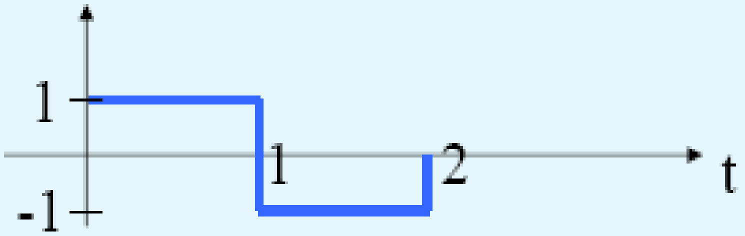
Fourier Series

Reference

– Chapter 2.1, Carlson, Communication Systems

Example

A signal *f(t)* is defined as



$$f(t) = \begin{cases} 1 & 0 < t < 1 \\ -1 & 1 < t < 2 \end{cases}$$

Now, we use a set of sinusoidal functions $\phi_n(t)$ to approximate the signal,

$$\phi_n(t) = \sin(n\pi t) \qquad n > 0$$

$$f(t) = \sum_{n=1}^{\infty} f_n \sin(n\pi t) \qquad (1)$$

The unknown coefficients can be found using the orthogonal property of the sinusoidal functions.

$$\int_0^2 \sin(n\pi t)\sin(m\pi t)dt = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

multiplying $\sin(m\pi t)$ to (1) and integrating from 0 to 2, we get

$$f(t) = \sum_{n=1}^{\infty} f_n \sin(n\pi t)$$

$$\Rightarrow \int_0^2 f(t)\sin(m\pi t)dt = \int_0^2 \left[\sum_{n=1}^{\infty} f_n \sin(n\pi t)\right]\sin(m\pi t)dt$$

$$\Rightarrow \int_0^2 f(t)\sin(m\pi t)dt = f_1\int_0^2 \sin(\pi t)\sin(m\pi t)dt$$

$$+ f_2\int_0^2 \sin(2\pi t)\sin(m\pi t)dt + \ldots$$

$$+ f_m\int_0^2 \sin(m\pi t)\sin(m\pi t)dt + \ldots$$

$$\Rightarrow \int_0^2 f(t)\sin(m\pi t)dt = f_m$$

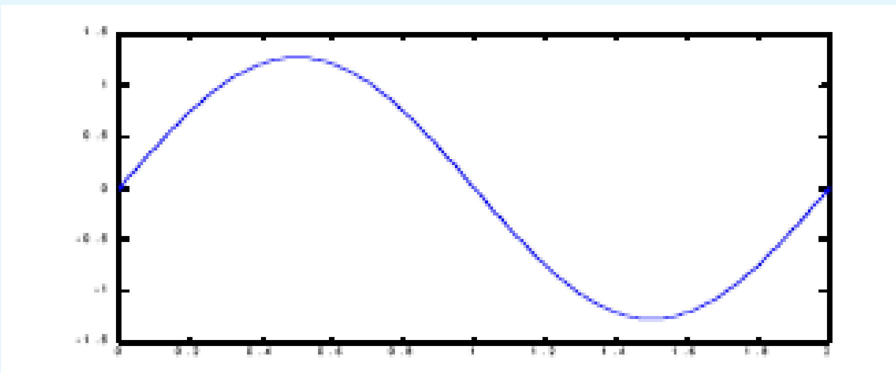
$$\Rightarrow \int_0^1 \sin(m\pi t)dt - \int_1^2 \sin(m\pi t)dt = f_m$$

$$\Rightarrow -\left.\frac{\cos(m\pi t)}{m\pi}\right|_0^1 + \left.\frac{\cos(m\pi t)}{m\pi}\right|_1^2 = f_m$$

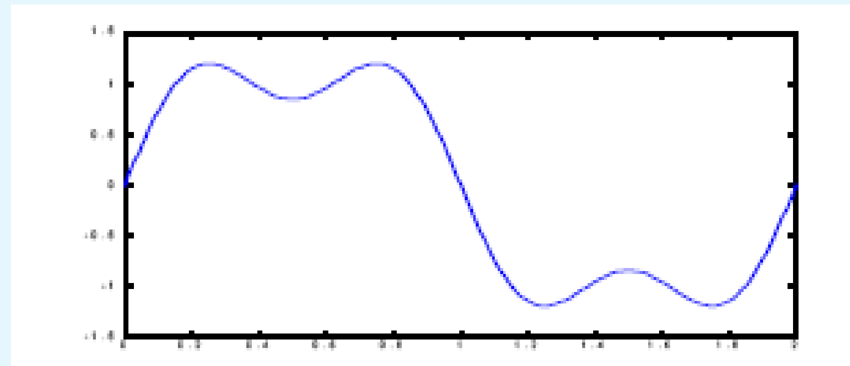
$$\therefore f_n = \begin{cases} \dfrac{4}{n\pi} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$

