Fundamentals of Reliability Engineering and Applications

Reliability Engineering Outline

- Reliability definition
- Reliability estimation
- System reliability calculations

Reliability Importance

- One of the most important characteristics of a product, it is a measure of its performance with time (Transatlantic and Transpacific cables)
- Products' recalls are common (only after time elapses). In October 2006, the Sony Corporation recalled up to 9.6 million of its personal computer batteries
- Products are discontinued because of fatal accidents (Pinto, Concord)
- Medical devices and organs (reliability of artificial organs)

Reliability Importance

- Business data

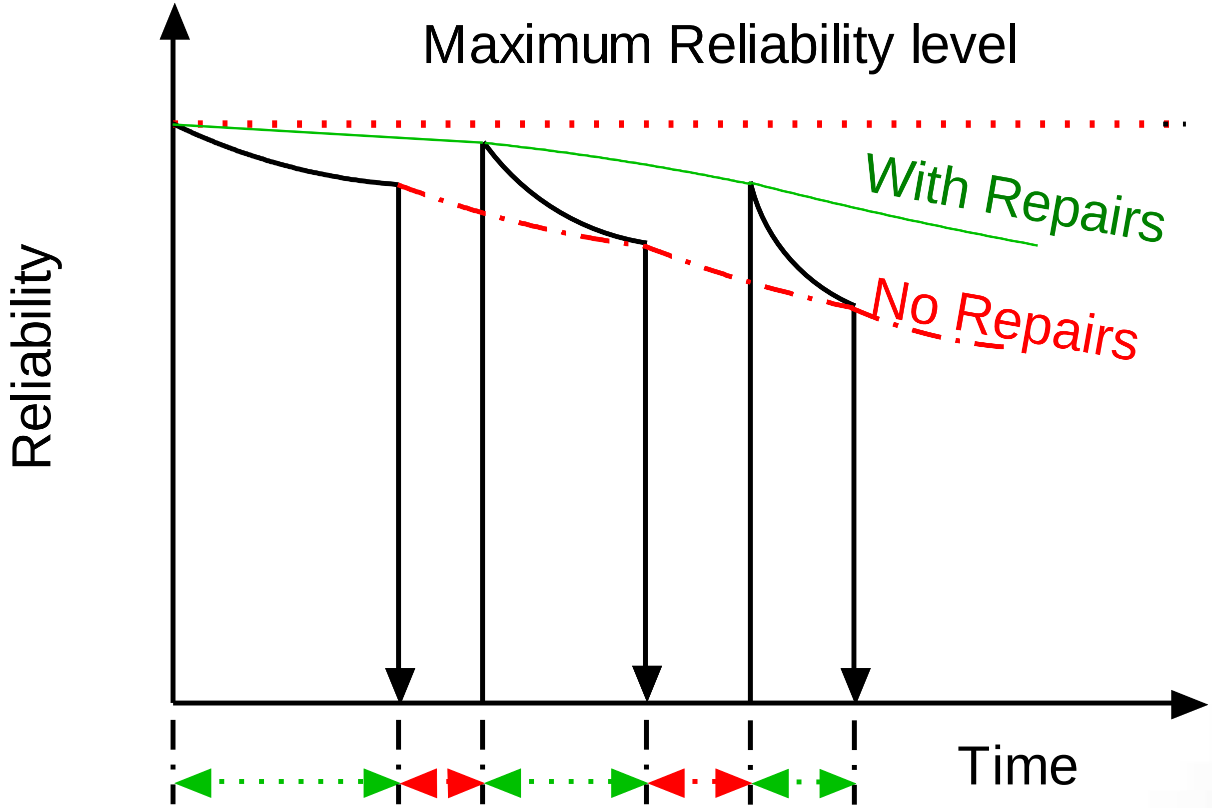
Table 1. Warranty Claims Paid by U.S. Manufacturers

Company	2006 Claims $ Million	2005 Claims $ Million
General Motors Corp.	$4,463	$4,696
Ford Motor Co.	$4,106	$3,986
Hewlett-Packard Co.	$2,346	$2,353
Dell Inc.	$1,775	$1,521
Motorola Inc.	$891	$716
IBM Corp.	$762	$831
Caterpillar Inc.	$745	$712
General Electric Co.	$665	$699
Deere & Co.	$509	$453
Whirlpool Corp.	$459	$294
Boeing Co.	$206	$146
Textron Inc.	$167	$149

Warranty costs measured in million dollars for several large American manufacturers in 2006 and 2005. (www.warrantyweek.com)

