}$$

VARIANCE AND STANDARD DEVIATION OF A RANDOM VARIABLE:

Let X be a random variable with mean μ and the probability distribution

x_1	X_2	x_3	…	x_n
$f(x_1)$	$f(x_2)$	$f(x_3)$	…	$f(x_n)$

The variance of X, measures the "spread" or "dispersion" of X from the mean μ and is denoted and defined as

$$\begin{aligned}\sigma_x^2 = \mathrm{Var}(X) &= \sum_{i=1}^{n}(x_i-\mu)^2 f(x_i)\\ &= E((X-\mu)^2)\\ &= E(X^2)-\mu^2\\ &= \sum x_i^2 f(x_i)-\mu^2\end{aligned}$$

The last expression is a more convenient form for computing Var(X).
The standard derivation of X, denoted by σ_X , is the non-negative square root of Var(X):

Where $\sigma_X = \sqrt{\mathrm{Var}(X)}$

EXERCISE:

Find the expectation μ, variance σ^2 and standard deviation σ of the distribution given in the following table.

x_i	1	3	4	5
$f(x_i)$	0.4	0.1	0.2	0.3

SOLUTION:

$\mu = E(X) = \sum x_i f(x_i)$
$= 1(0.4) + 3(0.1) + 4(0.2) + 5(0.3)$
$= 0.4 + 0.3 + 0.8 + 1.5$
$= 3.0$

Next
$E(X^2) = \sum x_i^2 f(x_i)$
$= 1^2 (0.4) + 3^2 (0.1) + 4^2 (0.2) + 5^2 (0.3)$
$= 0.4 + 0.9 + 3.2 + 7.5$
$= 12.0$

Hence
$\sigma^2 = \mathrm{Var}(X)= E(X^2) - \mu^2$
$= 12.0 - (3.0)^2 = 3.0$

and $\sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{3.0} \approx 1.7$

EXERCISE:

A pair of fair dice is thrown. Let X denote the maximum of the two numbers which appears.
(a) Find the distribution of X
(b) Find the μ, variance σ_x^2 = Var(X), and standard deviation σx of X

SOLUTION:

(a) The sample space S consist of the 36 pairs of integers (a,b) where a and b range from 1 to 6;

that is S = {(1,1), (1,2), …, (6,6)}
Since X assigns to each pair in S the larger of the two integers, the value of X are the integers from 1 to 6.
Note that:

$$f(1) = P(X = 1) = P(\{(1,1)\}) = \frac{1}{36}$$

$$f(2) = P(X = 2) = P(\{(2,1),(2,2),(1,2)\}) = \frac{3}{36}$$

$$f(3) = P(X = 3) = P(\{(3,1),(3,2),(3,3),(2,3),(1,3)\}) = \frac{5}{36}$$

$$f(4) = P(X = 4) = P(\{(4,1),(4,2),(4,3),(4,4),(3,4),(2,4),(1,4)\}) = \frac{7}{36}$$

Similarly

$$f(5) = P(X = 5) = \{(1,5),(2,5),(3,5),(4,5),(5,1),(5,2),(5,3),(5,4),(5,5)\}$$
$$= \frac{9}{36}$$

and

$$f(6) = P(X = 6) = \{(1,6),(2,6),(3,6),(4,6),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$
$$= \frac{11}{36}$$

Hence, the probability distribution of x is:

x_i	1	2	3	4	5	6
$f(x_i)$	1/36	3/36	5/36	7/36	9/36	11/36

(b)We find the expectation (mean) of X as

$$\begin{aligned}\mu = E(X) \quad &= \sum x_i \, f(x_i) \\ &= 1\cdot\frac{1}{36} + 2\cdot\frac{1}{36} + 3\cdot\frac{5}{36} + 4\cdot\frac{7}{36} + 5\cdot\frac{9}{36} + 6\cdot\frac{11}{36} \\ &= \frac{161}{36} \approx 4.5\end{aligned}$$

Next

$$
\begin{aligned}
E(X^2) &= \sum x_i^2 f(x_i) \\
&= 1^2 \cdot \frac{1}{36} + 2^2 \cdot \frac{1}{36} + 3^2 \cdot \frac{1}{36} + 4^2 \cdot \frac{7}{36} + 5^2 \cdot \frac{9}{36} + 6^2 \cdot \frac{11}{36} \\
&= \frac{791}{36} \approx 22.0
\end{aligned}
$$

Then

$$
\begin{aligned}
\sigma_x^2 = \mathrm{Var}(X) &= E(X^2) - \mu^2 \\
&= 22.0 - (4.5)^2 \\
&= 17.5
\end{aligned}
$$

and

$$\sigma_x = \sqrt{17.5} \approx 1.3$$

LECTURE 39

INTRODUCTION TO GRAPHS

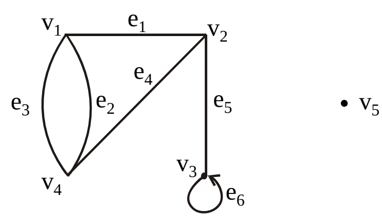
INTRODUCTION:

Graph theory plays an important role in several areas of computer science such as:

- switching theory and logical design
- artificial intelligence
- formal languages
- computer graphics
- operating systems
- compiler writing
- information organization and retrieval.

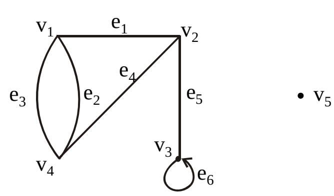
GRAPH:

A graph is a non-empty set of points called vertices and a set of line segments joining pairs of vertices called edges.
Formally, a graph G consists of two finite sets:
(i) A set V=V(G) of vertices (or points or nodes)
(ii)A set E=E(G) of edges; where each edge corresponds to a pair of vertices.



The graph G with
$V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and
$E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$

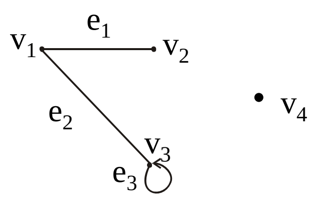
SOME TERMINOLOGY:



1. An edge connects either one or two vertices called its **endpoints** (edge e_1 connects vertices v_1 and v_2 described as $\{v_1, v_2\}$ i.e v_1 and v_2 are the endpoints of an edge e_1).
2. An edge with just one endpoint is called a **loop**. Thus a loop is an edge that connects a vertex to itself (e.g., edge e_6 makes a loop as it has only one endpoint v_3).
3. Two vertices that are connected by an edge are called **adjacent**; and a vertex that is an endpoint of a loop is said to be adjacent to itself.
4. An edge is said to be **incident** on each of its endpoints(i.e. e_1 is incident on v_1 and v_2).
5. A vertex on which no edges are incident is called **isolated** (e.g., v_5)
6. Two distinct edges with the same set of end points are said to be **parallel** (i.e. e_2 & e_3).

EXAMPLE:

Define the following graph formally by specifying its vertex set, its edge set, and a table giving the edge endpoint function.



SOLUTION:

Vertex Set = $\{v_1, v_2, v_3, v_4\}$

Edge Set = $\{e_1, e_2, e_3\}$

Edge - endpoint function is:

Edge	Endpoint
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_3\}$
e_3	$\{v_3\}$

EXAMPLE:

For the graph shown below

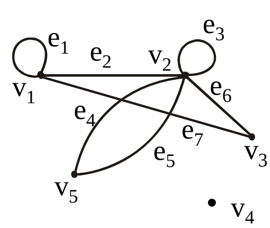
(i) find all edges that are incident on v_1;

(ii)find all vertices that are adjacent to v_3;

(iii)find all loops;

(iv)find all parallel edges;

(v)find all isolated vertices;



SOLUTION:

(i) v_1 is incident with edges e_1, e_2 and e_7

(ii) vertices adjacent to v_3 are v_1 and v_2

(iii) loops are e_1 and e_3

(iv) only edges e_4 and e_5 are parallel

(v) The only isolated vertex is v_4 in this Graph.

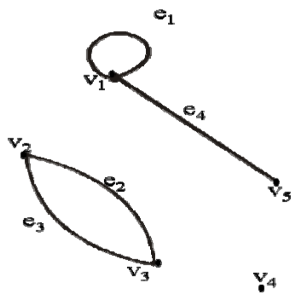
DRAWING PICTURE FOR A GRAPH:

Draw picture of Graph H having vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and edge set $\{e_1, e_2, e_3, e_4\}$ with edge endpoint function

Edge	Endpoint
e_1	$\{v_1\}$
e_2	$\{v_2,v_3\}$
e_3	$\{v_2,v_3\}$
e_4	$\{v_1,v_5\}$

SOLUTION:

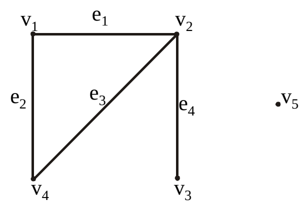
Given $V(H) = \{v_1, v_2, v_3, v_4, v_5\}$
and $E(H) = \{e_1, e_2, e_3, e_4\}$
with edge endpoint function



SIMPLE GRAPH

A simple graph is a graph that does not have any loop or parallel edges.

EXAMPLE:



It is a simple graph H
$V(H) = \{v_1, v_2, v_3, v_4, v_5\}$ & $E(H) = \{e_1, e_2, e_3, e_4\}$

EXERCISE:

Draw all simple graphs with the four vertices {u, v, w, x} and two edges, one of which is {u, v}.

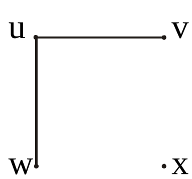
SOLUTION:

There are C(4,2) = 6 ways of choosing two vertices from 4 vertices. These edges may be listed as:

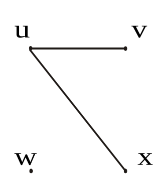
{u,v},{u,w},{u,x},{v,w}, {v,x},{w,x}

One edge of the graph is specified to be {u,v}, so any of the remaining five from this list may be chosen to be the second edge. This required graphs are:

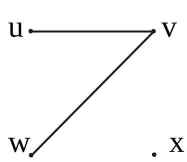
1.



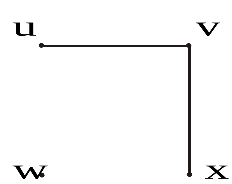
2.



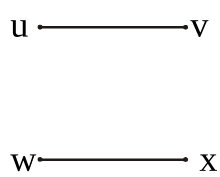
3.



4.



5.



DEGREE OF A VERTEX:

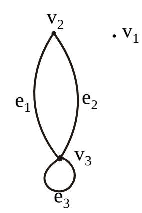
Let G be a graph and ***v*** a vertex of G. The degree of ***v***, denoted **deg(*v*),** equals the number of edges that are incident on ***v***, with an edge that is a loop counted twice.

Note:(i)The total degree of G is the sum of the degrees of all the vertices of G.

(ii) The **degree** of a loop is counted twice.

EXAMPLE:

For the graph shown



deg (v_1) = 0, since v_1 is isolated vertex.
deg (v_2) = 2, since v_2 is incident on e_1 and e_2 .
deg (v_3) = 4, since v_3 is incident on e_1,e_2 and the loop e_3.
Total degree of G = $\deg(v_1) + \deg(v_2) + \deg(v_3)$
$= 0 + 2 + 4$
$= 6$

REMARK:

The total degree of G, which is 6, equals twice the number of edges of G, which is 3.

THE HANDSHAKING THEOREM:

If G is any graph, then the sum of the degrees of all the vertices of G equals twice the number of edges of G.

Specifically, if the vertices of G are v_1, v_2, …, v_n, where n is a positive integer, then
the total degree of G $= \deg(v_1) + \deg(v_2) + \ldots + \deg(v_n)$
$= 2$. (the number of edges of G)

PROOF:

Each edge "e" of G connects its end points v_i and v_j. This edge, therefore contributes 1 to the degree of v_i and 1 to the degree of v_j.

If "e" is a loop, then it is counted twice in computing the degree of the vertex on which it is incident.

Accordingly, each edge of G contributes 2 to the total degree of G.

Thus,

the total degree of G = 2. (the number of edges of G)

COROLLARY:

The total degree of G is an even number

EXERCISE:

Draw a graph with the specified properties or explain why no such graph exists.

(i) Graph with four vertices of degrees 1, 2, 3 and 3
(ii) Graph with four vertices of degrees 1, 2, 3 and 4

(iii)Simple graph with four vertices of degrees 1, 2, 3 and 4

SOLUTION:

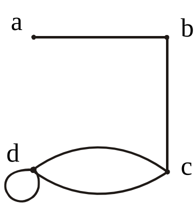
(i) Total degree of graph = 1 + 2 + 3 + 3
= 9 an odd integer

Since, the total degree of a graph is always even, hence no such graph is possible.

Note: As we know that "for any graph,the sum of the degrees of all the vertices of G equals twice the number of edges of G **or** the total degree of G is an even number".

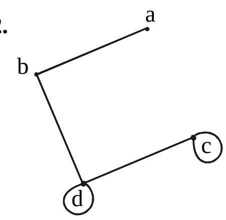
(ii) Two graphs with four vertices of degrees 1, 2, 3 & 4 are

1.



or

2.