Finally, we have

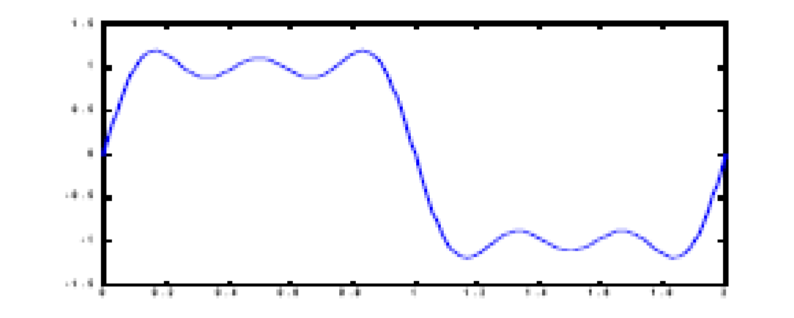
$$f(t)=\frac{4}{\pi}\left(\sin(\pi t)+\frac{1}{3}\sin(3\pi t)+\frac{1}{5}\sin(5\pi t)+\ldots+\frac{1}{n}\sin(n\pi t)+\ldots\right)$$



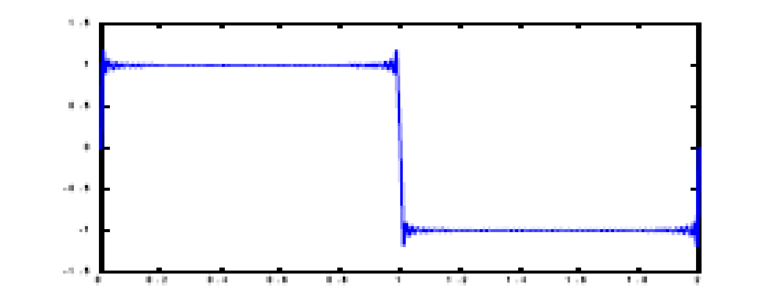
$$\frac{4}{\pi}\sin(\pi t)$$



$$\frac{4}{\pi}\left(\sin(\pi t)+\frac{1}{3}\sin(3\pi t)\right)$$



$$\frac{4}{\pi}\left(\sin(\pi t)+\frac{1}{3}\sin(3\pi t)+\frac{1}{5}\sin(5\pi t)\right)$$



$$f(t)=\frac{4}{\pi}\left(\sin(\pi t)+\frac{1}{3}\sin(3\pi t)+\frac{1}{5}\sin(5\pi t)+\ldots+\frac{1}{99}\sin(99\pi t)\right)$$

The example shows that the signal *f(t)* can be considered as a infinite sum of sinusoidal signals.

Exponential Fourier series

$e^{j\omega t}$

In circuit analysis, the phasor form ($v(t)=Re[Ve^{j\omega t}]$) is often used to represent a sinusoidal voltage source ($v(t)=V\cos(\omega t)$) because differentiating $e^{j\omega t}$ remains as an exponential function.

In signal analysis, the exponential functions are also used to expand a signal.

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_o t} \qquad t_1 < t < t_2$$

where $$F_n = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} f(t) e^{-jn\omega_o t} dt \qquad \omega_o = \frac{2\pi}{t_2 - t_1}$$

Exponential Fourier series

Proof:

Consider a signal *f(t)* represented by a linear combination of complex exponential functions over an finite interval (t_1, t_2).

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_o t} \qquad t_1 < t < t_2$$

Multiplying $e^{-jm\omega_o t}$ to both sides and integrating from t_1 to t_2, we have

$$\int_{t_1}^{t_2} f(t) e^{-jm\omega_o t} dt = \sum_{n=-\infty}^{\infty} F_n \int_{t_1}^{t_2} e^{jn\omega_o t} e^{-jm\omega_o t} dt \qquad (2)$$

If we choose $\omega_o = 2\pi/(t_2 - t_1)$

For $m \neq n$

$$\int_{t_1}^{t_2} e^{jn\omega_o t} e^{-jm\omega_o t} dt = \int_{t_1}^{t_2} e^{\frac{j(n-m)2\pi}{t_2-t_1}t} dt$$

$$= \frac{1}{\frac{j(n-m)2\pi}{t_2-t_1}} \left[e^{\frac{j(n-m)2\pi}{t_2-t_1}t_1} - e^{\frac{j(n-m)2\pi}{t_2-t_1}t_2} \right]$$

$$= 0$$

For $m = n$

$$\int_{t_1}^{t_2} e^{jn\omega_o t} e^{-jm\omega_o t} dt = \int_{t_1}^{t_2} 1 dt$$

$$= t_2 - t_2$$

Therefore, (2) becomes

$$\int_{t_1}^{t_2} f(t)e^{-jm\omega_o t}\,dt = F_m(t_2 - t_1)$$

or $$F_m = \frac{1}{(t_2 - t_1)}\int_{t_1}^{t_2} f(t)e^{-jm\omega_o t}\,dt$$

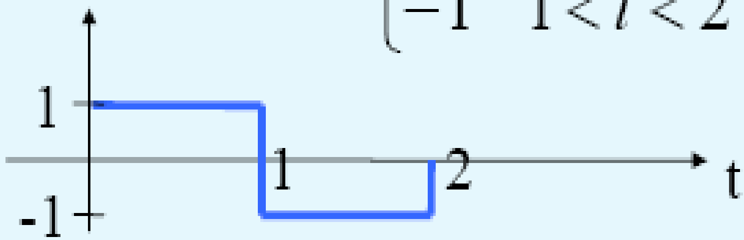
$$\boxed{\begin{aligned} & f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_o t} \qquad\qquad t_1 < t < t_2 \\ & \text{where} \qquad F_n = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} f(t)e^{-jn\omega_o t}\,dt \qquad\qquad \omega_o = \frac{2\pi}{t_2 - t_1} \end{aligned}}$$

known as the **exponential Fourier series** representation of *f(t)* over the interval (t_1, t_2)