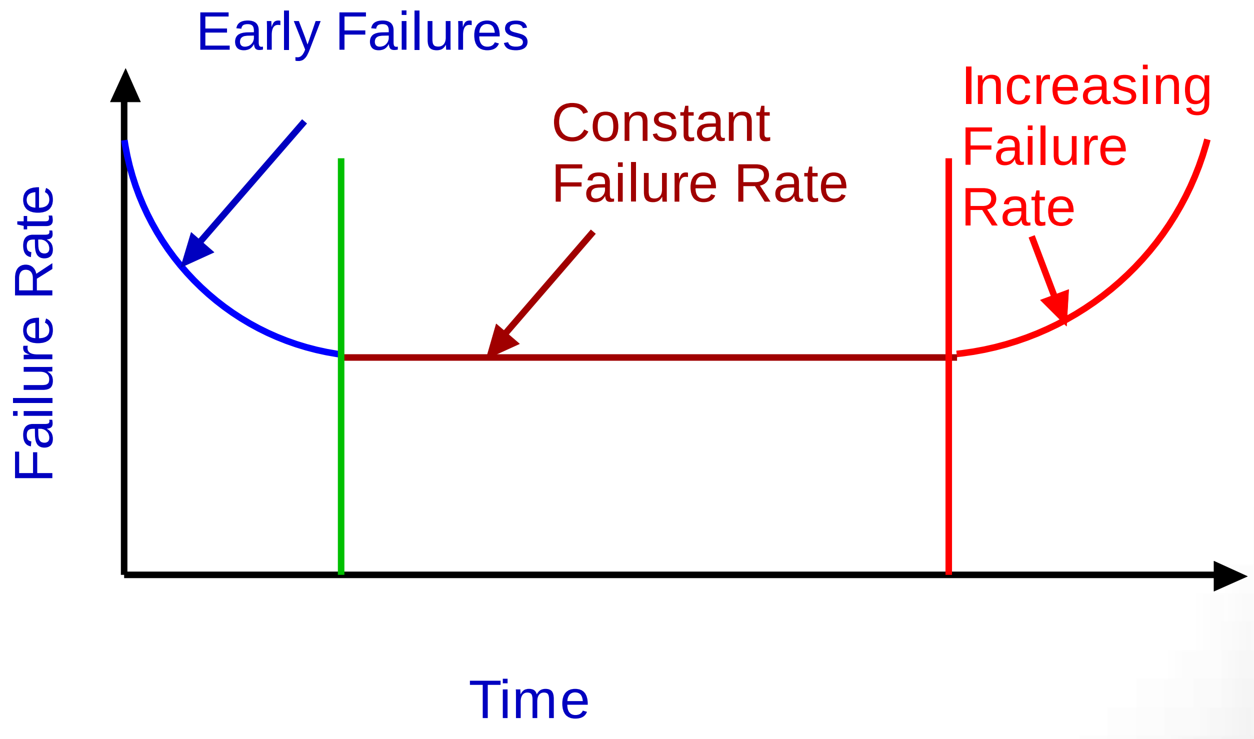
Some Initial Thoughts
Repairable and Non-Repairable

Another measure of reliability is availability (probability that the system provides its functions when needed).



Some Initial Thoughts
Warranty

- Will you buy additional warranty?
- Burn in and removal of early failures. (Lemon Law).



Reliability Definitions

Reliability is a time dependent characteristic.

- It can only be determined after an elapsed time but can be predicted at any time.
- It is the probability that a product or service will operate properly for a specified period of time (design life) under the design operating conditions without failure.

Other Measures of Reliability

Availability is used for repairable systems

- It is the probability that the system is operational at any random time t.
- It can also be specified as a proportion of time that the system is available for use in a given interval (0,T).

Other Measures of Reliability

Mean Time To Failure (MTTF): *It is the average time that elapses until a failure occurs.*

It does not provide information about the distribution of the TTF, hence we need to estimate the variance of the TTF.

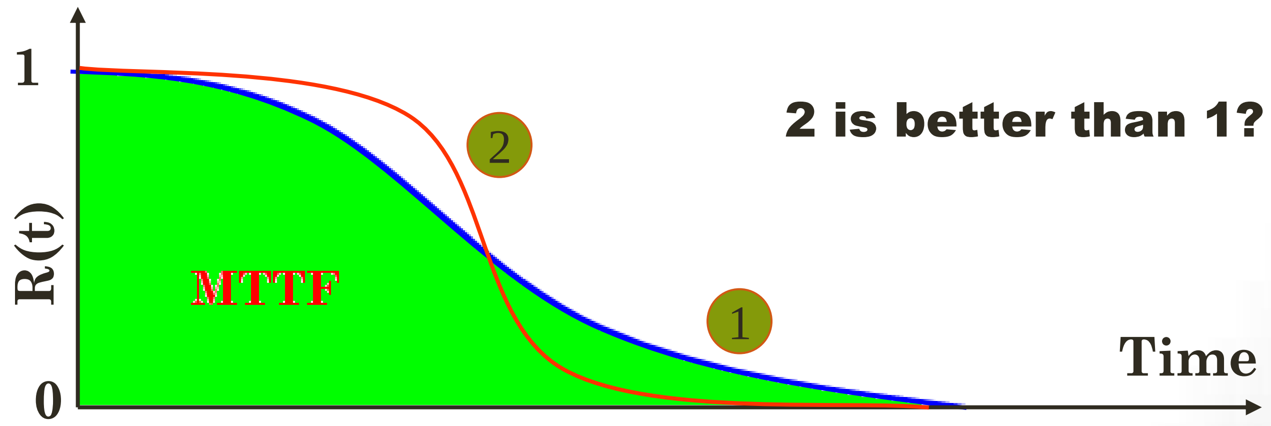
Mean Time Between Failure (MTBF): It is the average time between successive failures.

It is used for repairable systems.