The vertices a, b, c, d have degrees 1,2,3, and 4 respectively(i.e graph exists).

(iii) Suppose there was a simple graph with four vertices of degrees 1, 2, 3, and 4. Then the vertex of degree 4 would have to be connected by edges to four distinct vertices other than itself because of the assumption that the graph is simple (and hence has no loop or parallel edges.) This contradicts the assumption that the graph has four vertices in total. Hence there is no simple graph with four vertices of degrees 1, 2, 3, and 4, so simple graph is not possible in this case.

EXERCISE:

Suppose a graph has vertices of degrees 1, 1, 4, 4 and 6. How many edges does the graph have?

SOLUTION:

The total degree of graph = 1 + 1 + 4 + 4 + 6
= 16

Since, the total degree of graph = 2.(number of edges of graph) [by using Handshaking theorem]

$\Rightarrow$ 16 = 2.(number of edges of graph)

$\Rightarrow$ Number of edges of graph = $\frac{16}{2} = 8$

EXERCISE:

In a group of 15 people, is it possible for each person to have exactly 3 friends?

SOLUTION:

Suppose that in a group of 15 people, each person had exactly 3 friends. Then we could draw a graph representing each person by a vertex and connecting two vertices by an edge if the corresponding people were friends.

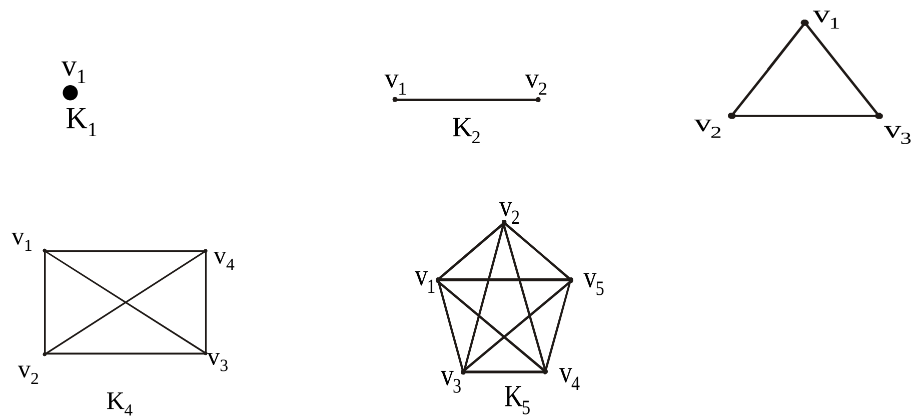
But such a graph would have 15 vertices each of degree 3, for a total degree of 45 (not even) which is not possible.

Hence, in a group of 15 people it is not possible for each to have exactly three friends.

COMPLETE GRAPH:

A complete graph on n vertices is a simple graph in which each vertex is connected to every other vertex and is denoted by K_n (K_n means that there are n vertices).

The following are complete graphs K_1, K_2,K_3, K_4 and K_5.



EXERCISE:

For the complete graph K_n, find

(i) the degree of each vertex

(ii)the total degrees

(iii)the number of edges

SOLUTION:

(i) Each vertex v is connected to the other (n-1) vertices in K_n; hence deg (v) = n - 1 for every v in K_n.

(ii)Each of the n vertices in K_n has degree n - 1; hence, the total degree in

K_n = (n - 1) + (n - 1) + … + (n - 1) n times

= n (n - 1)

(iii)Each pair of vertices in K_n determines an edge, and there are C(n, 2) ways of selecting two vertices out of n vertices. Hence,

Number of edges in K_n = C(n, 2)

$$= \frac{n(n-1)}{2}$$

Alternatively,

The total degrees in graph K_n = 2 (number of edges in K_n)

$\Rightarrow$ n(n-1) = 2(number of edges in K_n)

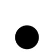
$\Rightarrow$ Number of edges in K_n $= \frac{n(n-1)}{2}$

REGULAR GRAPH:

A graph G is regular of degree k or k-regular if every vertex of G has degree k.

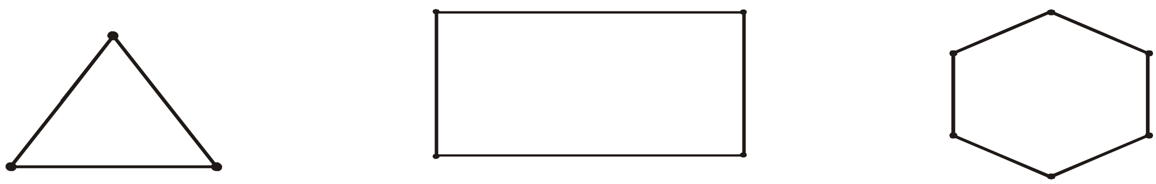
In other words, a graph is regular if every vertex has the same degree.

Following are some regular graphs.



(i) 0-regular

(ii) 1-regular



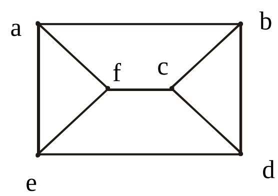
(iii) 2-regular

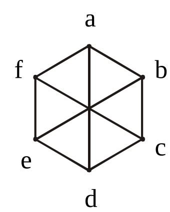
REMARK: The complete graph K_n is (n-1) regular.

EXERCISE:

Draw two 3-regular graphs with six vertices.

SOLUTION:

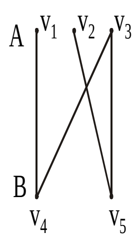


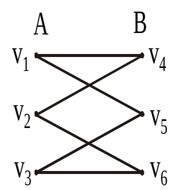


BIPARTITE GRAPH:

A bipartite graph G is a simple graph whose vertex set can be partitioned into two mutually disjoint non empty subsets A and B such that the vertices in A may be connected to vertices in B, but no vertices in A are connected to vertices in A and no vertices in B are connected to vertices in B.

The following are bipartite graphs





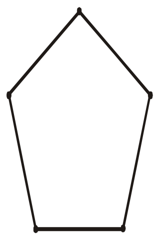
DETERMINING BIPARTITE GRAPHS:

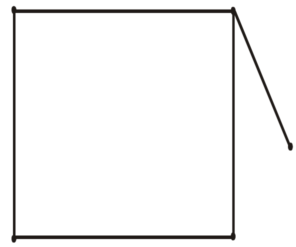
The following labeling procedure determines whether a graph is bipartite or not.

1. Label any vertex **a**
2. Label all vertices adjacent to **a** with the label **b**.
3. Label all vertices that are adjacent to a vertex just labeled **b** with label **a**.
4. Repeat steps 2 and 3 until all vertices got a distinct label (a bipartite graph) or there is a conflict i.e., a vertex is labeled with a and b (not a bipartite graph).

EXERCISE:

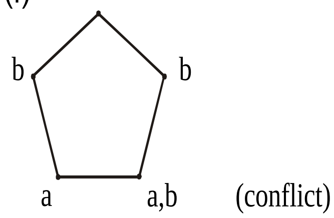
Find which of the following graphs are bipartite. Redraw the bipartite graph so that its bipartite nature is evident.





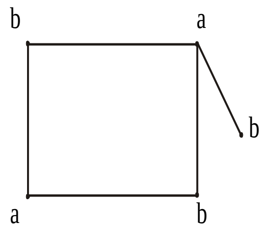
SOLUTION:

(i)

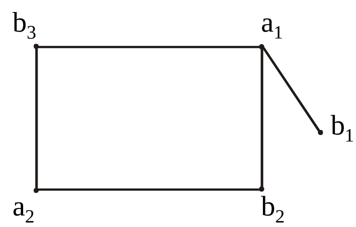


The graph is not bipartite.

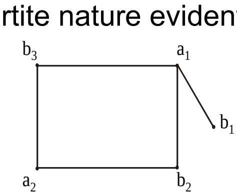
(ii)

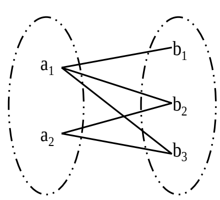


By labeling procedure, each vertex gets a distinct label. Hence the graph is bipartite. To redraw the graph we mark labels a's as a_1, $a_2$and b's as b_1, b_2,



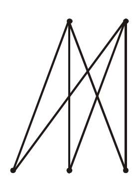
Redrawing graph with bipartite nature evident.



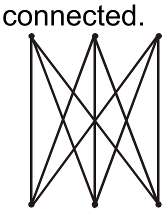


COMPLETE BIPARTITE GRAPH:

A complete bipartite graph on (m+n) vertices denoted $K_{m,n}$ is a simple graph whose vertex set can be partitioned into two mutually disjoint non empty subsets A and B containing m and n vertices respectively, such that each vertex in set A is connected (adjacent) to every vertex in set B, but the vertices within a set are not connected.



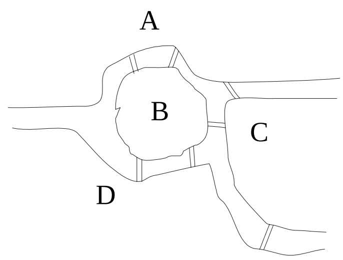
$K_{2,3}$



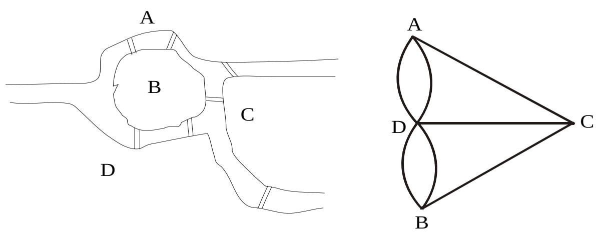
$K_{3,3}$

LECTURE 40
PATHS AND CIRCUITS

KONIGSBERG BRIDGES PROBLEM



It is possible for a person to take a walk around town, starting and ending at the same location and crossing each of the seven bridges exactly once?



Is it possible to find a route through the graph that starts and ends at some vertex A, B, C or D and traverses each edge exactly once?

Equivalently:

Is it possible to trace this graph, starting and ending at the same point, without ever lifting your pencil from the paper?

DEFINITIONS:

Let G be a graph and let v and w be vertices in graph G.

1. WALK

A walk from v to w is a finite alternating sequence of adjacent vertices and edges of G.

Thus a walk has the form

$$v_0e_1v_1e_2 \dots v_{n-1}e_nv_n$$

where the v's represent vertices, the e's represent edges v_0=v , v_n=w, and for all i = 1, 2 … n, v_{i-1} and v_i are endpoints of e_i.

The trivial walk from v to v consists of the single vertex v.

2. CLOSED WALK

A closed walk is a walk that starts and ends at the same vertex.

3. CIRCUIT

A circuit is a closed walk that does not contain a repeated edge. Thus a circuit is a walk of the form

$$v_0e_1v_1e_2 \dots v_{n-1}\ e_n\ v_n$$

where $v_0 = v_n$ and all the e_i,s are distinct.

4. SIMPLE CIRCUIT

A simple circuit is a circuit that does not have any other repeated vertex except the first and last.
Thus a simple circuit is a walk of the form

$$v_0e_1v_1e_2 \dots v_{n-1}e_nv_n$$

where all the e_i,s are distinct and all the v_j,s are distinct except that $v_0 = v_n$

5. PATH

A path from v to w is a walk from v to w that does not contain a repeated edge.
Thus a path from v to w is a walk of the form

$$v = v_0e_1v_1e_2 \dots v_{n-1}e_nv_n = w$$

where all the e_i,s are distinct (that is $e_i \neq e_k$ for any $i \neq k$).

6. SIMPLE PATH

A simple path from v to w is a path that does not contain a repeated vertex.
Thus a simple path is a walk of the form

$$v = v_0e_1v_1e_2 \dots v_{n-1}e_nv_n= w$$

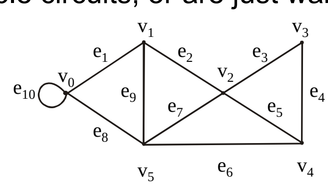
where all the e_i,s are distinct and all the v_j,s are also distinct (that is, $v_j \neq v_m$ for any $j \neq m$).

SUMMARY

	Repeated Edge	Repeated Vertex	Starts and Ends at Same Point
walk	allowed	Allowed	allowed
closed walk	allowed	Allowed	yes(means, where it starts also ends at that point)
circuit	no	Allowed	yes
simple circuit	no	first and last only	yes
path	no	Allowed	allowed
simple path	no	no	No

EXERCISE:

In the graph below, determine whether the following walks are paths, simple paths, closed walks, circuits, simple circuits, or are just walks.