Example

Expand the signal in B.1 using the exponential Fourier series,

$$f(t)=\begin{cases}1 & 0<t<1\\ -1 & 1<t<2\end{cases}$$

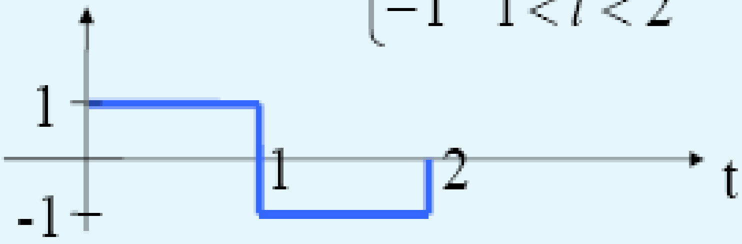


$$F_n=\frac{1}{(t_2-t_1)}\int_{t_1}^{t_2} f(t)e^{-jn\omega_o t}\,dt$$

$$=\frac{1}{2}\int_0^2 f(t)e^{-jn\pi t}\,dt$$

$$=\begin{cases}2/jn\pi & n \quad \text{odd}\\ 0 & n \quad \text{even}\end{cases}$$

$$f(t)=\begin{cases}1 & 0<t<1\\ -1 & 1<t<2\end{cases}$$



$$f(t)=\sum_{n=-\infty}^{\infty}F_n e^{jn\omega_o t}$$

$$=\frac{2}{j\pi}\left(e^{j\pi t}+\frac{1}{3}e^{j3\pi t}+\ldots-e^{-j\pi t}-\frac{1}{3}e^{-j3\pi t}-\ldots\right)$$

$$=\frac{4}{\pi}\left(\sin(\pi t)+\frac{1}{3}\sin(3\pi t)+\ldots\right) \qquad \because \sin x=\frac{e^{jx}-e^{-jx}}{2j}$$

Fourier series expansion of periodic signals

If a periodic signal has only finite average power, it can be represented by as series of complex exponential functions.

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_o t} \qquad t_o < t < T + t_o$$

where $$F_n = \frac{1}{T}\int_{t_o}^{t_o+T} f(t) e^{-jn\omega_o t}\, dt \qquad \omega_o = \frac{2\pi}{T}$$

Fourier series expansion of periodic signals

– The complex exponential functions are periodic with period T

$$e^{jn\omega_o(t+T)} \equiv e^{jn\omega_o t} \cdot e^{jn\omega_o T}$$

$$= e^{jn\omega_o t} \cdot e^{jn\left(\frac{2\pi}{T}\right)T}$$

$$= e^{jn\omega_o t}$$

– The lower limit t_o is arbitrary.

– It is often convenient to take t_o equal to $-T/2$.

– The representation of the periodic signal converges in a mean square sense.

Examples

$$Sa(x) = \sin(x)/x$$

Periodic waveform

F_n

Symmetric square wave

$$f(t) = \begin{cases} 1 & |t| < T/4 \\ -1 & T/4 \le |t| < T/2 \end{cases}$$

$$F_n = \begin{cases} Sa(n\pi/2) & n \neq 0 \\ 0 & n = 0 \end{cases}$$

Rectangular pulse train

$$f(t) = \begin{cases} 1 & |t| < \tau/2 \\ 0 & \tau/2 \le |t| < T/2 \end{cases}$$

$$F_n = \frac{\tau}{T} Sa(n\pi\tau/T)$$

Symmetric triangular wave

$$f(t) = 1 - 4|t|/T \qquad |t| < T/2$$

$$F_n = \begin{cases} Sa^2(n\pi/2) & n \neq 0 \\ 0 & n = 0 \end{cases}$$

Half-wave rectified sinusoid

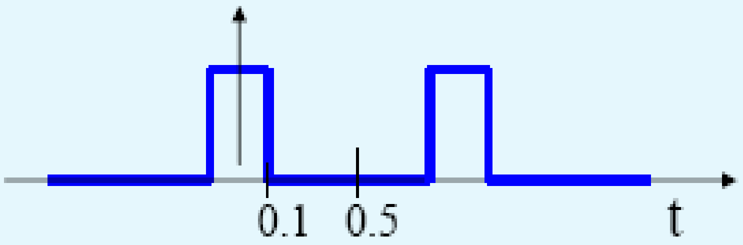
$$f(t) = \begin{cases} \sin\omega_o t & 0 \le t < T/2 \\ 0 & -T/2 \le t < 0 \end{cases}$$