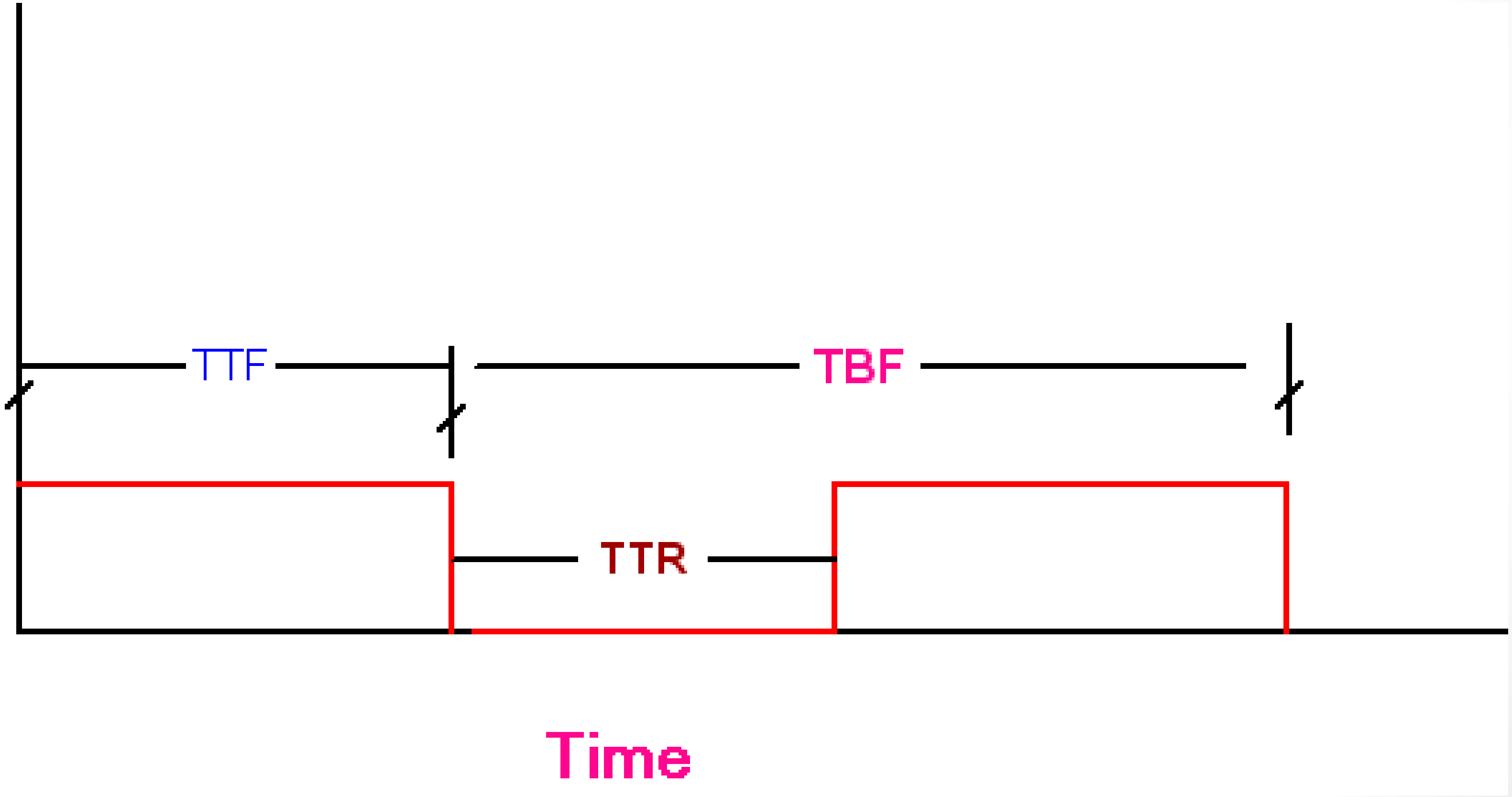
Mean Time to Failure: MTTF

$$MTTF = \int_0^\infty t f(t)dt = \int_0^\infty R(t)dt$$

$$MTTF = \frac{1}{n}\sum_{i=1}^{n} t_i$$



Mean Time Between Failure: MTBF



Other Measures of Reliability

Mean Residual Life (MRL): It is the expected remaining life, T-t, given that the product, component, or a system has survived to time t.

$$L(t) = E[T - t \mid T \geq t] = \frac{1}{R(t)} \int_t^{\infty} \tau f(\tau) d\tau - t$$

Failure Rate (FITs failures in 10^9 hours): The failure rate in a time interval [t_1, t_2] is the probability that a failure per unit time occurs in the interval given that no failure has occurred prior to the beginning of the interval.

Hazard Function: It is the limit of the failure rate as the length of the interval approaches zero.

Basic Calculations

Suppose n_0 identical units are subjected to a test. During the interval (t, t+Δt), we observed $n_f(t)$ failed components. Let $n_s(t)$ be the surviving components at time t, then the MTTF, failure density, hazard rate, and reliability at time t are:

$$MTTF = \frac{\sum_{i=1}^{n_0} t_i}{n_0}, \qquad \hat{f}(t) = \frac{n_f(t)}{n_0 \Delta t}$$

$$\hat{\lambda}(t) = \frac{n_f(t)}{n_s(t)\Delta t}, \qquad \hat{R}(t) = P_r(T > t) = \frac{n_s(t)}{n_0}$$