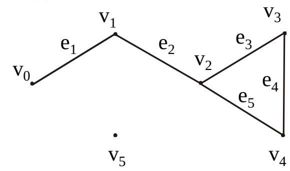


(a) $v_1e_2v_2e_3v_3e_4v_4e_5v_2e_2v_1e_1v_0$
(b) $v_1v_2v_3v_4v_5v_2$
(c) $v_4v_2v_3v_4v_5v_2v_4$
(d) $v_2v_1v_5v_2v_3v_4v_2$
(e) $v_0v_5v_2v_3v_4v_2v_1$
(f) $v_5v_4v_2v_1$

SOLUTION:

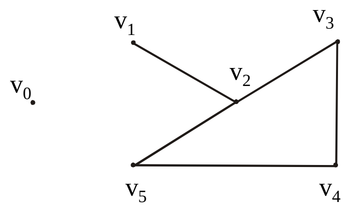
(a) $v_1e_2v_2e_3v_3e_4v_4e_5v_2e_2v_1e_1v_0$

(a)



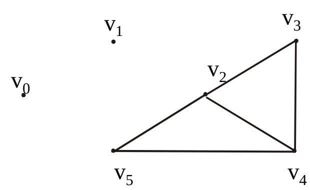
This graph starts at vertex v_1,then goes to v_2 along edge e_2,and moves continuously, at the end it goes from v_1 to v_0 along e_1.Note it that the vertex v_2 and the edge e_2 is repeated twice, and starting and ending, not at the same points.Hence The graph is just a walk.

(b) $v_1v_2v_3v_4v_5v_2$



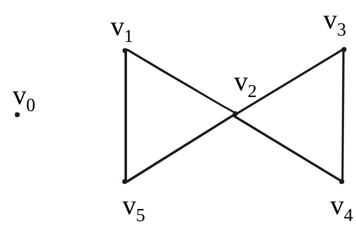
In this graph vertex v_2 is repeated twice.As no edge is repeated so the graph is a path.

(c) $v_4v_2v_3v_4v_5v_2v_4$



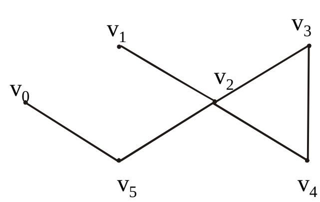
As vertices v_2 & v_4 are repeated and graph starts and ends at the same point v_4,also the edge(i.e. e_5)connecting v_2 & v_4 is repeated, so the graph is a closed walk.

(d) $v_2v_1v_5v_2v_3v_4v_2$



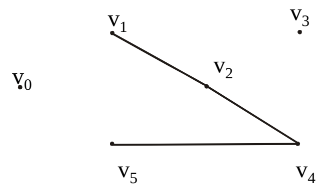
In this graph, vertex v_2 is repeated and the graph starts and end at the same vertex (i.e. at v_2) and no edge is repeated, hence the above graph is a circuit.

(e) $v_0v_5v_2v_3v_4v_2v_1$



Here vertex v_2 is repeated and no edge is repeated so the graph is a path.

(f) $v_5v_4v_2v_1$



Neither any vertex nor any edge is repeated so the graph is a simple path**.**

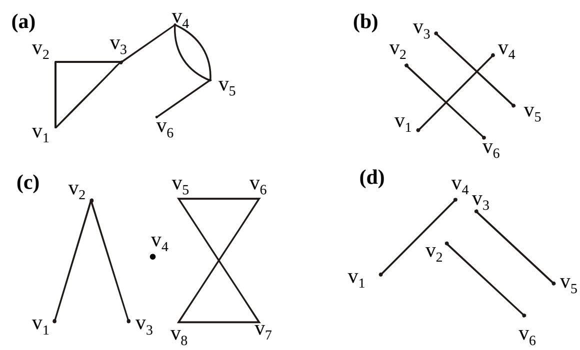
CONNECTEDNESS:

Let G be a graph. Two vertices *v* and *w* of G are connected if, and only if, there is a walk from *v* to *w*. The graph G is connected if, and only if, given any two vertices *v* and *w* in G, there is a walk from *v* to *w*. Symbolically:

G is connected $\Leftrightarrow$ $\forall$ vertices v, w $\in$ V (G), $\exists$ a walk from v to w:

EXAMPLE:

Which of the following graphs have a connectedness**?**



EULER CIRCUITS

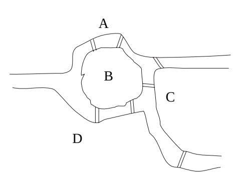
DEFINITION:

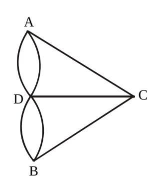
Let G be a graph. An Euler circuit for G is a circuit that contains every vertex and every edge of G. That is, an Euler circuit for G is sequence of adjacent vertices and edges in G that starts and ends at the same vertex uses every vertex of G at least once, and used every edge of G exactly once.

THEOREM:

A graph G has an Euler circuit if, and only if, G is connected and every vertex of G has an even degree.

KONIGSBERG BRIDGES PROBLEM



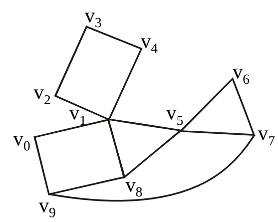


We try to solve Konigsberg bridges problem by Euler method.

Here deg(a)=3,deg(b)=3,deg(c)=3 and deg(d)=5 as the vertices have odd degree so there is no possibility of an Euler circuit.

EXERCISE:

Determine whether the following graph has an Euler circuit**.**

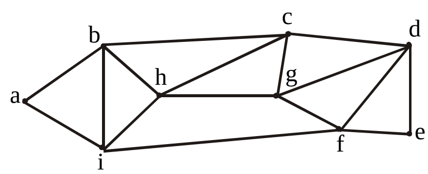


SOLUTION:

As deg (v_1) =5, an odd degree so the following graph has not an Euler circuit.

EXERCISE:

Determine whether the following graph has Euler circuit.



SOLUTION:

From above clearly deg(a)=2, deg(b)=4, deg(c)=4, deg(d)=4, deg(e)=2, deg(f)=4, deg(g)=4, deg(h)=4, deg(i)=4

Since the degree of each vertex is even, and the graph has Euler Circuit. One such circuit is:

a b c d e f g d f i h c g h b i a

EULER PATH

DEFINITION:

Let G be a graph and let v and w be two vertices of G. An Euler path from v to w is a sequence of adjacent edges and vertices that starts at v, end and w, passes through every vertex of G at least once, and traverses every edge of G exactly once.

COROLLARY

Let G be a graph and let v and w be two vertices of G. There is an Euler path from v to w if, and only if, G is connected, v and w have odd degree and all other vertices of G have even degree.

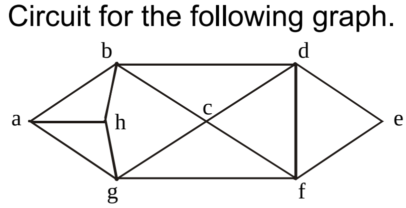
HAMILTONIAN CIRCUITS

DEFINITION:

Given a graph G, a Hamiltonian circuit for G is a simple circuit that includes every vertex of G. That is, a Hamiltonian circuit for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once.

EXERCISE:

Find Hamiltonian Circuit for the following graph.



SOLUTION:

The Hamiltonian Circuit for the following graph is:

a b d e f c g h a

Another Hamiltonian Circuit for the following graph could be:

a b c d e f g h a

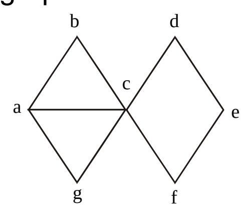
PROPOSITION:

If a graph G has a Hamiltonian circuit then G has a sub-graph H with the following properties:

1. H contains every vertex G
2. H is connected
3. H has the same number of edges as vertices
4. Every vertex of H has degree 2

EXERCISE:

Show that the following graph does not have a Hamiltonian circuit.



Here deg(c)=5,if we remove 3 edges from vertex c then deg(b)< 2 , deg(g)< 2
or deg(f)< 2,deg(d)< 2.

It means that this graph does not satisfy the desired properties as above, so the graph does not have a Hamiltonian circuit.

LECTURE 41
MATRIX REPRESENTATIONS OF GRAPHS

MATRIX:
An m × n matrix A over a set S is a rectangular array of elements of S arranged into m rows and n columns:

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

← ith row of A

↑
jth column of A

Briefly, it is written as:

A = [a_{ij}]$_{m\times n}$

EXAMPLE:

$$A=\begin{bmatrix} 4 & -2 & 0 & 6 \\ 2 & -3 & 1 & 9 \\ 0 & 7 & 5 & -1 \end{bmatrix}$$

A is a matrix having 3 rows and 4 columns. We call it a 3 × 4 matrix, or matrix of size 3 × 4(or we say that a matrix having an order 3 × 4).
Note it that
a_{11} = 4 (11 means 1st row and 1st column), a_{12} = -2 (12 means 1st row and 2nd column),
a_{13} = 0, a_{14} = 6
a_{21} = 2, a_{22} = -3, a_{23} = 1, a_{24} =9 etc.

SQUARE MATRIX:
A matrix for which the number of rows and columns are equal is called a square matrix. A square matrix A with m rows and n columns (size m × n) but m=n (i.e of order n × n) has the form:

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ii} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ni} & \cdots & a_{nn} \end{bmatrix}$$

Diagonal entries

Note:
The main diagonal of A consists of all the entries
$a_{11}, a_{22}, a_{33}, \ldots, a_{ii},\ldots, a_{nn}$

TRANSPOSE OF A MATRIX:

The transpose of a matrix A of size m × n, is the matrix denoted by A^t of size n × m, obtained by writing the rows of A, in order, as columns.(**Or** we can say that transpose of a matrix means "write the rows instead of colums or write the columns instead of rows". Thus if

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad then \;\; A^t = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

EXAMPLE:

$$A = \begin{bmatrix} 4 & -2 & 0 & 6 \\ 2 & -3 & 1 & 9 \\ 0 & 7 & 5 & -1 \end{bmatrix}$$

Then

$$A^t = \begin{bmatrix} 4 & 2 & 0 \\ -2 & -3 & 7 \\ 0 & 1 & 5 \\ 6 & 9 & -1 \end{bmatrix}$$

SYMMETRIC MATRIX:

A square matrix A = $[a_{ij}]$ of size n $\times$ n is called symmetric if, and only if, $A^t = A$
i.e., for all i, j = 1, 2, …, n, $a_{ij} = a_{ji}$

EXAMPLE:

$$\text{Let} \quad A = \begin{bmatrix} 1 & 3 & 7 \\ 5 & 2 & 9 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 4 & 2 & 0 \\ 2 & -3 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

$$\text{Then} \quad A^t = \begin{bmatrix} 1 & 5 \\ 3 & 2 \\ 7 & 9 \end{bmatrix}, \quad \text{and} \quad B^t = \begin{bmatrix} 4 & 2 & 0 \\ 2 & -3 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

Note that $B^t = B$, so that B is a symmetric matrix.

MATRIX MULTIPLICATION:

Suppose A and B are two matrices such that the number of columns of A is equal to the number of rows of B, say A is an m $\times$ p matrix and B is a p $\times$ n matrix. Then the product of A and B, written AB, is the m $\times$ n matrix whose ijth entry is obtained by multiplying the elements of the ith row of A by the corresponding elements of the jth column of B and then adding;

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ip} \\ \vdots & \vdots & & \vdots \\ a_{mi} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \begin{bmatrix} \mathrm{b}_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ \mathrm{b}_{21} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & & & & \\ \mathrm{b}_{\mathrm{p}1} & \cdots & b_{pj} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1n} \\ \vdots & & & & \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{in} \\ \vdots & & & & \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mn} \end{bmatrix}$$

where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}$$

REMARK:

If the number of columns of A is not equal to the number of rows of B, then the product AB is not defined.

EXAMPLE:

Find the product AB and BA of the matrices

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & -4 \\ 3 & -2 & 6 \end{bmatrix}$$

SOLUTION:

Size of A is 2 × 2 and of B is 2 × 3, the product AB is defined as a 2 × 3 matrix. But BA is not defined, because no. of columns of B = 3 ≠ 2 = no. of rows of A.

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -4 \\ 3 & -2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(2)+(3)(3) & (1)(0)+(3)(6) & (1)(-4)+(3)(6) \\ (2)(2)+(-1)(3) & (2)(0)+(-1)(-2) & (2)(-4)+(-2)(6) \end{bmatrix} = \begin{bmatrix} 11 & -6 & 14 \\ 1 & 2 & -14 \end{bmatrix}$$

EXERCISE:

Find AA^t and A^tA, where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix}$$

SOLUTION:

A^t is obtained from A by rewriting the rows of A as columns:

i.e $A^t = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{bmatrix}$

Now

$$AA^t = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1+4+0 & 3-2+0 \\ 3-2+0 & 9+1+16 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 26 \end{bmatrix}$$

and

$$A^tA = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+9 & 2-3 & 0+12 \\ 2-3 & 4+1 & 0-4 \\ 0+12 & 0-4 & 0+16 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -1 & 12 \\ -1 & 5 & -4 \\ 12 & -4 & 16 \end{bmatrix}$$

ADJACENCY MATRIX OF A GRAPH:

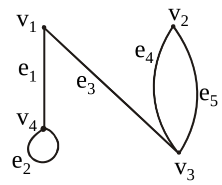
Let G be a graph with ordered vertices $v_1, v_2, \ldots, v_n$.The adjacency matrix of G is the matrix A = $[a_{ij}]$ over the set of non-negative integers such that

a_{ij} = the number of edges connecting v_i and v_j for all i, j = 1, 2, …, n.
OR
The adjancy matrix say A= $[a_{ij}]$ is also defined as

$$a_{ij} = \begin{cases} 1 \;\; if\ \{v_{i,}v_j\}\ is\ an\ edge\ of\ G \\ 0 \;\; otherwise \end{cases}$$

EXAMPLE:

A graph with it's adjacency matrix is shown.