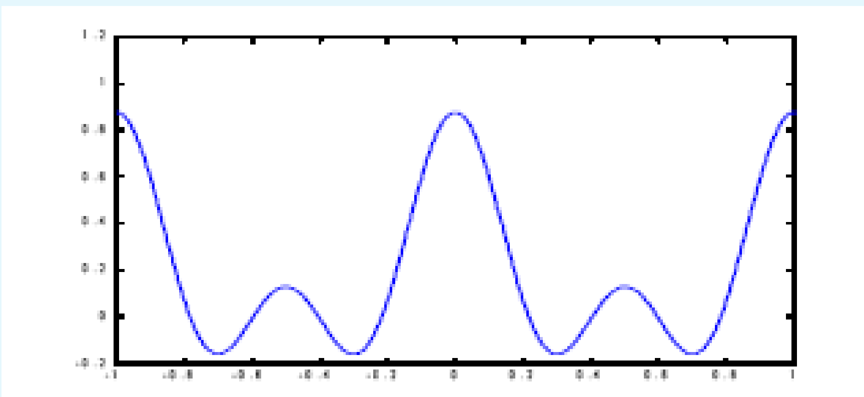
$$F_n = \begin{cases} \dfrac{1}{\pi\left(1 - n^2\right)} & n \quad \text{even} \\ \mp j/4 & n = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

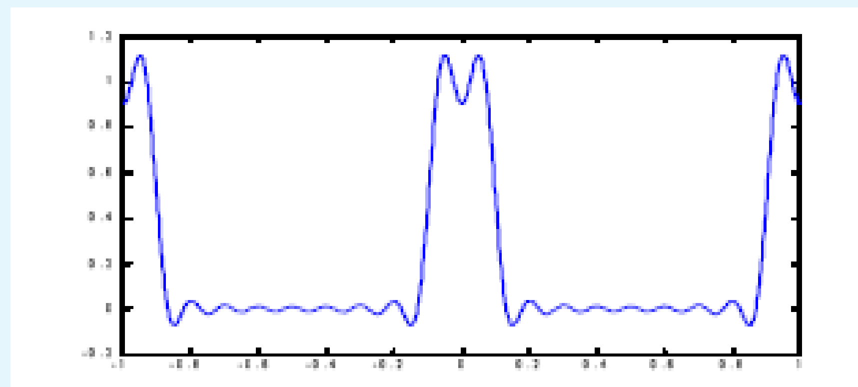
Example

Consider a rectangular pulse train $f(t)=\begin{cases}1 & |t|<0.1\\ 0 & 0.1\le |t|<0.5\end{cases}$

$$f(t)=\sum_{n=-\infty}^{\infty}\frac{1}{5}Sa\left(\frac{n\pi}{5}\right)e^{j4n\pi t}$$

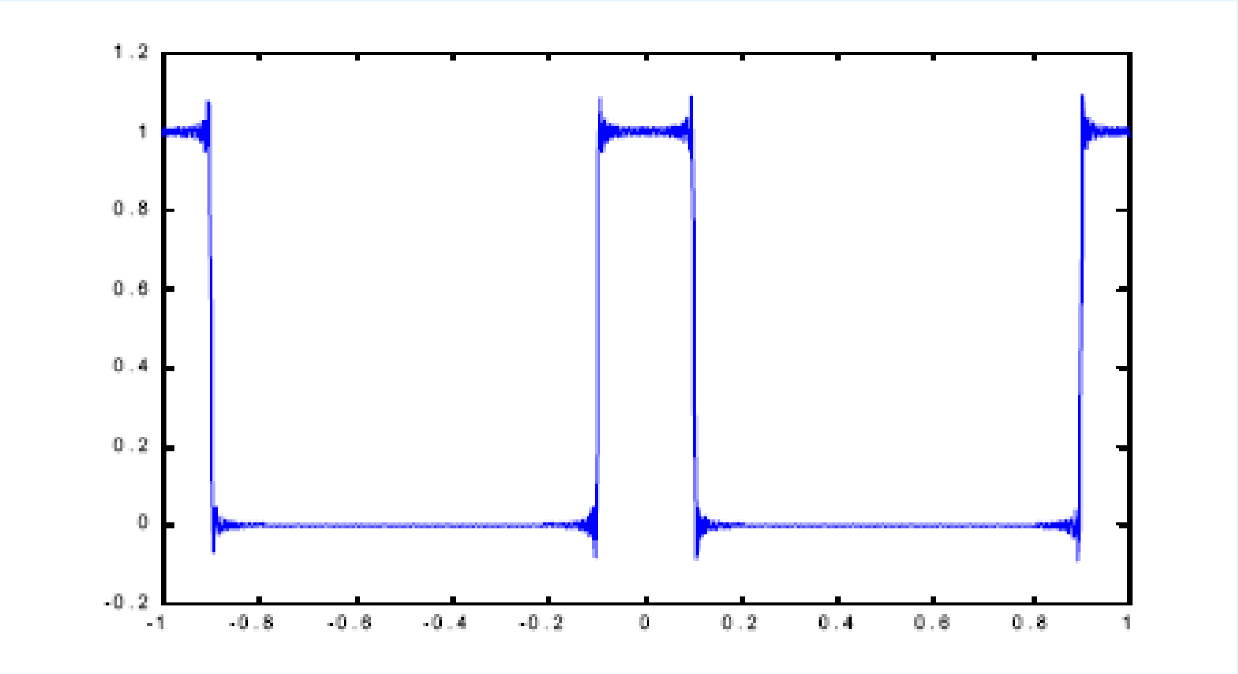






$$f(t)=\sum_{n=-2}^{2}\frac{1}{5}Sa\left(\frac{n\pi}{5}\right)e^{j4n\pi t}$$

$$f(t)=\sum_{n=-10}^{10}\frac{1}{5}Sa\left(\frac{n\pi}{5}\right)e^{j4n\pi t}$$



$$f(t)=\sum_{n=-100}^{100}\frac{1}{5}Sa\left(\frac{n\pi}{5}\right)e^{j4n\pi t}$$

Parseval's theorem

The average power of a signal can be calculated by summing the square of the magnitude of the Fourier coefficients.