Basic Definitions Cont'd

The unreliability $F(t)$ is

$$F(t) = 1 - R(t)$$

Example: 200 light bulbs were tested and the failures in 1000-hour intervals are

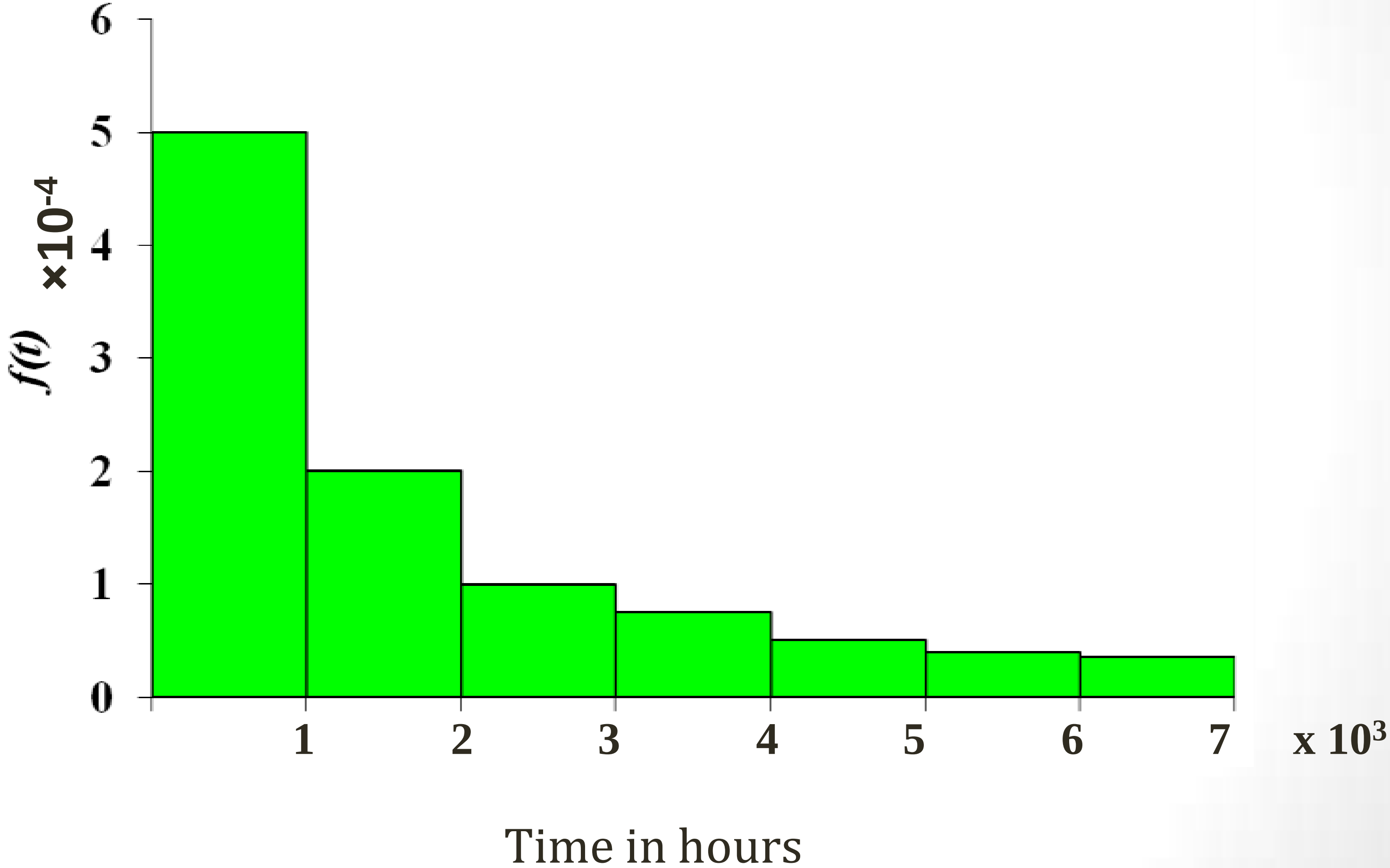
Time Interval (Hours)	Failures in the interval
0-1000	100
1001-2000	40
2001-3000	20
3001-4000	15
4001-5000	10
5001-6000	8
6001-7000	7
Total	200

Calculations

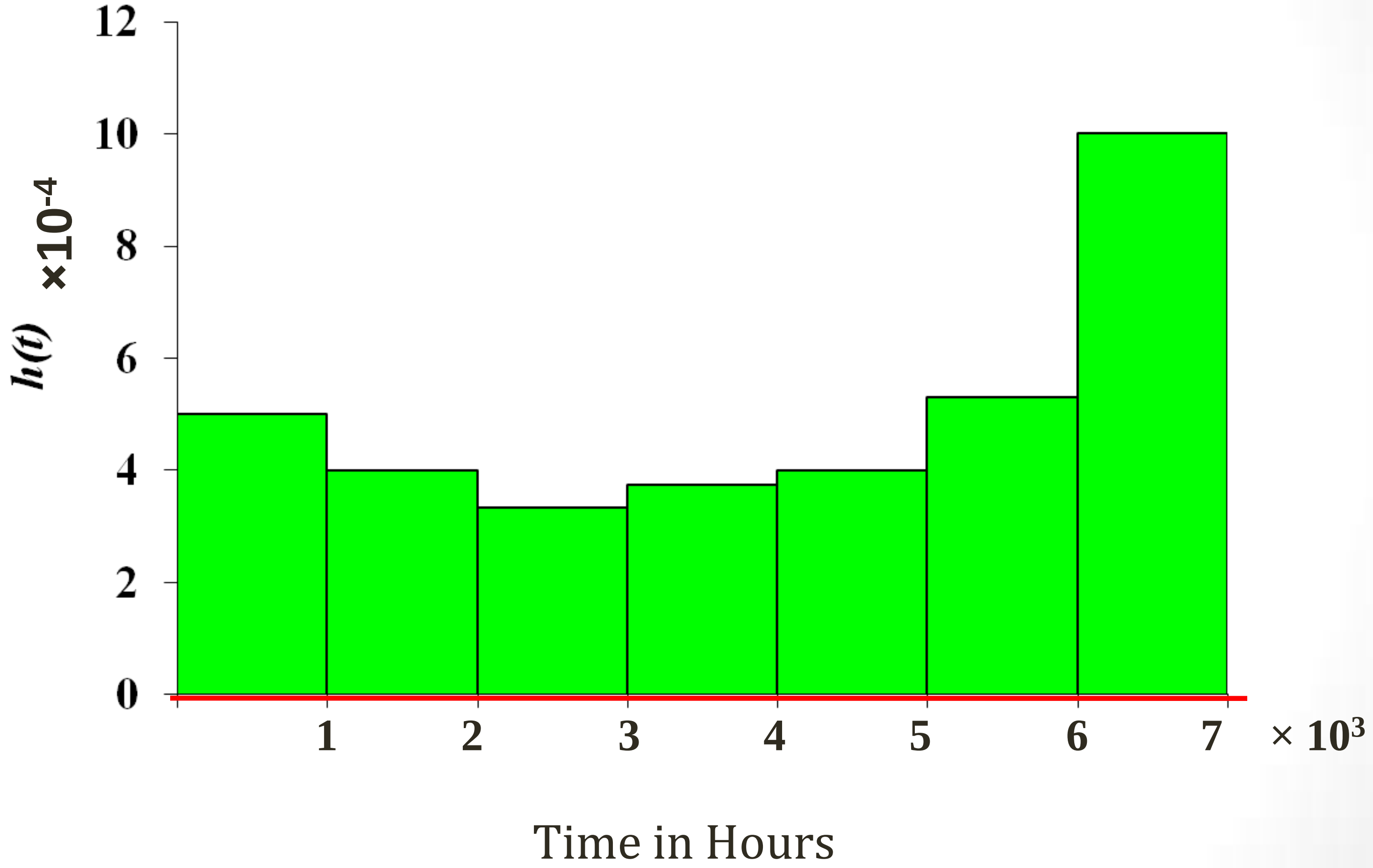
Time Interval (Hours)	Failures in the interval
0-1000	100
1001-2000	40
2001-3000	20
3001-4000	15
4001-5000	10
5001-6000	8
6001-7000	7
Total	200