$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Note that the nonzero entries along the main diagonal of A indicate the presence of loops and entries larger than 1 correspond to parallel edges.
Also note A is a symmetric matrix.

EXERCISE:

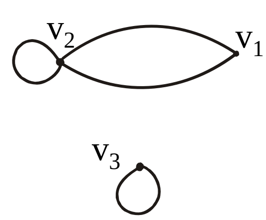
Find a graph that have the following adjacency matrix.

$$\begin{bmatrix} 0 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

SOLUTION:

Let the three vertices of the graph be named v_1, v_2 and v_3. We label the adjacency matrix across the top and down the left side with these vertices and draw the graph accordingly(as from v_1 to v_2 there is a value "2",it means that two parallel edges between v_1 and v_2 and same condition occurs between v_2 and v_1 and the value "1" represent the loops of v_2 and v_3).

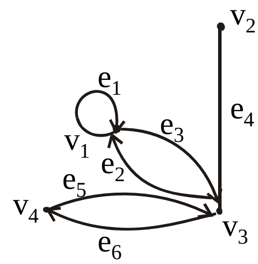
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 0 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$



DIRECTED GRAPH:

A directed graph or digraph, consists of two finite sets: a set V(G) of vertices and a set D(G) of directed edges, where each edge is associated with an ordered pair of vertices called its end points.
If edge e is associated with the pair (v, w) of vertices, then e is said to be the directed edge from v to w and is represented by drawing an arrow from v to w.

EXAMPLE OF A DIGRAPH:



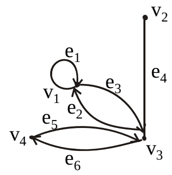
ADJACENCY MATRIX OF A DIRECTED GRAPH:

Let G be a graph with ordered vertices $v_1, v_2, \dots, v_n$.
The adjacency matrix of G is the matrix $A = [a_{ij}]$ over the set of non-negative integers such that

a_{ij} = the number of arrows from v_i to v_j for all $i, j = 1, 2, \dots, n$.

EXAMPLE:

A directed graph with its adjacency matrix is shown



$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

is the adjacency matrix

EXERCISE:

Find directed graph that has the adjacency matrix

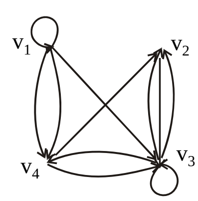
$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

SOLUTION:

The 4×4 adjacency matrix shows that the graph has 4 vertices say v_1, v_2, v_3 and v_4 labeled across the top and down the left side of the matrix.

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

A corresponding directed graph is



It means that a loop exists from v_1 and v_3, two arrows go from v_1 to v_4 and two from v_3 and v_2 and one arrow go from v_1 to v_3, v_2 to v_3, v_3 to v_4, v_4 to v_2 and v_3.

THEOREM

If G is a graph with vertices $v_1, v_2, \dots, v_m$ and A is the adjacency matrix of G, then for each positive integer n,
the ijth entry of A^n = the number of walks of length n from v_i to v_j
for all integers $i,j = 1, 2, \dots, n$

PROBLEM:

Let $$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

be the adjacency matrix of a graph G with vertices v_1, v_2, and v_3. Find
(a) the number of walks of length 2 from v_2 to v_3
(b) the number of walks of length 3 from v_1 to v_3
Draw graph G and find the walks by visual inspection for (a)

SOLUTION:

(a)

$$A^2 = AA = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 2 & 2 \\ 3 & 2 & 5 \end{bmatrix} \longrightarrow$$ it shows the entry (2,3) from v_2 to v_3

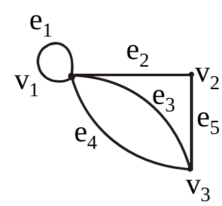
Hence, number of walks of length 2(means "multiply matrix A two times") from v_2 to v_3 = the entry at (2,3) of A^2 = 2

(b)

$$A^3 = AA^2 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 6 & 3 & 3 \\ 3 & 2 & 2 \\ 3 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 15 & 9 & 15 \\ 9 & 5 & 8 \\ 15 & 8 & 8 \end{bmatrix} \longrightarrow$$ it shows the entry (1,3) from v_1 to v_3

Hence, number of walks of length 3 from v_1 to v_3 = the entry at (1,3) of A^3 = 15
Walks from v_2 to v_3 by visual inspection of graph is

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$



so in part (a)two Walks of length 2 from v_2 to v_3 are
(i) $v_2\ e_2\ v_1\ e_3\ v_3$ (by using the above theorem).
(ii) $v_2\ e_2\ v_1\ e_4\ v_3$

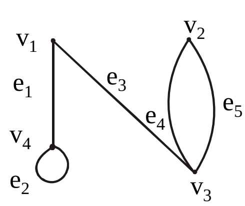
INCIDENCE MATRIX OF A SIMPLE GRAPH:

Let G be a graph with vertices $v_1, v_2, \ldots, v_n$ and edges $e_1, e_2, \ldots, e_n$. The incidence matrix of G is the matrix $M = [m_{ij}]$ of size $n \times m$ defined by

$$m_{ij} = \begin{cases} 1 & \text{if the vertex } v_i \text{ is incident on the edge } e_j \\ 0 & \text{otherwise} \end{cases}$$

EXAMPLE:

A graph with its incidence matrix is shown.



$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

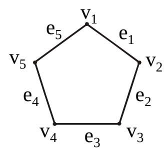
REMARK:

In the incidence matrix
1. Multiple edges are represented by columns with identical entries (in this matrix e_4 & e_5 are multiple edges).

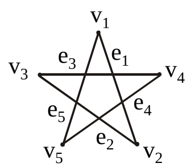
2. Loops are represented using a column with exactly one entry equal to 1, corresponding to the vertex that is incident with this loop and other zeros (here e_2 is only a loop).

LECTURE 42
ISOMORPHISM OF GRAPHS

Here we have a graph



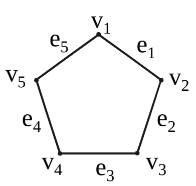
Which can also be defined as



Its vertices and edges can be written as:
$V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G) = \{e_1, e_2, e_3, e_4, e_5\}$
Edge endpoint function is:

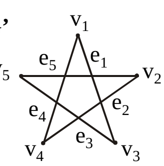
Edge	Endpoints
E_1	$\{v_1,v_2\}$
E_2	$\{v_2,v_3\}$
E_3	$\{v_3,v_4\}$
E_4	$\{v_4,v_5\}$
E_5	$\{v_5,v_1\}$

G



Another graph G' is

G'



Edge endpoint function of G is:

Edge	Endpoints
e_1	$\{v_1,v_2\}$
e_2	$\{v_2,v_3\}$
e_3	$\{v_3,v_4\}$
e_4	$\{v_4,v_5\}$
e_5	$\{v_5,v_1\}$

Edge endpoint function of G' is:

Edge	Endpoints
e_1	$\{v_1,v_3\}$
e_2	$\{v_2,v_4\}$
e_3	$\{v_3,v_5\}$
e_4	$\{v_1,v_4\}$
e_5	$\{v_2,v_5\}$

Two graphs (G and G') that are the same except for the labeling of their vertices are not considered different.

GRAPHS OF EDGE POINT FUNCTIONS

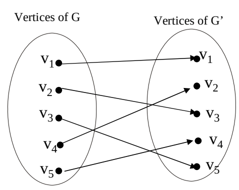
Edge point function of G is:

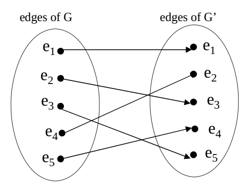
Edge point function of G' is:

Edge	Endpoints
e_1	$\{v_1,v_2\}$
e_2	$\{v_2,v_3\}$
e_3	$\{v_3,v_4\}$
e_4	$\{v_4,v_5\}$
e_5	$\{v_5,v_1\}$

Edge	Endpoints
e_1	$\{v_1,v_3\}$
e_2	$\{v_2,v_4\}$
e_3	$\{v_3,v_5\}$
e_4	$\{v_1,v_4\}$
e_5	$\{v_2,v_5\}$

Note it that the graphs G and G' are looking different because in G the end points of e_1 are v_1,v_2 but in G' are v_1, v_3 etc.
Buts G' is very similar to G ,if the vertices and edges of G' are relabeled by the function shown below, then G' becomes same as G:





It shows that if there is one-one correspondence between the vertices of G and G', then also one-one correspondence between the edges of G and G'.

ISOMORPHIC GRAPHS:

Let G and G' be graphs with vertex sets V(G) and V(G') and edge sets E(G) and E(G'), respectively.
G is isomorphic to G' if, and only if, there exist one-to-one correspondences $g: V(G)\rightarrow V(G')$ and $h: E(G) \rightarrow E(G')$ that preserve the edge-endpoint functions of G and G' in the sense that for all $v \in V(G)$ and $e \in E(G)$.
v is an endpoint of e $\Leftrightarrow$ g(v) is an endpoint of h(e).

EQUIVALENCE RELATION:

Graph isomorphism is an equivalence relation on the set of graphs.
1. Graphs isomorphism is Reflexive (It means that the graph should be isomorphic to itself).
2. Graphs isomorphism is Symmetric (It means that if G is isomorphic to G' then G' is also isomorphic to G).
3. Graphs isomorphism is Transitive (It means that if G is isomorphic to G' and G' is isomorphic to G'', then G is isomorphic to G'').

ISOMORPHIC INVARIANT:

A property P is called an isomorphic invariant if, and only if, given any graphs G and G', if G has property P and G' is isomorphic to G, then G' has property P.

THEOREM OF ISOMORPHIC INVARIANT:

Each of the following properties is an invariant for graph isomorphism, where n, m and k are all non-negative integers, if the graph:
1. has n vertices.
2. has m edges.
3. has a vertex of degree k.
4. has m vertices of degree k.
5. has a circuit of length k.
6. has a simple circuit of length k.
7. has m simple circuits of length k.
8. is connected.
9. has an Euler circuit.

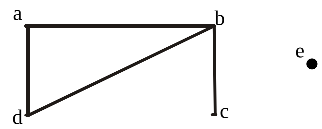
10. has a Hamiltonian circuit.

DEGREE SEQUENCE:

The degree sequence of a graph is the list of the degrees of its vertices in non-increasing order.

EXAMPLE:

Find the degree sequence of the following graph.



SOLUTION:

Degree of a = 2, Degree of b = 3, Degree of c = 1,
Degree of d = 2, Degree of e = 0

By definition, degree of the vertices of a given graph should be in decreasing (non-increasing) order.

Therefore   Degree sequence is: 3, 2, 2, 1, 0

GRAPH ISOMORPHISM FOR SIMPLE GRAPHS:

If G and G' are simple graphs (means the "graphs which have no loops or parallel edges") then G is isomorphic to G' if, and only if, there exists a one-to-one correspondence (1-1 and onto function) g from the vertex set V (G) of G to the vertex set
V (G') of G' that preserves the edge-endpoint functions of G and G' in the sense that for all vertices u and v of G,

{u, v} is an edge in G ⇔ {g(u), g(v)} is an edge in G'.

OR

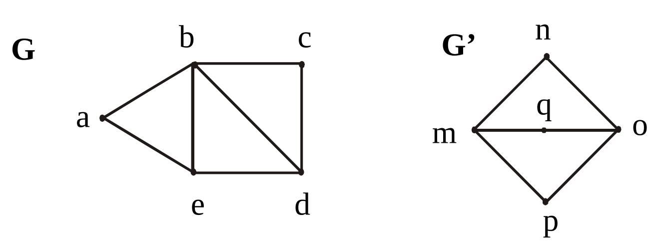
You can say that with the property of one-one correspondence, u and v are adjacent in graph G ⇔ if g (u) and g (v) are adjacent in G'.

Note:

It should be noted that unfortunately, there is no efficient method for checking that whether two graphs are isomorphic(methods are there but take so much time in calculations).Despite that there is a simple condition. Two graphs are isomorphic if they have the same number of vertices(as there is a 1-1 correspondence between the vertices of both the graphs) and the same number of edges(also vertices should have the same degree.

EXERCISE:

Determine whether the graph G and G' given below are isomorphic.



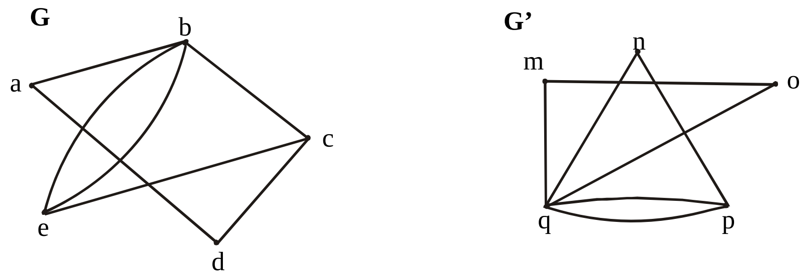
SOLUTION:

As both the graphs have the same number of vertices. But the graph G has 7 edges and the graph G' has only 6 edges. Therefore the two graphs are not isomorphic.