The average power developed across a 1Ω resistance is

$$P = \frac{1}{T}\int_{-T/2}^{T/2} f(t)f^*(t)dt \qquad \text{* denotes complex conjugate}$$

$$= \frac{1}{T}\int_{-T/2}^{T/2}\left[\sum_{m=-\infty}^{\infty} F_m e^{jm\omega_o t}\right]\left[\sum_{n=-\infty}^{\infty} F_n^* e^{-jn\omega_o t}\right]dt$$

$$= \sum_{n=-\infty}^{\infty} F_n^* \left\{\sum_{m=-\infty}^{\infty} F_m \left[\frac{1}{T}\int_{-T/2}^{T/2} e^{j(m-n)\omega_o t}dt\right]\right\}$$

$$= \sum_{n=-\infty}^{\infty} F_n^* F_n \qquad \because \int_{-T/2}^{T/2} e^{j(m-n)\omega_o t}dt = \begin{cases} 0 & m \neq n \\ T & m = n \end{cases}$$

Parseval's theorem

Therefore, we have

$$\boxed{P = \frac{1}{T}\int_{-T/2}^{T/2} \left| f(t) \right|^2 dt = \sum_{n=-\infty}^{\infty} \left| F_n \right|^2}$$

Example

Determine the average power of $f(t)$=2sin100t+sin200t.

$$P=\frac{1}{T}\int_{-T/2}^{T/2}\left|f(t)\right|^2dt$$

$$=\frac{1}{(2\pi/100)}\int_{-\pi/100}^{\pi/100}\left|2\sin 100t+\sin 200t\right|^2dt$$

$$=2W+0.5W=2.5W$$

The Fourier coefficients of $f(t)$ are $F_1=-j$, $F_{-1}=j$, $F_2=-j/2$, $F_{-2}=j/2$, $F_n=0$ for other n.

Using Parseval's theorem,

$$P=\sum_{n=-\infty}^{\infty}\left|F_n\right|^2=\left|F_1\right|^2+\left|F_{-1}\right|^2+\left|F_2\right|^2+\left|F_{-2}\right|^2=\boxed{1+1}+\boxed{0.25+0.25}=2.5W$$

Power contained in 2sin100*t*

Power contained in sin200*t*

Steady-state response

$$f(t) \longrightarrow \boxed{H(\omega)} \longrightarrow g(t)$$

If the input signal to a linear time-invariant system H(ω) is

$$f(t) = Ae^{j(\omega_1 t + \phi_1)}$$

the output is

$$g(t) = AH(\omega_1)e^{j(\omega t_1 + \phi_1)}$$

If the input signal is written as an exponential Fourier series,

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_o t}$$

the output is $$g(t) = \sum_{n=-\infty}^{\infty} H(n\omega_o) F_n e^{jn\omega_o t}$$

Steady-state response

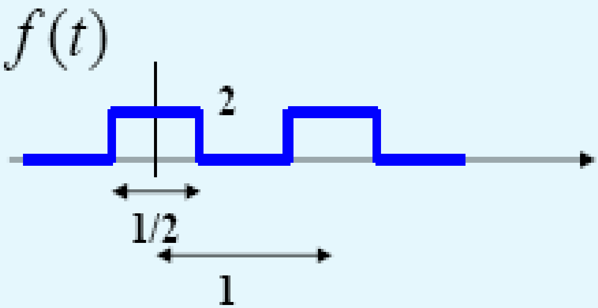
Similarly, the input and output average powers are

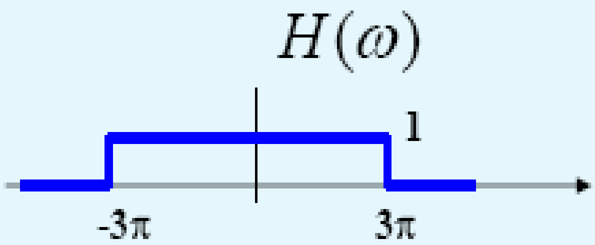
$$P_f = \sum_{n=-\infty}^{\infty} |F_n|^2$$

$$P_g = \sum_{n=-\infty}^{\infty} |H(n\omega_o)|^2 |F_n|^2$$

Example

Determine the output of a linear time-invariant system (lowpass filter) whose input and frequency transfer function are