Time Interval	Failure Density $f(t)$ x 10^{-4}	Hazard rate $h(t)$ x 10^{-4}
0–1000	$\frac{100}{200\times10^3}=5.0$	$\frac{100}{200\times10^3}=5.0$
1001–2000	$\frac{40}{200\times10^3}=2.0$	$\frac{40}{100\times10^3}=4.0$
2001–3000	$\frac{20}{200\times10^3}=1.0$	$\frac{20}{60\times10^3}=3.33$
……	……..	……
6001–7000	$\frac{7}{200\times10^3}=0.35$	$\frac{7}{7\times10^3}=10$

Failure Density vs. Time



Hazard Rate vs. Time

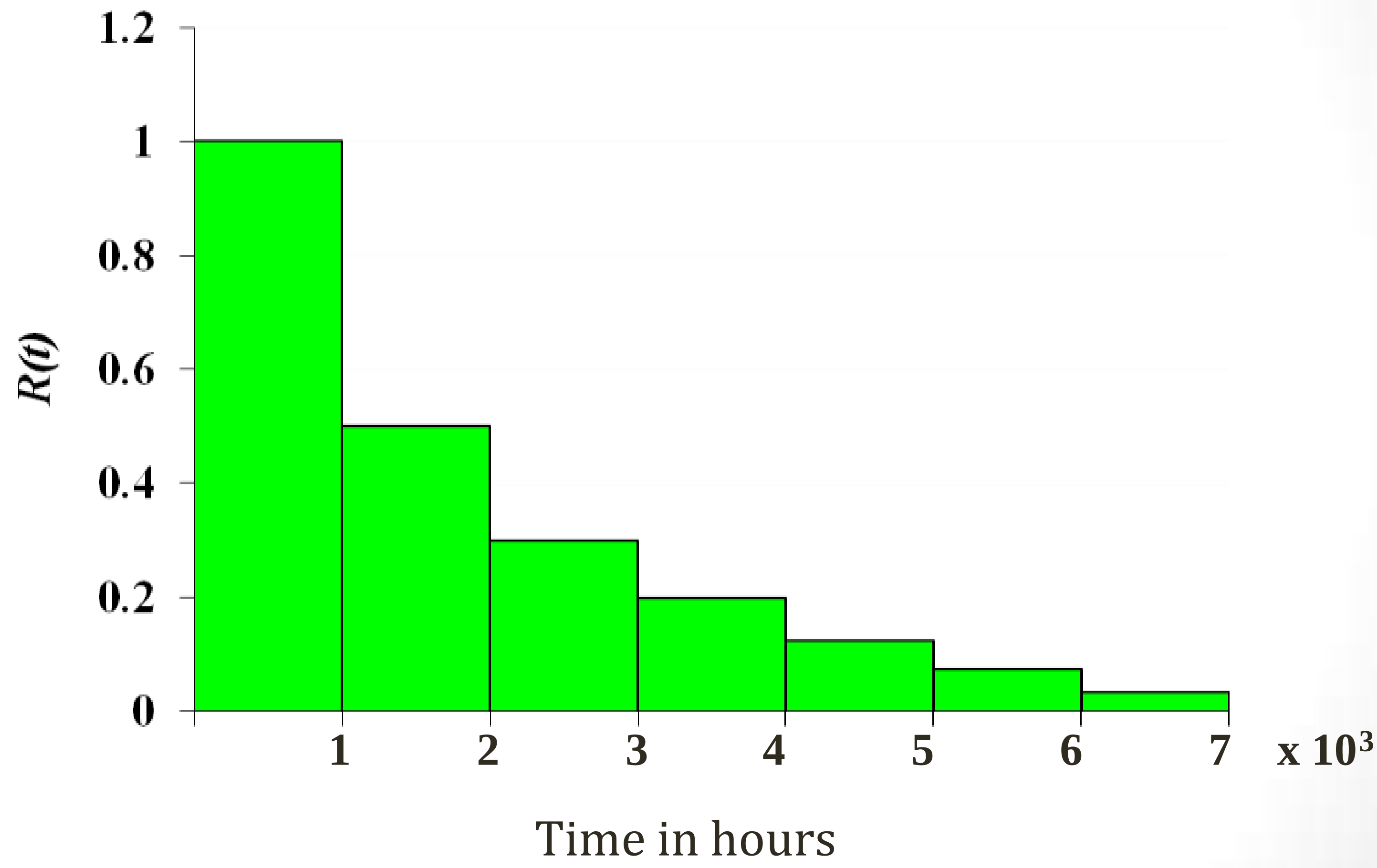


Calculations

Time Interval (Hours)	Failures in the interval
0-1000 1001-2000 2001-3000 3001-4000 4001-5000 5001-6000 6001-7000	100 40 20 15 10 8 7
Total	200

Time Interval	Reliability $R(t)$
0–1000	200/200=1.0
1001–2000	100/200=0.5
2001–3000	60/200=0.33
……	……
6001–7000	0.35/10=.035

Reliability vs. Time



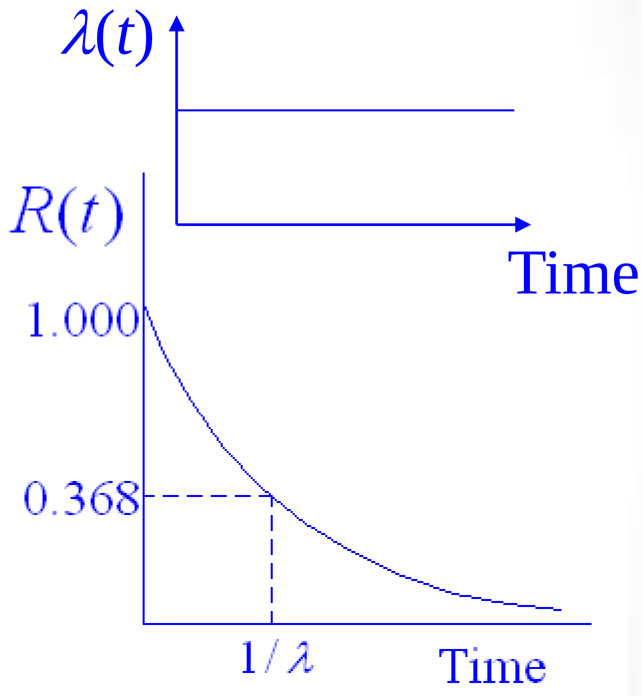
Exponential Distribution

Definition

$$\lambda(t) = \lambda \qquad \lambda > 0,\ \ t \geq 0$$

$$f(t) = \lambda \exp(-\lambda t)$$

$$R(t) = \exp(-\lambda t) = 1 - F(t)$$



Exponential Model Cont'd

Statistical Properties

$$MTTF = \frac{1}{\lambda}$$

$\lambda = 5 \times 10^{-6}$ Failures/hr

MTTF=200,000 hrs or 20 years