Note: As the edges of both the graphs G and G' are not same then how the one-one correspondence is possible ,that the reason the graphs G and G' are not isomorphic.

EXERCISE:

Determine whether the graph G and G’ given below are isomorphic.

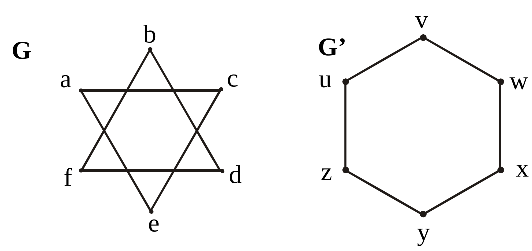


SOLUTION:

Both the graphs have 5 vertices and 7 edges. The vertex q of G’ has degree 5. However G does not have any vertex of degree 5 (so one-one correspondence is not possible). Hence, the two graphs are not isomorphic.

EXERCISE:

Determine whether the graph G and G’ given below are isomorphic.

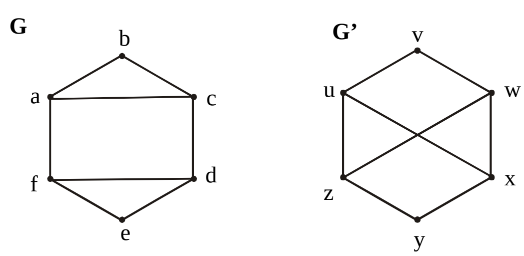


SOLUTION:

Clearly the vertices of both the graphs G and G’ have the same degree (i.e “2”) and having the same number of vertices and edges but isomorphism is not possible. As the graph G’ is a connected graph but the graph G is not connected due to have two components (eca and bdf). Therefore the two graphs are non isomorphic.

EXERCISE:

Determine whether the graph G and G’ given below are isomorphic.



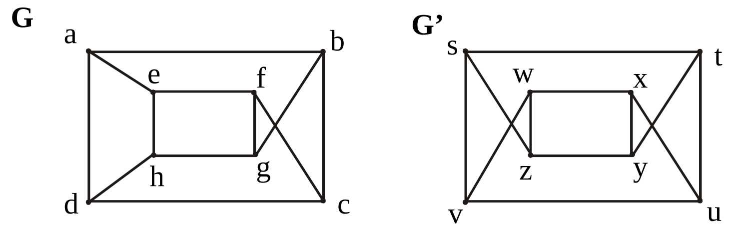
SOLUTION:

Clearly G has six vertices,G’ also has six vertices. And the graph G has two simple circuits of length 3; one is abca and the other is defd.But G’ does not have any simple circuit of length 3(as one simple circuit in G’ is uxwv of length 4).Therefore the two graphs are non-isomorphic.

Note: A simple circuit is a circuit that does not have any other repeated vertex except the first and last.

EXERCISE:

Determine whether the graph G and G' given below are isomorphic.

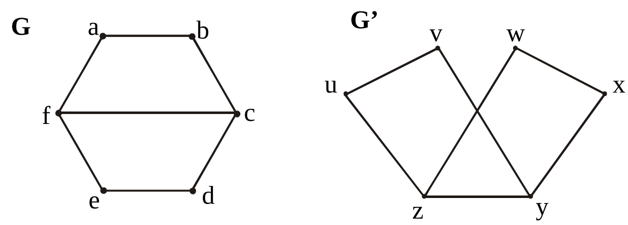


SOLUTION:

Both the graph G and G' have 8 vertices and 12 edges and both are also called regular graph(as each vertex has degree 3).The graph G has two simple circuits of length 5; abcfea(i.e starts and ends at a) and cdhgfc(i.e starts and ends at c). But G' does not have any simple circuit of length 5 (it has simple circuit tyxut,vwxuv of length 4 etc). Therefore the two graphs are non-isomorphic.

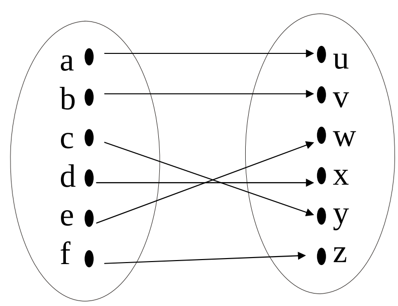
EXERCISE:

Determine whether the graph G and G' given below are isomorphic.



SOLUTION:

We note that all the isomorphism invariants seems to be true.
We shall prove that the graphs G and G' are isomorphic.
Here G has four vertices of degree "2" and two vertices of degree"3". Similar case in G'. Also G and G' have circuits of length 4.As a is adjacent to b and f in graph G.In graph G' u is adjacent to v and z. And as a and u has degree 2 so both are mapped. And b mapped with v, f mapped with z(as both have the same degree also a is adjacent to f and u is to z), and as we moves further we get the 1-1 correspondence.
Define a function f: V(G) $\rightarrow$V(G') as follows.

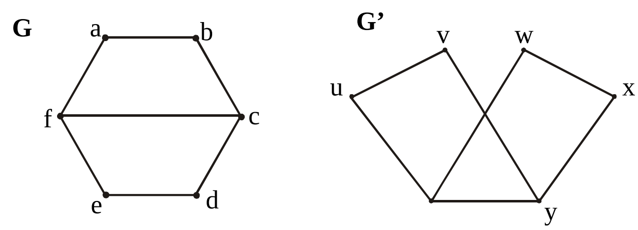


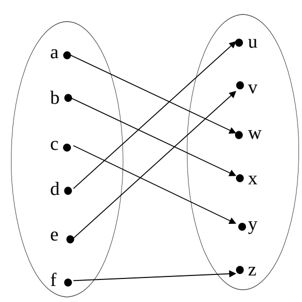
Clearly the above function is one and onto that is a bijective mapping. Note that I write the above mapping by keeping in mind the invariants of isomorphism as well as the fact that the mapping should preserve edge end point function. Also you should note that the mapping is not unique.
f is clearly a bijective function. The fact that f preserves the edge endpoint functions of G and G' is shown below.

Edges of G	Edges of G'
{a, b}	{u, v} = {g(a), g(b)}
{b, c}	{v, y} = {g(b), g(c)}
{c, d}	{y, x} = {g(c), g(d)}
{d, e}	{x, w} = {g(d), g(e)}
{e, f}	{w, z} = {g(e), g(f)}
{a, f}	{u, z} = {g(a), g(f)}
{c, f}	{y, z} = {g(c), g(f)}

ALTERNATIVE SOLUTION:

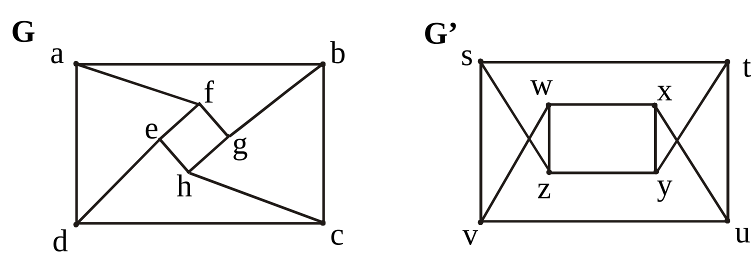
We shall prove that the graphs G and G' are isomorphic. Define a function f: V(G) $\rightarrow$V(G') as follows.





EXERCISE:

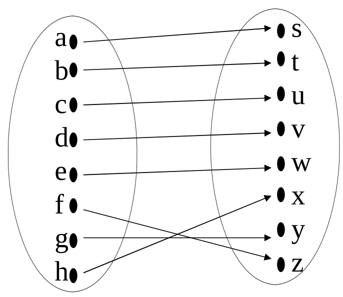
Determine whether the graph G and G' given below are isomorphic.



SOLUTION:

We shall prove that the graphs G and G' are isomorphic.
Clearly the isomorphism invariants seems to be true between G and G'.

Define a function f: V(G) →V(G') as follows.



f is clearly a bijective function(as it satisfies conditions the one-one and onto function clearly). The fact that f preserves the edge endpoint functions of G and G' is shown below.

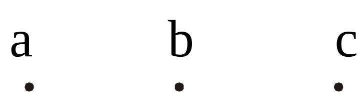
Edges of G	Edges of G'
{a, b}	{s, t} = {f(a), f(b)}
{b, c}	{t, u} = {f(b), f(c)}
{c, d}	{u, v} = {f(c), f(d)}
{a,d}	{s, v} = {f(a), f(d)}
{a, f}	{s, z} = {f(a), f(f)}
{b, g}	{t, y} = {f(b), f(g)}
{c, h}	{u, x} = {f(c), f(h)}
{d, e}	{v, w} = {f(d), f(e)}
{e, f}	{w, z} = {f(e), f(f)}
{f, g}	{z, y} = {f(f), f(g)}
{g, h}	{y, x} = {f(g), f(h)}
{h, e}	{x, w} = {f(h), f(e)}

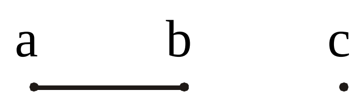
EXERCISE:

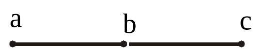
Find all non isomorphic simple graphs with three vertices.

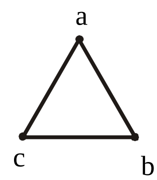
SOLUTION:

There are four simple graphs with three vertices as given below(which are non-isomorphic simple graphs).







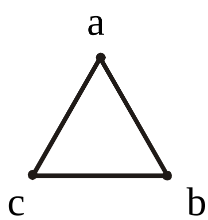


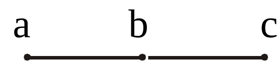
EXERCISE:

Find all non isomorphic simple connected graphs with three vertices.

SOLUTION:

There are two simple connected graphs with three vertices as given below(which are non-isomorphic connected simple graphs).



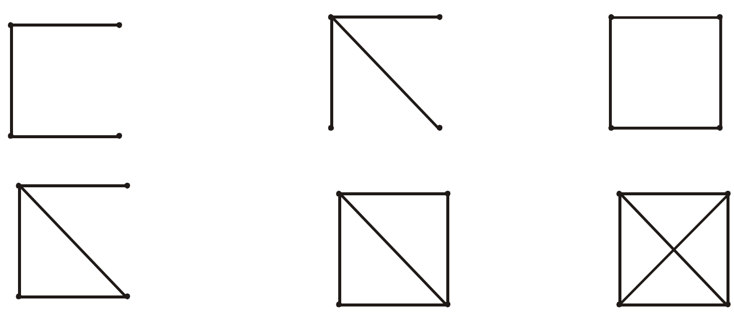


EXERCISE:

Find all non isomorphic simple connected graphs with four vertices.

SOLUTION:

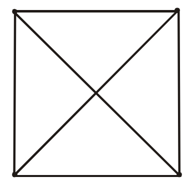
There are six simple connected graphs with four vertices as given below.



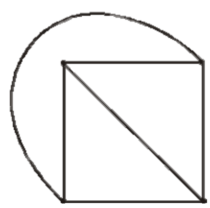
LECTURE 43

PLANAR GRAPHS
GRAPH COLORING

In this lecture, we will study that whether any graph can be drawn in the plane (means "a flat surface") without crossing any edges.



It is a graph on 4 vertices and written as K_4. Each vertex is connected to every other vertex. Note it that here edges are crossed. Also the above graph can also be drawn as



In this graph, note it that each vertex is connected to every other vertex, but no edge is crossed.

Note: The graphs shown above are complete graphs with four vertices (denoted by K_4).

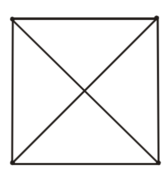
DEFINITION:

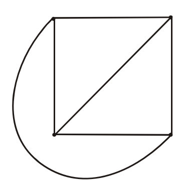
A graph is called planar if it can be drawn in the plane without any edge crossed (crossing means the intersection of lines). Such a drawing is called a plane drawing of the graph.

OR

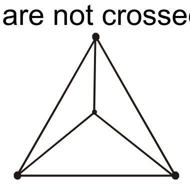
You can say that a graph is called planar in which the graph crossing number is "0".

EXAMPLES:





The graphs given above are planar .In the first figure edges are crossed but it can be redrawn in second figure where edges are not crossed, so called planar.



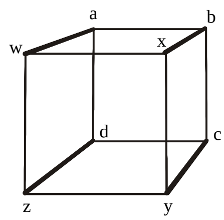
It is also a graph on 4 vertices (written as K_4) with no edge crossed, hence called planar.

Note: The graphs given above are also complete graphs (except second; are those where each vertex is connected to every other vertex) on 4 vertices and is written as K_4 **.**

Note: Complete graphs are planar only for $n \leq 4$.

EXAMPLE:

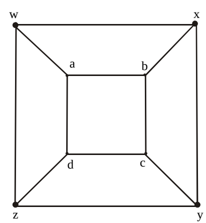
Show that the graph below is planar.



SOLUTION:

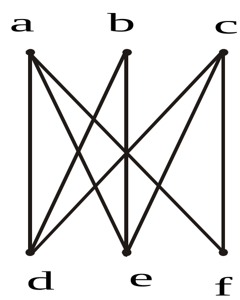
This graph has 8 vertices and 12 edges, and is called **3-cube** and is denoted **Q_3**.