The Fourier series of the input is $f(t) = \sum_{n=-\infty}^{\infty} \frac{\sin(n\pi/2)}{(n\pi/2)} e^{jn2\pi t}$ (refer to B.9) and then the output is

$$g(t) = \sum_{n=-\infty}^{\infty} H(n\omega_o) F_n e^{jn2\pi t}$$

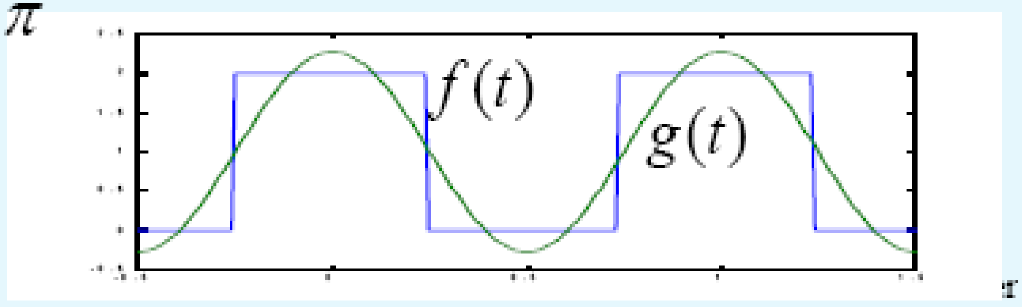
$$= \sum_{n=-\infty}^{\infty} H(2\pi n) \frac{\sin(n\pi/2)}{(n\pi/2)} e^{jn2\pi t} \qquad \because \omega_o = 2\pi$$

$$= H(2\pi(0)) \frac{\sin((0)\pi/2)}{((0)\pi/2)} e^{j(0)2\pi t} + H(2\pi(1)) \frac{\sin((1)\pi/2)}{((1)\pi/2)} e^{j(1)2\pi t}$$

$$+ H(2\pi(-1)) \frac{\sin((-1)\pi/2)}{((-1)\pi/2)} e^{j(-1)2\pi t} + 0$$

$$= 1 + \frac{2}{\pi} e^{j2\pi t} + \frac{2}{\pi} e^{-j2\pi t}$$

$$= 1 + \frac{4}{\pi} \cos 2\pi t$$



Fourier spectrum

- The exponential Fourier series is composed of a summation of complex exponentials with the F_n representing the magnitudes and initial phase angles of the harmonically related rotating phasors. The resultant phasor is found by adding the individual phasors vectorially. However, the addition of a series of phasors of each instant of time turns out to be inconvenient way to describe a signal.

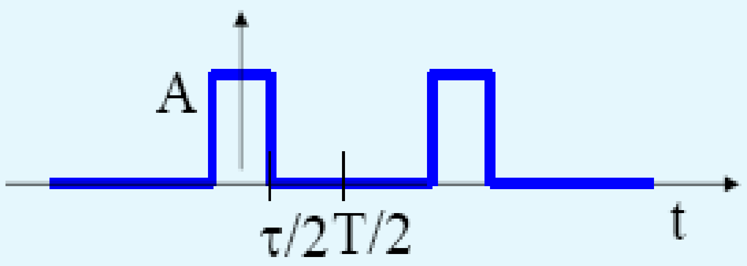
- Instead of looking at every instant of a signal, the Fourier coefficient is plot as a function of the frequency. This plot is called the Fourier spectrum (or simply spectrum) of $f(t)$.

Fourier spectrum

- For a periodic signal, the Fourier spectrum exists only at $\omega = 0, \pm\omega_o, \pm 2\omega_o, \ldots$. It is therefore a discrete spectrum, sometimes referred to as a line spectrum.
- In general, the F_n are complex-valued. To describe the coefficients then requires two graphs, the magnitude spectrum and phase spectrum.

Example

Consider a rectangular pulse train (refer to B.16-17)



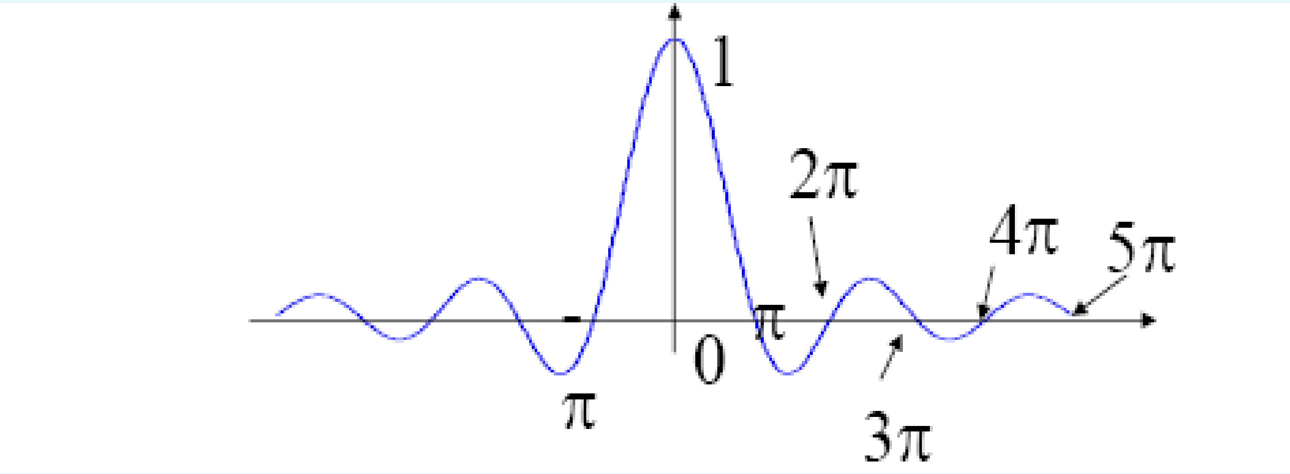
– The Fourier series of *f(t)* and F_n are

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{A\tau}{T} Sa(n\omega_o \tau / 2) e^{jn\omega_o t} \qquad F_n = \frac{A\tau}{T} Sa(n\omega_o \tau / 2)$$

– $Sa(x) = \sin(x)/x$

- the amplitude of the function oscillates, decaying in either direction of x and approaching zero as $|x| \to \infty$.