$$Variance = \frac{1}{\lambda^2}$$

$$\text{Median life} = (\ln 2)\frac{1}{\lambda}$$

Median life =138,626 hrs or 14 years

Empirical Estimate of *F(t)* and *R(t)*

When the exact failure times of units is known, we use an empirical approach to estimate the reliability metrics. The most common approach is the Rank Estimator. Order the failure time observations (failure times) in an ascending order:

$$t_1 \leq t_2 \leq \ldots \leq t_{i-1} \leq t_i \leq t_{i+1} \leq \ldots \leq t_{n-1} \leq t_n$$

Empirical Estimate of *F(t)* and *R(t)*

$F(t_i)$ is obtained by several methods

1. Uniform "naive" estimator $\frac{i}{n}$
2. Mean rank estimator $\frac{i}{n+1}$
3. Median rank estimator (Bernard) $\frac{i-0.3}{n+0.4}$
4. Median rank estimator (Blom) $\frac{i-3/8}{n+1/4}$

Empirical Estimate of *F(t)* and *R(t)*

Assume that we use the mean rank estimator

$$\hat{F}(t_i) = \frac{i}{n+1}$$

$$\hat{R}(t_i) = \frac{n+1-i}{n+1} \qquad t_i \le t \le t_{i+1} \qquad i = 0, 1, 2, \ldots, n$$

Since *f*(*t*) is the derivative of *F*(*t*), then

$$\hat{f}(t_i) = \frac{\hat{F}(t_{i+1}) - \hat{F}(t_i)}{\Delta t_i \,.(n+1)} \qquad \Delta t_i = t_{i+1} - t_i$$