The above representation includes many "edge crossing." A plane drawing of the graph in which no two edges cross is possible and shown below.



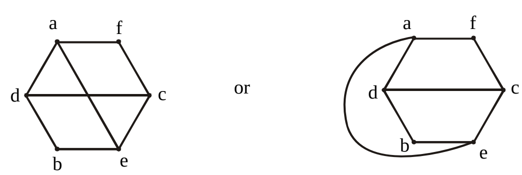
EXERCISE:

Determine whether the graph below is planar. If so, draw it so that no edges cross.



SOLUTION:

The graph given above is bipartite graph denoted by K_3. It also has a circuit afcebda. This graph can be re-drawn as

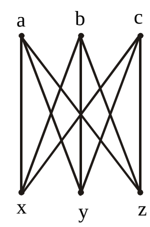


Hence the given graph is planar

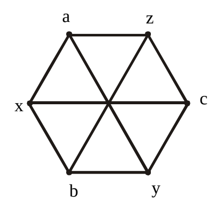
THEOREM:

Show that $K_{3,3}$ is not planar.

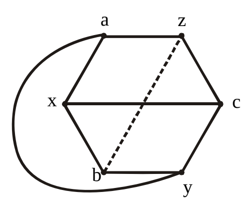
PROOF:



Clearly it is a complete bipartite graph (means bipartite graph, but the vertices within a set are not connected) denoted by $\mathbf{K_{3,3}}$**.** Now $\mathbf{K_{3,3}}$ can be re-drawn as



We re-draw the edge ***ay*** so that it does not cross any other edge like that.



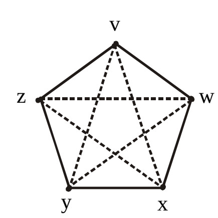
Note it that ***bz*** cannot be drawn without crossings. Hence, $K_{3,3}$ is not planar.
Similary if ay can be drawn inside(i.e drawn with crossing) and bz drawn outside, then same result exits.

THEOREM:

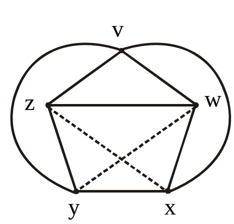
Show that K_5 is non-planar.

PROOF:

Graph K_5 (means a “complete graph” in which every vertex is connected to every other vertex) can be drawn as



To show that K_5 is non-planar, it can be re-drawn as

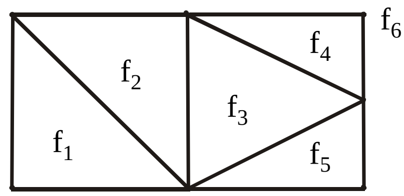


But still edges wy and zx contain the lines which crossed each other. Hence called non-planar.

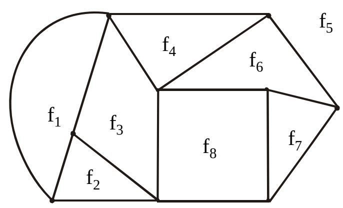
DEFINITION:

A plane drawing of a planar graph divides the plane into regions, including an unbounded region, called **faces**.

The unbounded region is called the **infinite face**.



Here we have 6 faces,7 vertices and 10 edges.f_6 is the unbounded region or called the infinite face because f_6 is outside of the graph.



In this graph, it has 8 faces,9 vertices and 14 edges. Here f_5 is the infinite face or unbounded region.

EULER'S FORMULA

THEOREM:

Let G be a connected planar simple graph with *e* edges and *v* vertices. Let *f* be the number of faces in a plane drawing of G. Then

$$f = e - v + 2$$

EXERCISE:

Suppose that a connected planar simple graph has 30 edges. If a plane drawing of this graph has 20 faces, how many vertices does this graph have?

SOLUTION:

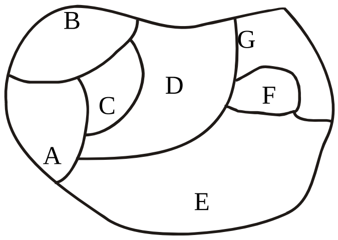
Given that $e = 30$, and $f = 20$. Substituting these values in the Euler's Formula $f = e - v + 2$, we get

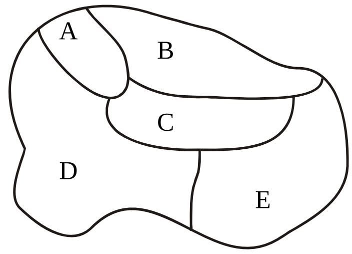
$$20 = 30 - v + 2$$

Hence,

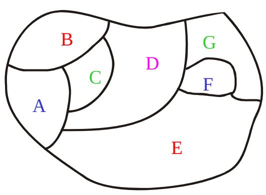
$$v = 30 - 20 + 2 = 12$$

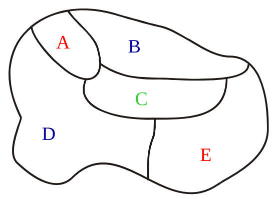
GRAPH COLORING





We also have to face many problems in the form of maps (maps like the parts of the world), which have generated many results in graph theory. Note it that in any graph, many regions are there, but two adjacent regions can't have the same color. And we have to choose a small number of color whenever possible.
Given two graphs above, our problem is to determine the least number of colors that can be used to color the map so that no adjacent regions have the same color.
In the first map given above, 4 colors are necessary, but three colors are not enough. In the second graph, 3 colors are necessary but 2 colors are not enough.



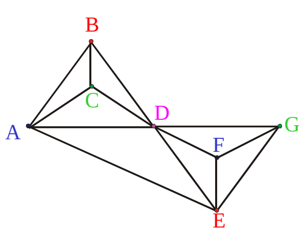


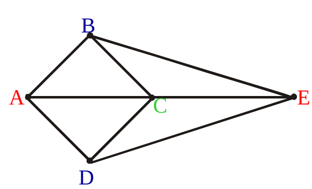
As in the 1st graph, four colors (red, pink, green, blue) are used like that adjacent regions not have the same color. In 2nd graph, three colors (red, blue, green) are used in the same manner.

HOW TO DRAW A GRAPH FROM A MAP:

1. Each map in the plane can be represented by a graph.
2. Each region is represented by a vertex (in 1st map as there are 7 regions, so 7 vertices are used in drawing a graph, similarly we can see 2nd map).
3. If the regions connected by these vertices have the common border, then edge connect two vertices.
4. Two regions that touch at only one point are not adjacent.

So apply these rules, we have (first graph drawn from first map given above, second graph from second map).



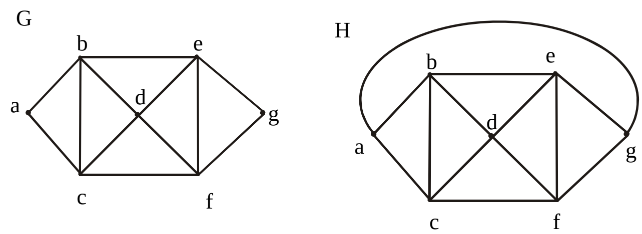


DEFINITION:

1. A **coloring** of a **simple graph** is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.
2. The **chromatic number** of a graph is the least (minimum) number of colors for coloring of this graph.

EXAMPLE:

What is the chromatic number of the graphs G and H shown below?



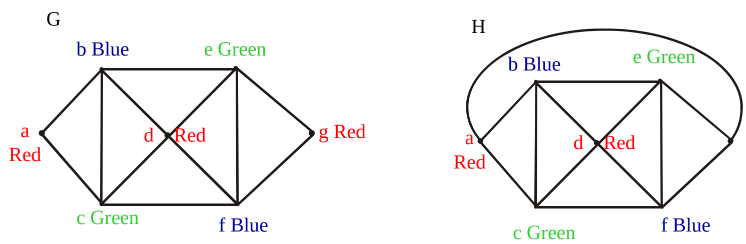
SOLUTION:

Clearly the chromatic number of G is 3 and chromatic number of H is 4(by using the above definition).

In graph G,

As vertices a, b and c are adjacent to each other so assigned different colors. So we assign red color to vertex a, blue to b and green to vertex c. Then no more colors we choose (due to above definition).Now vertex d must be colored red because it is adjacent to vertex b(with blue color) and c(with green color). And e must be colored green because it is adjacent to vertex b(blue color) and vertex d(red color). And f must be colored blue as it is adjacent to red and green color. At last,vertex g must be colored red as it is adjacent to green and blue color.

Same process is used in Graph H.



THE FOUR COLOR THEOREM:

The chromatic number of a simple planar graph is no greater than four.

APPLICATION OF GRAPH COLORING

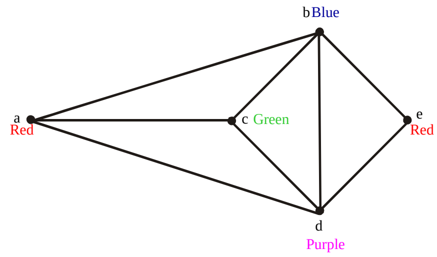
EXAMPLE:

Suppose that a chemist wishes to store five chemicals *a, b, c, d* and *e* in various areas of a warehouse. Some of these chemicals react violently when in contact, and so must be kept in separate areas. In the following table, an asterisk indicates those pairs of chemicals that must be separated. How many areas are needed?

	a	*b*	*c*	*d*	*e*
a	–	*	*	*	–
b	*	–	*	*	*
c	*	*	–	*	–
d	*	*	–	–	*
e	–	*	*	*	–

SOLUTION:

We draw a graph whose vertices correspond to the five chemicals, with two vertices adjacent whenever the corresponding chemicals are to be kept apart.



Clearly the chromatic number is 4 and so four areas are needed.

LECTURE 44

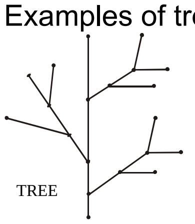
TREES

APPLICATION AREAS:

Trees are used to solve problems in a wide variety of disciplines. In computer science trees are employed to
1) construct efficient algorithms for locating items in a list.
2) construct networks with the least expensive set of telephone lines linking distributed computers.
3) construct efficient codes for storing and transmitting data.
4) model procedures that are carried out using a sequence of decisions, which are valuable in the study of sorting algorithms.

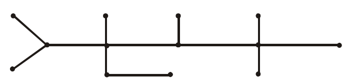
TREE:

A tree is a connected graph that does not contain any non-trivial circuit. (i.e. it is circuit-free). A trivial circuit is one that consists of a single vertex.
Examples of tree are



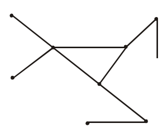
TREE

TREE

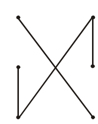


TREE

EXAMPLES OF NON TREES



(a) Graph with a circuit



(b) Disconnected graph

(c) Graph with a circuit

In graph (a), there exists circuit, so not a tree.
In graph (b), there exists no connectedness, so not a tree.
In graph (c), there exists a circuit (also due to loop), so not a tree (because trees have to be a circuit free).

SOME SPECIAL TREES

1. TRIVIAL TREE:

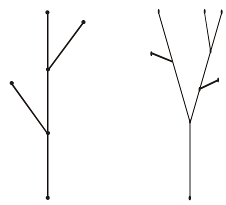
A graph that consists of a single vertex is called a trivial tree or degenerate tree.

2. EMPTY TREE

A tree that does not have any vertices or edges is called an empty tree.

3. **FOREST**

A graph is called a forest if, and only if, it is circuit-free.
OR "Any non-connected graph that contains no circuit is called a forest."
Hence, it clears that the connected components of a forest are trees.



A forest

As in both the graphs above, there exists no circuit, so called forest.

PROPERTIES OF TREES:

1. A tree with n vertices has n - 1 edges (where n ≥0).
2. Any connected graph with n vertices and n - 1 edges is a tree.
3. A tree has no non-trivial circuit; but if one new edge (but no new vertex) is added to it, then the resulting graph has exactly one non-trivial circuit.
4. A tree is connected, but if any edge is deleted from it, then the resulting graph is not connected.
5. Any tree that has more than one vertex has at least two vertices of degree 1.
6. A graph is a tree iff there is a unique path between any two of its vertices.

EXERCISE:

Explain why graphs with the given specification do not exist.
1. Tree, twelve vertices, fifteen edges.
2. Tree, five vertices, total degree 10.

SOLUTION:

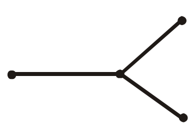
1. Any tree with 12 vertices will have 12 - 1 = 11 edges, not 15.
2. Any tree with 5 vertices will have 5 - 1 = 4 edges.
Since, total degree of graph = 2 (No. of edges)
= 2(4) = 8
Hence, a tree with 5 vertices would have a total degree 8, not 10.

EXERCISE:

Find all non-isomorphic trees with four vertices.

SOLUTION:

Any tree with four vertices has (4-1=3) three edges. Thus, the total degree of a tree with 4 vertices must be 6 [by using total degree=2(total number of edges)].
Also, every tree with more than one vertex has at least two vertices of degree 1, so the only possible combinations of degrees for the vertices of the trees are 1, 1, 1, 3 and 1, 1, 2, 2.
The corresponding trees (clearly non-isomorphic, by definition) are



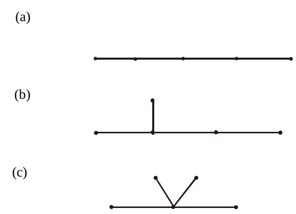
and

EXERCISE:

Find all non-isomorphic trees with five vertices.