- The maximum value of this function occurs as x approaches zero, for $\sin(x)/x \to 1$ as $x \to 0$.



Spectrum

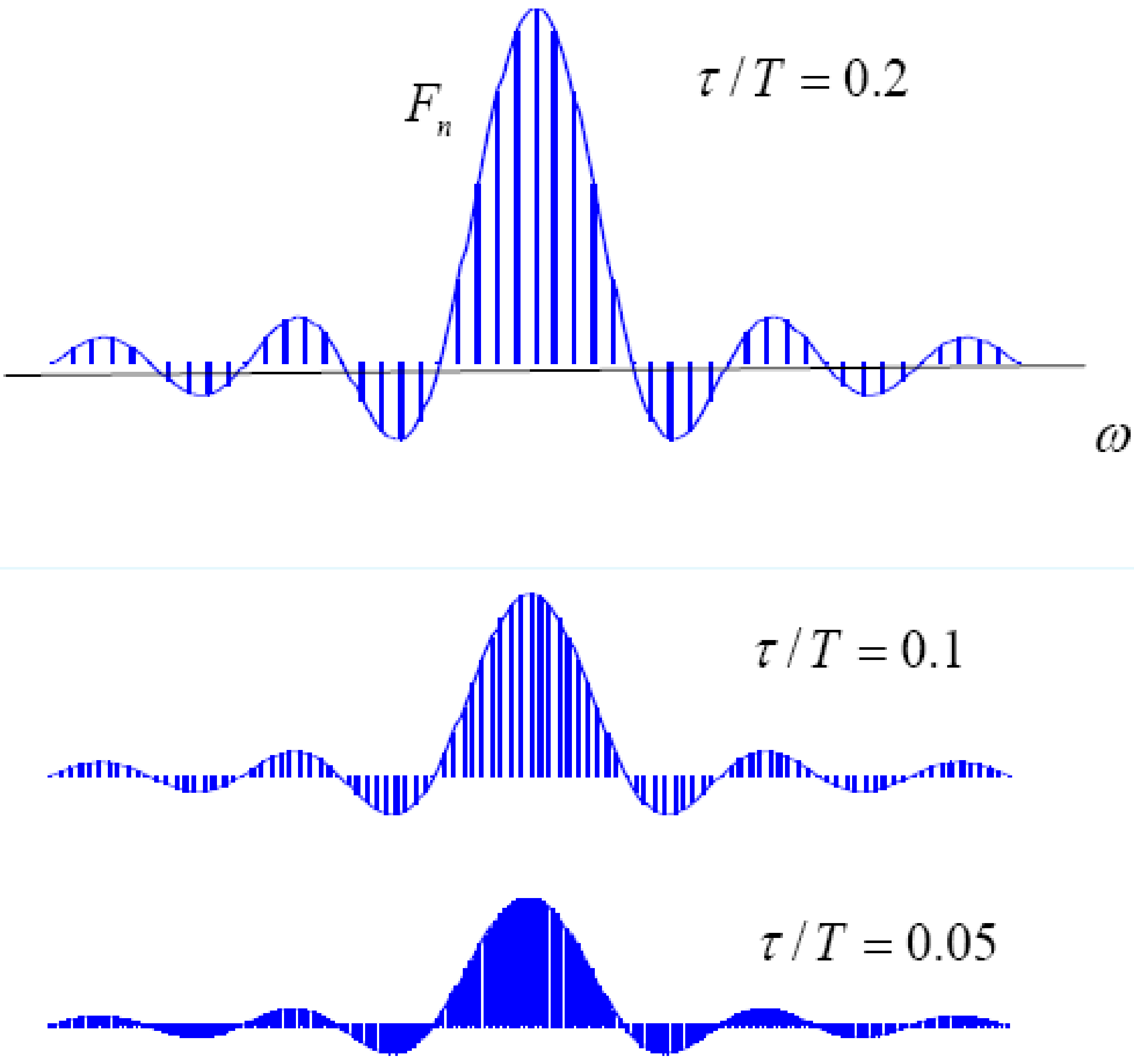
$$F_n = \frac{A\tau}{T} Sa(n\omega_o\tau/2)$$