$$\hat{f}(t_i) = \frac{1}{\Delta t_i \,.(n+1)}$$

Empirical Estimate of *F(t)* and *R(t)*

$$\hat{\lambda}(t_i) = \frac{1}{\Delta t_i .(n+1-i)}$$

$$\hat{H}(t_i) = -\ln(\hat{R}(t_i))$$

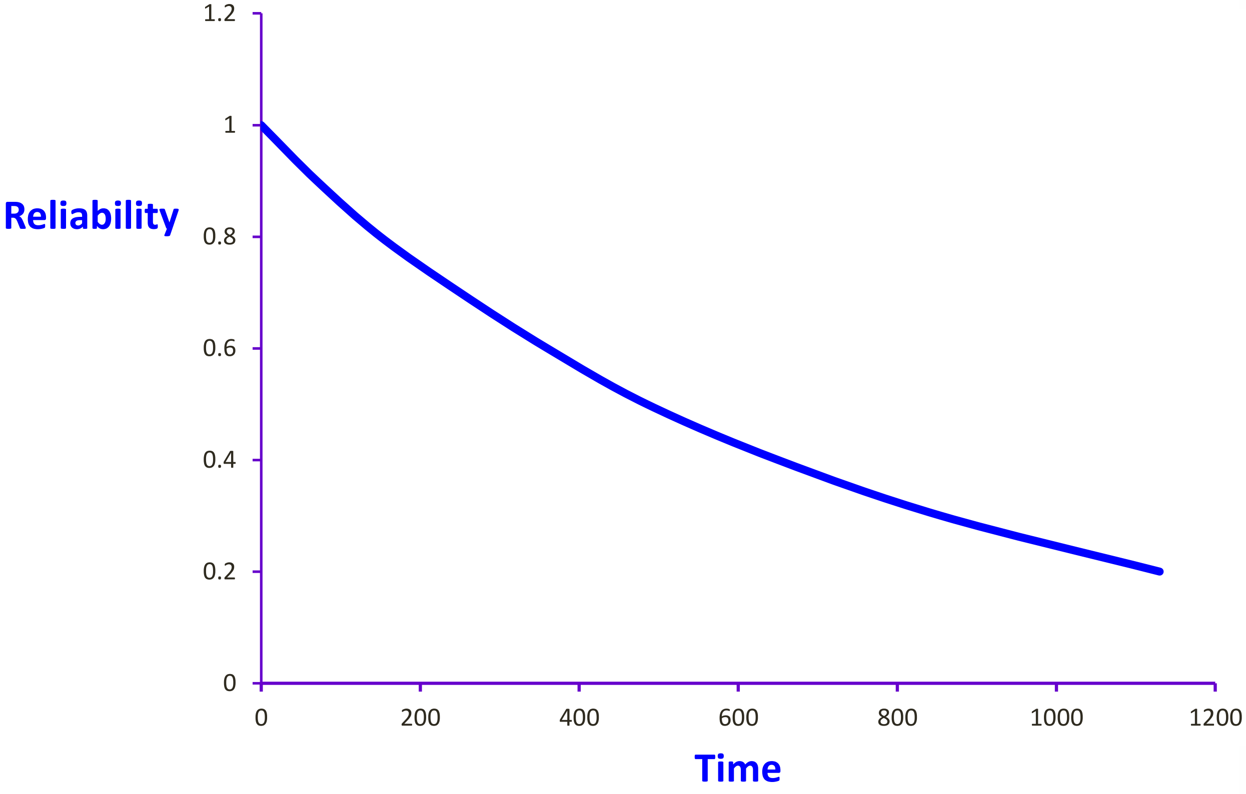
Example:

Recorded failure times for a sample of 9 units are observed at *t*=70, 150, 250, 360, 485, 650, 855, 1130, 1540. Determine F(t), R(t), f(t), $\lambda(t)$,H(t)