SOLUTION:

There are three non-isomorphic trees with five vertices as shown (where every tree with five vertices has 5-1=4 edges).



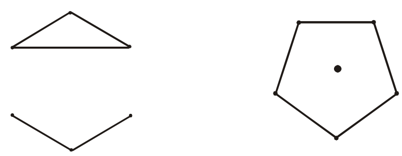
In part (a), tree has 2 vertices of degree ‘1’ and 3 vertices of degree ‘2’.
In part (b), 3 vertices have degree ‘1’, 1 has degree ‘2’and 1 vertex has degree ‘3’.
In part (c), possible combinations of degree are 1, 1, 1, 1, 4.

<u>EXERCISE:</u>

Draw a graph with six vertices, five edges that is not a tree.

<u>SOLUTION:</u>

Two such graphs are:



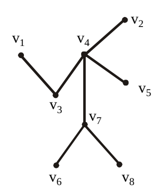
First graph is not a tree; because it is not connected also there exists a circuit.
Similarly, second graph not a tree.

<u>DEFINITION:</u>

A vertex of degree 1 in a tree is called a ***terminal vertex*** or a leaf and a vertex of degree greater than 1 in a tree is called an ***internal vertex*** or a branch vertex.

<u>EXAMPLE:</u>

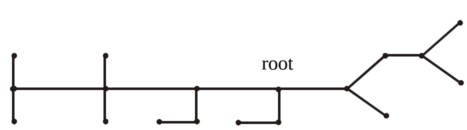
The terminal vertices of the tree are v_1, v_2, v_5, v_6 and v_8 and internal vertices are v_3, v_4, v_7.



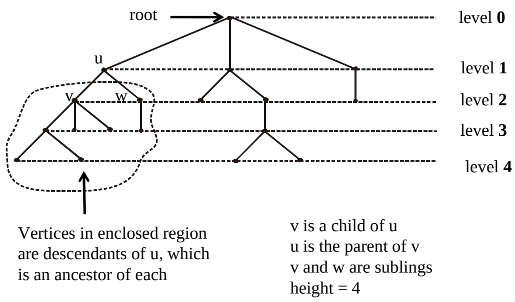
<u>ROOTED TREE:</u>

A **rooted tree** is a tree in which one vertex is distinguished from the others and is called the ***root.***
The **level** of a vertex is the number of edges along the unique path between it and the root.
The **height** of a rooted tree is the maximum level to any vertex of the tree.
The **children** of any internal vertex v are all those vertices that are adjacent to v and are one level farther away from the root than v.
If **w** is a **child** of **v**, then **v** is called the **parent** of **w**.
Two vertices that are both children of the same parent are called **siblings**.
Given vertices v and w, if v lies on the unique path between w and the root, then v is an **ancestor** of w and w is a **descendant** of v.

<u>EXAMPLE:</u>



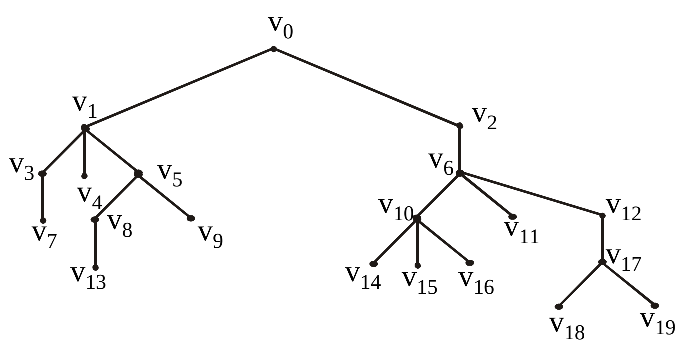
We redraw the tree as and see what the relations are



EXERCISE:

Consider the rooted tree shown below with root v_0

a. What is the level of v_8?
b. What is the level of v_0?
c. What is the height of this tree?
d. What are the children of v_{10}?
e. What are the siblings of v_1?
f. What are the descendants of v_{12}?



SOLUTION:

As we know that "Level means the total number of edges along the unique path between it and the root".

(a). As v_0 is the root so the level of v_8 (from the root v_0 along the unique path) is 3,because it covers the 3 edges.

(b).The level of v_0 is 0(as no edge cover from v_0 to v_0).

(c).The height of this tree is 5.

Note: As levels are 0, 1, 2, 3, 4, 5 but to find height we have to take the maximum level.

(d).The children of v_{10} are v_{14}, v_{15} and v_{16} .

(e).The siblings of v_1 are v_3 , v_4 , and v_5.

(f).The descendants of v_{12} are v_{17}, v_{18}, and v_{19}.

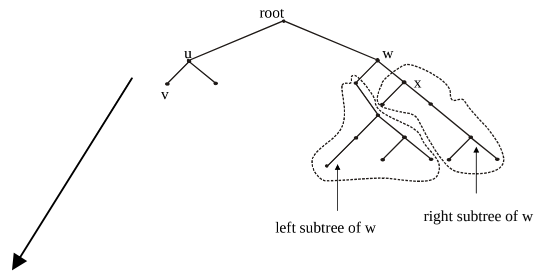
BINARY TREE

A ***binary tree*** is a rooted tree in which every internal vertex has at most two children.

Every child in a binary tree is designated either a left child or a right child (but not both).

A ***full binary tree*** is a binary tree in which each internal vertex has exactly two children.

EXAMPLE:



v is the left child of u.

THEOREMS:

1. If k is a positive integer and T is a full binary tree with k internal vertices, then T has a total of 2k + 1 vertices and has k + 1 terminal vertices.

2. If T is a binary tree that has t terminal vertices and height h, then $t \leq 2^h$
Equivalently,

$$\log_2 t \leq h$$

Note: The maximum number of terminal vertices of a binary tree of height h is 2^h.

EXERCISE:

Explain why graphs with the given specification do not exist.
1. full binary tree, nine vertices, five internal vertices.
2. binary tree, height 4, eighteen terminal vertices.

SOLUTION:

1. Any full binary tree with five internal vertices has six terminal vertices, for a total of eleven vertices (according to 2(5) +1=11), not nine vertices in all.

OR

As total vertices=2k+1=9
$\Rightarrow$ k=4(internal vertices)
but given internal vertices=5 ,which is a contradiction.
Thus there is no full binary tree with the given properties.

2. Any binary tree of height 4 has at most 2^4 = 16 terminal vertices.
Hence, there is no binary tree that has height 4 and eighteen terminal vertices.

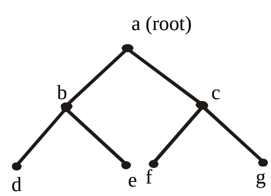
EXERCISE:

Draw a full binary tree with seven vertices.

SOLUTION:

Total vertices= 2k + 1 = 7 (by using the above theorem)
$\Rightarrow$ k = 3

Hence, total number of internal vertices (i.e. a vertex of degree greater than 1)=k=3
and total number of terminal vertices(i.e. a vertex of degree 1 in a tree) =k+1=3+1=4
Hence, a full binary tree with seven vertices is

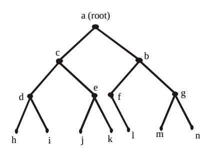


EXERCISE:

Draw a binary tree with height 3 and having seven terminal vertices.

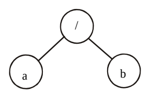
SOLUTION:

Given height=h=3
Any binary tree with height 3 has atmost 2^3=8 terminal vertices.
But here terminal vertices are 7
and Internal vertices=k=6 so binary tree exists and is as fellows:



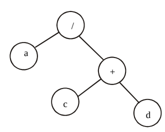
REPRESENTATION OF ALGEBRAIC EXPRESSIONS BY BINARY TREES

Binary trees are specially used in computer science to represent algebraic expression with Arbitrary nesting of balanced parentheses.



Binary tree for a/b

The above figure represents the expression a/b. Here the operator (/) is the root and b are the left and right children.



Binary tree for a/(c+d)

The second figure represents the expression a/(c+d).Here the operator (/) is the root.
Here the terminal vertices are variables (here a, c and d), and the internal vertices are arithmetic operators (+ and /).

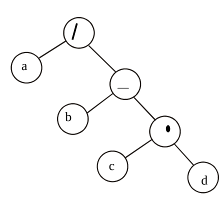
EXERCISE:

Draw a binary tree to represent the following expression

a/(b-c.d)

SOLUTION:

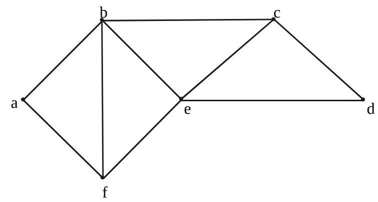
Note that the internal vertices are arithmetic operators, the terminal vertices are variables and the operator at each vertex acts on its left and right sub trees in left-right order.



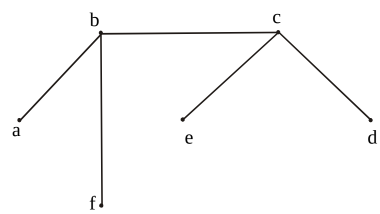
LECTURE 45

SPANNING TREES:

Suppose it is required to develop a system of roads between six major cities.
A survey of the area revealed that only the roads shown in the graph could be constructed.



For economic reasons, it is desired to construct the least possible number of roads to connect the six cities.
One such set of roads is



Note that the subgraph representing these roads is a tree, it is connected & circuit-free (six vertices and five edges)

SPANNING TREE:

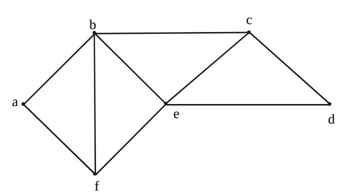
A spanning tree for a graph G is a subgraph of G that contains every vertex of G and is a tree.

REMARK:

1. Every connected graph has a spanning tree.
2. A graph may have more than one spanning trees.
3. Any two spanning trees for a graph have the same number of edges.
4. If a graph is a tree, then its only spanning tree is itself.

EXERCISE:

Find a spanning tree for the graph below:



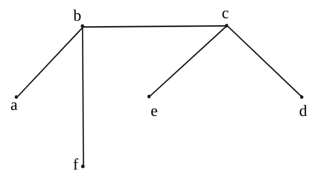
SOLUTION:

The graph has 6 vertices (a, b, c, d, e, f) & 9 edges so we must delete 9 - 6 + 1 = 4 edges (as we have studied in lecture 44 that a tree of vertices n has n-1 edges). We delete an edge in each cycle.

1. Delete af 2. Delete fe
3. Delete be 4. Delete ed

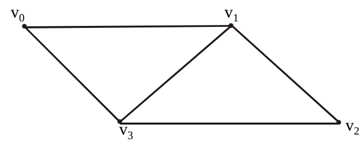
Note it that we can construct road from vertex a to b, but can't go from "a to e", also from "a to d" and from "a to c ", because there is no path available.

The associated spanning tree is



EXERCISE:

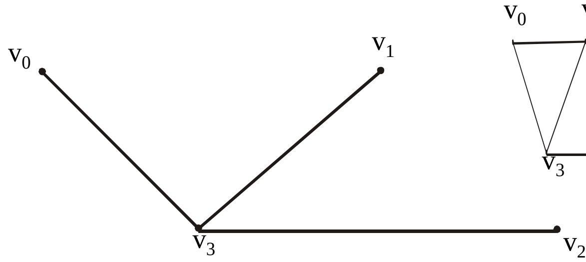
Find all the spanning trees of the graph given below.

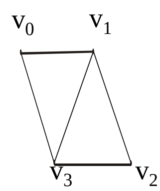


SOLUTION:

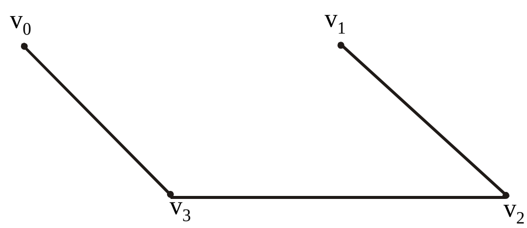
The graph has n = 4 vertices and e = 5 edges. So we must delete e - v + 1 = 5 - 4 + 1 = 2 edges from the cycles in the graph to obtain a spanning tree.