$$= \frac{A\tau}{T} Sa(n\pi\tau/T) \qquad \because \omega_o = 2\pi/T$$

The pulse duration τ is fixed. When the period of the signal (T) increases,

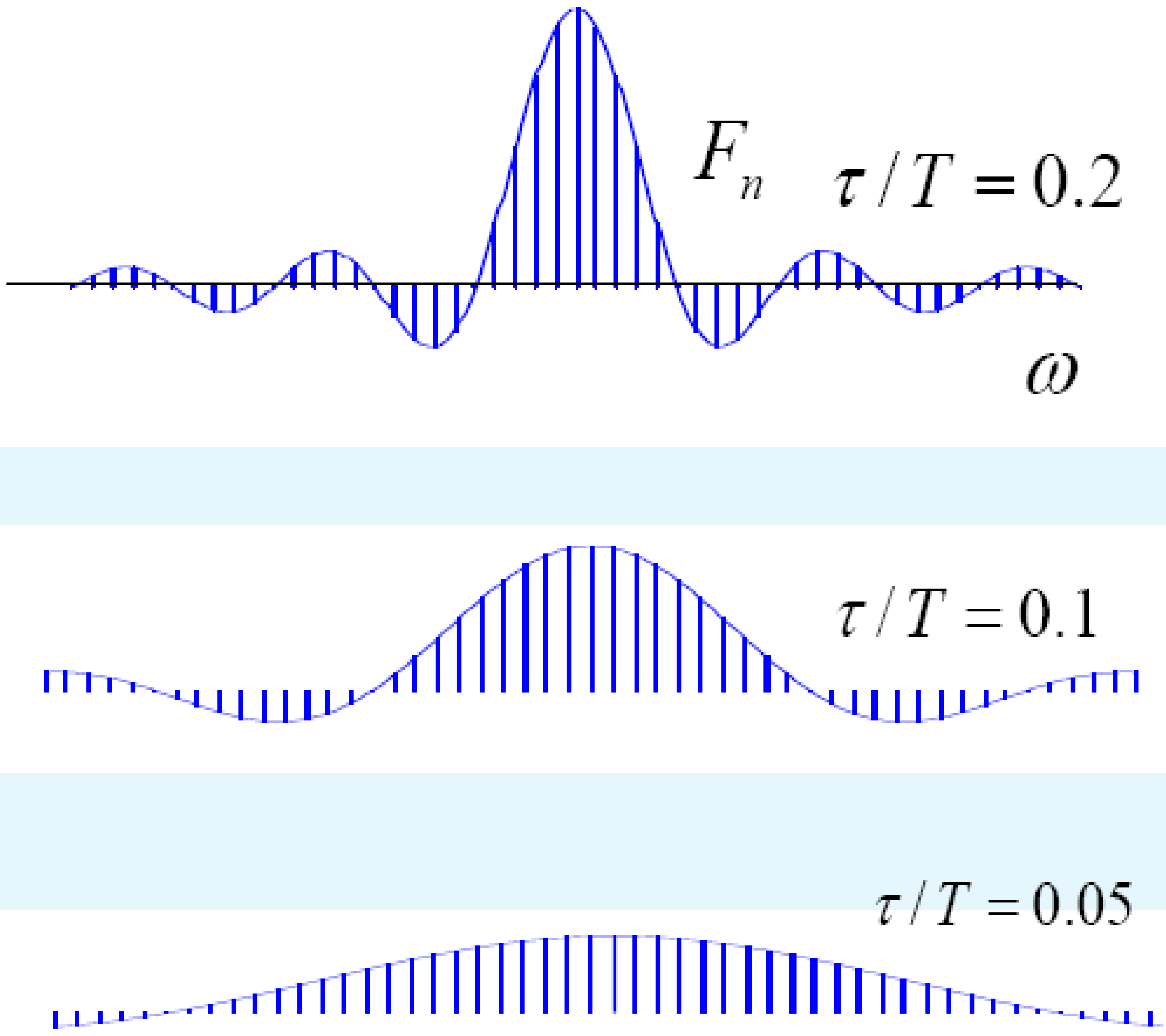
- the amplitude of the spectrum decreases (amplitude = $A\tau/T$) and
- the spacing between lines decreases. (spacing = $2\pi/T$)

The shape of the spectrum is independent of the period (T).


F_n
$\tau / T = 0.2$
ω
$\tau / T = 0.1$
$\tau / T = 0.05$

The period T is fixed. If the pulse duration increases,

- the amplitude of the spectrum increases proportional to τ (amplitude = $A\tau/T$) and
- the frequency content of the signal is compressed within an increasingly narrower range of frequencies (inverse relationship between pulse with in time and the frequency 'spread' of the spectrum)


F_n
$\tau / T = 0.2$
ω
$\tau / T = 0.1$
$\tau / T = 0.05$

Frequency Spectrum of Electronic Signals

- Nonrepetitive signals have continuous spectra often occupying a broad range of frequencies
- Fourier theory tells us that repetitive signals are composed of a set of sinusoidal signals with distinct amplitude, frequency, and phase.
- The set of sinusoidal signals is known as a **Fourier series**.
- The frequency spectrum of a signal is the amplitude and phase components of the signal versus frequency.

Frequencies of Some Common Signals

Audible sounds	20 Hz - 20 KHz
Baseband TV	0 - 4.5 MHz
FM Radio	88 - 108 MHz
Television (Channels 2-6)	54 - 88 MHz
Television (Channels 7-13)	174 - 216 MHz
Maritime and Govt. Comm.	216 - 450 MHz
Cell phones	1710 - 2690 MHz
Satellite TV	3.7 - 4.2 GHz