(1) Delete v_0v_1 & v_1v_2 to get

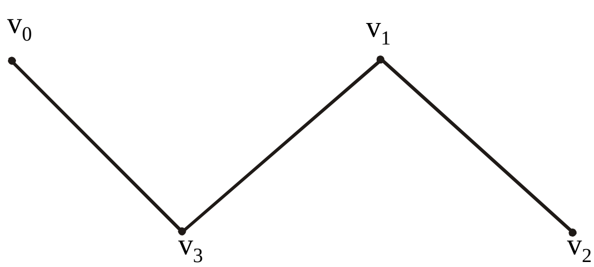


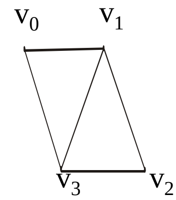


(2) Delete v_0v_1 & v_1v_3 to get

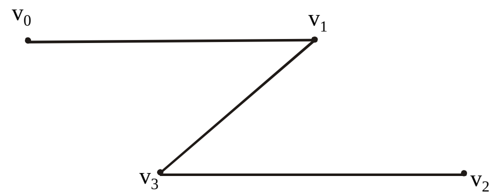


(3) Delete v_0v_1 & v_2v_3 to get

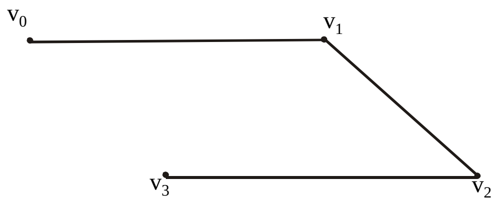


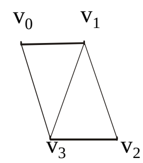


(4) Delete v_0v_3 & v_1v_2 to get

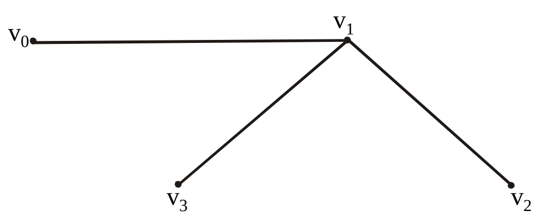


(5) Delete v_0v_3 & v_1v_3 to get

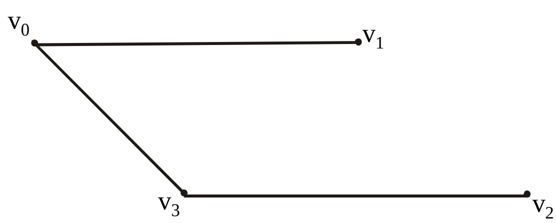


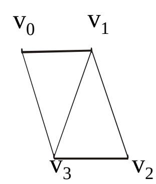


(6) Delete v_0v_3 & v_2v_3 to get

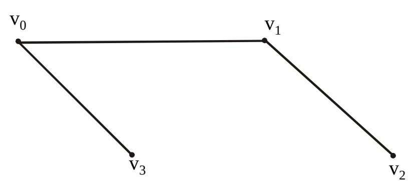


(7) Delete v_1v_3 & v_1v_2 to get





(8) Delete v_1v_3 & v_2v_3 to get



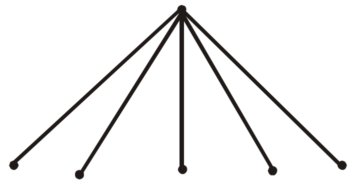
EXERCISE:

Find a spanning tree for each of the following graphs.

(a) $k_{1,5}$ (b) k_4

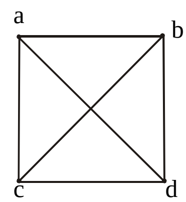
SOLUTION:

(a). $k_{1,5}$ represents a complete bipartite graph on (1,5) vertices, drawn below:



Clearly the graph itself is a tree (six vertices and five edges). Hence the graph is itself a spanning tree.

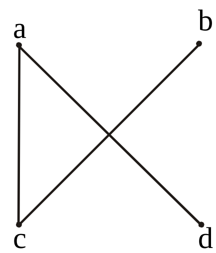
(b) k_4 represents a complete graph on four vertices.



Now
number of vertices = n = 4 and number of edges = e = 6
Hence we must remove

$$e - v + 1 = 6 - 4 + 1 = 3$$

edges to obtain a spanning tree.
Let ab, bd & cd edges are removed. The associated spanning tree is

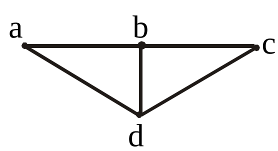


KIRCHHOFF'S THEOREM OR MATRIX - TREE THEOREM

Let M be the matrix obtained from the adjacency matrix of a connected graph G by changing all 1's to -1's and replacing each diagonal 0 by the degree of the corresponding vertex. Then the number of spanning trees of G is equal to the value of any cofactor of M.

EXAMPLE:

Find the number of spanning trees of the graph G.



SOLUTION:

The adjacency matrix of G is

$$A(G) = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The matrix specified in Kirchhoff's theorem is

$$M = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

Now cofactor of the element at (1,1) in M is

$$\begin{vmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{vmatrix}$$

Expanding by first row, we get

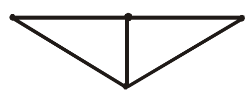
$$= 3\begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ -1 & 3 \end{vmatrix} + (-1)\begin{vmatrix} -1 & 2 \\ -1 & -1 \end{vmatrix}$$

$$= 3(6-1) + (-3-1) + (-1)(1+2)$$

$$= 15 - 4 - 3 = 8$$

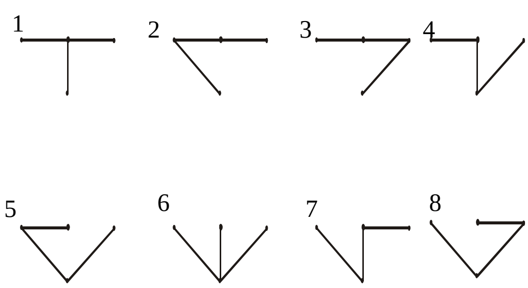
EXERCISE:

How many non-isomorphic spanning trees does the following simple graph has?



SOLUTION:

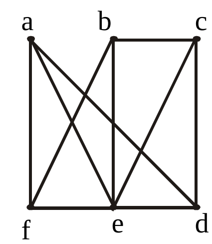
There are eight spanning tree of the graph



Clearly 1 & 6 are isomorphic, and 2, 3, 4, 5, 7, 8 are isomorphic. Hence there are only two non-isomorphic spanning trees of the given graph.

EXERCISE:

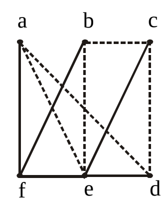
Suppose an oil company wants to build a series of pipelines between six storage facilities in order to be able to move oil from one storage facility to any of the other five. For environmental reasons it is not possible to build a pipeline between some pairs of storage facilities. The possible pipelines that can be build are.



Because the construction of a pipeline is very expensive, construct as few pipelines as possible.
(The company does not mind if oil has to be routed through one or more intermediate facilities)

SOLUTION:

The task is to find a set of edges which together with the incident vertices from a connected graph containing all the vertices and having no cycles. This will allow oil to go from any storage facility to any other without unnecessary building costs. Thus, a tree containing all the vertices of the graph is to be soughed. One selection of edges is

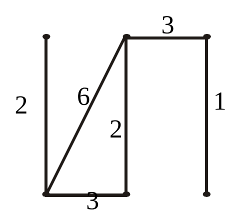


DEFINITION:

A weighted graph is a graph for which each edge has an associated real number weight.
The sum of the weights of all the edges is the total weight of the graph.

EXAMPLE:

The figure shows a weighted graph



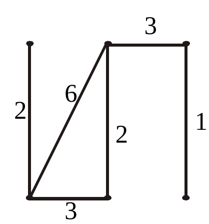
with total weight is 2 + 6 + 3 + 2 + 3 + 1 = 17

MINIMAL SPANNING TREE:

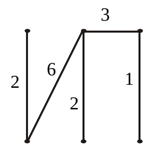
A minimal spanning tree for a weighted graph is a spanning tree that has the least possible total weight compared to all other spanning trees of the graph.
If G is a weighted graph and e is an edge of G then w(e) denotes the weight of e and w(G) denotes the total weight of G.

EXERCISE:

Find the three spanning trees of the weighted graph below. Also indicate the minimal spanning tree.

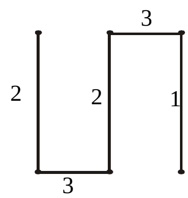


SOLUTION:



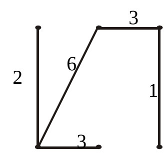
T1

w(T1)=14



w(T2)=11

T2



w(T3)=15

T3

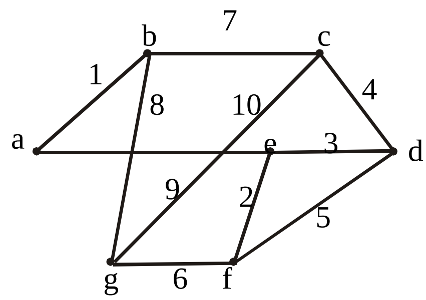
T_2 is the minimal spanning tree, since it has the minimum weight among the spanning trees.

KRUSKAL'S ALGORITHM:

Input: G [a weighted graph with n vertices]
Algorithm:
1. Initialize T(the minimal spanning tree of G) to have all the vertices of G and no edges.
2. Let E be the set of all edges of G and let m: = 0.
3. While (m < n - 1)
3a. Find an edge e in E of least weight.
3b. Delete e from E.
3c. If addition of e to the edge set of T does not produce a circuit then add e to the edge set of T and set m: = m + 1
end while
Output T
end Algorithm

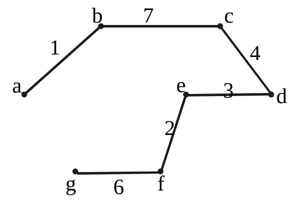
EXERCISE:

Use Kruskal's algorithm to find a minimal spanning tree for the graph below. Indicate the order in which edges are added to form the tree.



SOLUTION:

Minimal spanning tree:



Order of adding the edges:
{a,b}, {e,f}, {e,d}, {c,d}, {g,f}, {b,c}

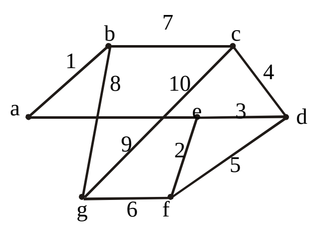
PRIM'S ALGORITHM:

Input: G[a weighted graph with n vertices]
Algorithm Body:
1. Pick a vertex v of G and let T be the graph with one vertex, v, and no edges.
2. Let V be the set of all vertices of G except v
3. for i: = 1 to n - 1
3a. Find an edge e of G such that
(1) e connects T to one of the vertices in V and
(2)e has the least weight of all edges connecting T to a vertex in V.
Let w be the end point of e that is in V.
3b. Add e and w to the edge and vertex sets of T and delete w from V.

next i
Output: T
end Algorithm

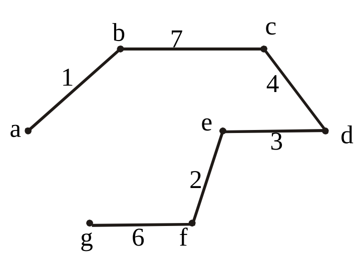
EXERCISE:

Use Prim's algorithm starting with vertex ***a*** to find a minimal spanning tree of the graph below. Indicate the order in which edges are added to form the tree.



SOLUTION:

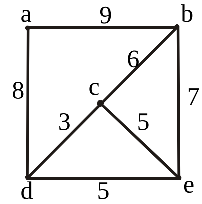
Minimal spanning tree is



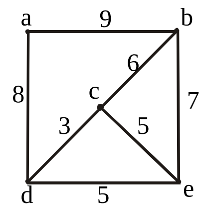
Order of adding the edges:
{a,b}, {b.c}, {c,d}, {d,e}, {e,f}, {f,g}

EXERCISE:

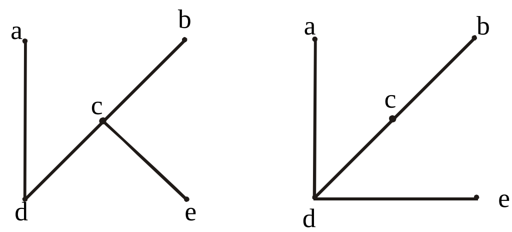
Find all minimal spanning trees that can be obtained using
(a) Kruskal's algorithm
(b) Prim's algorithm starting with vertex a



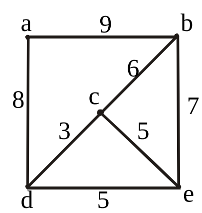
SOLUTION:
Given :



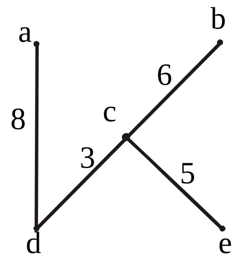
(a) When Kruskal's algorithm is applied, edges are added in one of the following two orders:
1. {c,d}, {c,e}, {c,b}, {d,a}
2. {c,d}, {d,e}, {c,b}, {d,a}
Thus, there are two distinct minimal spanning trees:

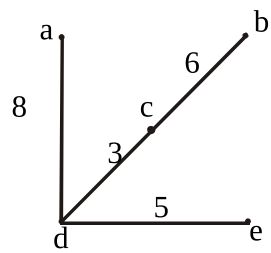


(b)



When Prim's algorithm is applied starting at a, edges are added in one of the following two orders:
1. {a,d}, {d,c}, {c,e}, {c,b}
2. {a,d}, {d,c}, {d,e}, {c,b}
Thus, the two distinct minimal spanning trees are:


a
b
c
d
e
8
6
3
5


a
b
c
d
e
8
6
3
5