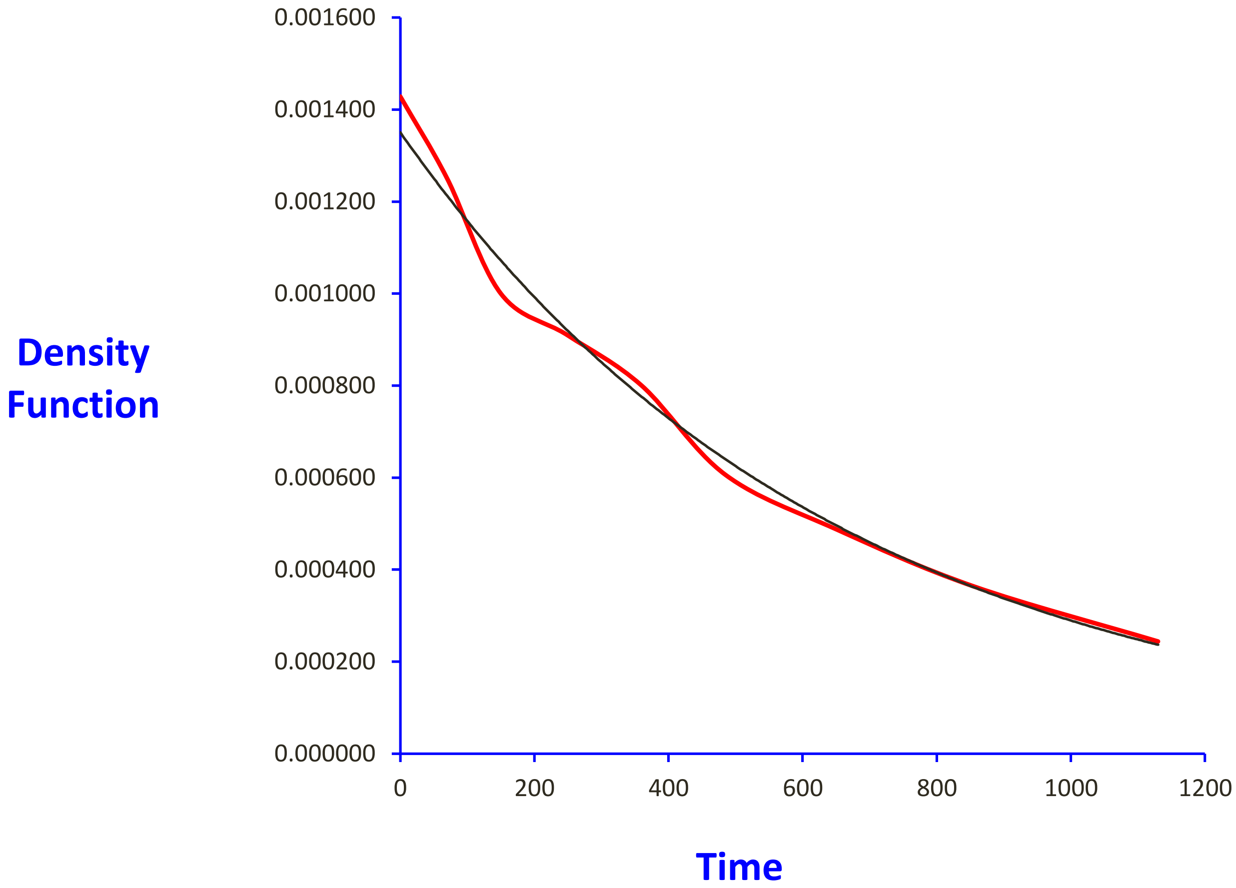
Calculations

i	t (i)	t(i+1)	F=i/10	R=(10-i)/10	f=0.1/Δt	λ =1/(Δt.(10-i))	H(t)
0	0	70	0	1	0.001429	0.001429	0
1	70	150	0.1	0.9	0.001250	0.001389	0.10536052
2	150	250	0.2	0.8	0.001000	0.001250	0.22314355
3	250	360	0.3	0.7	0.000909	0.001299	0.35667494
4	360	485	0.4	0.6	0.000800	0.001333	0.51082562
5	485	650	0.5	0.5	0.000606	0.001212	0.69314718
6	650	855	0.6	0.4	0.000488	0.001220	0.91629073
7	855	1130	0.7	0.3	0.000364	0.001212	1.2039728
8	1130	1540	0.8	0.2	0.000244	0.001220	1.60943791
9	1540	-	0.9	0.1			2.30258509

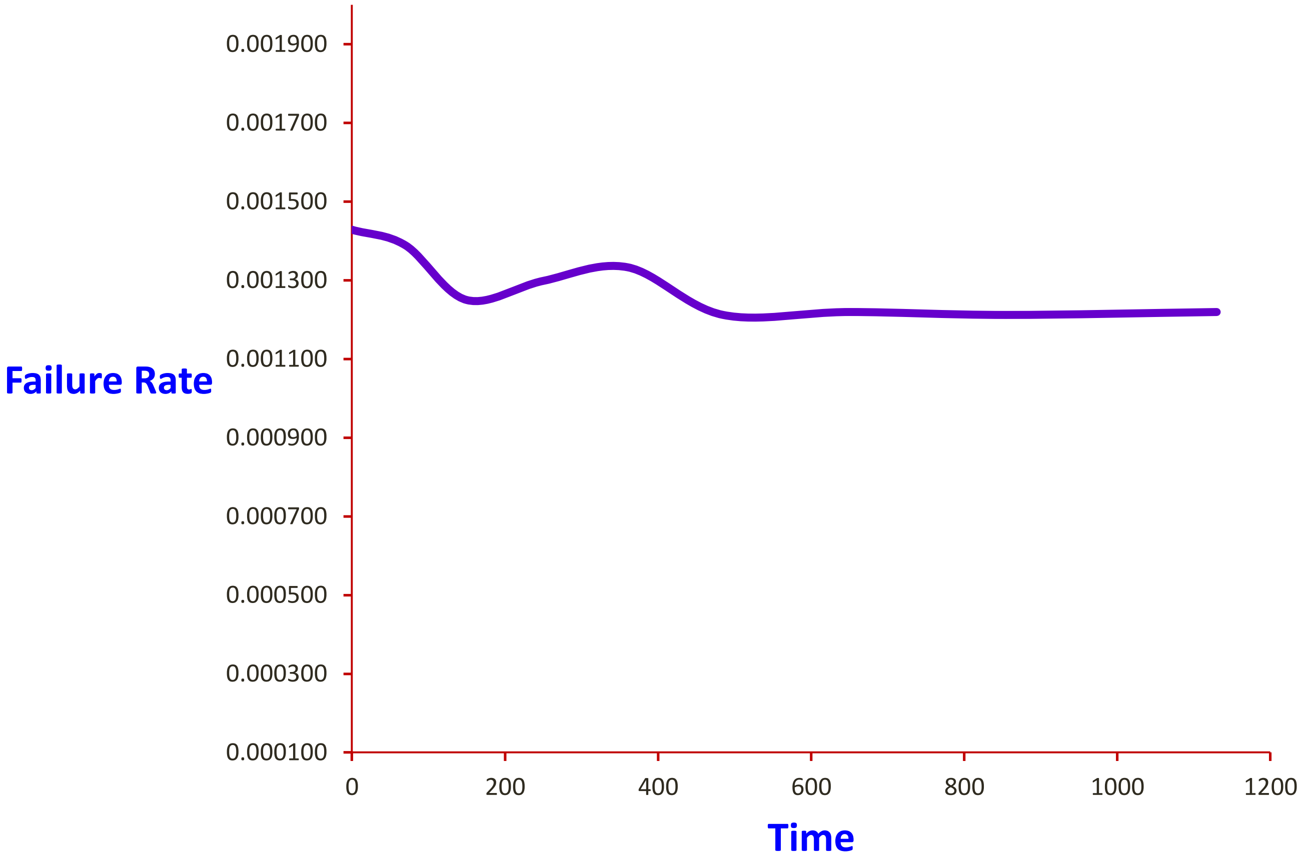
Reliability Function



Probability Density Function



Failure Rate Constant



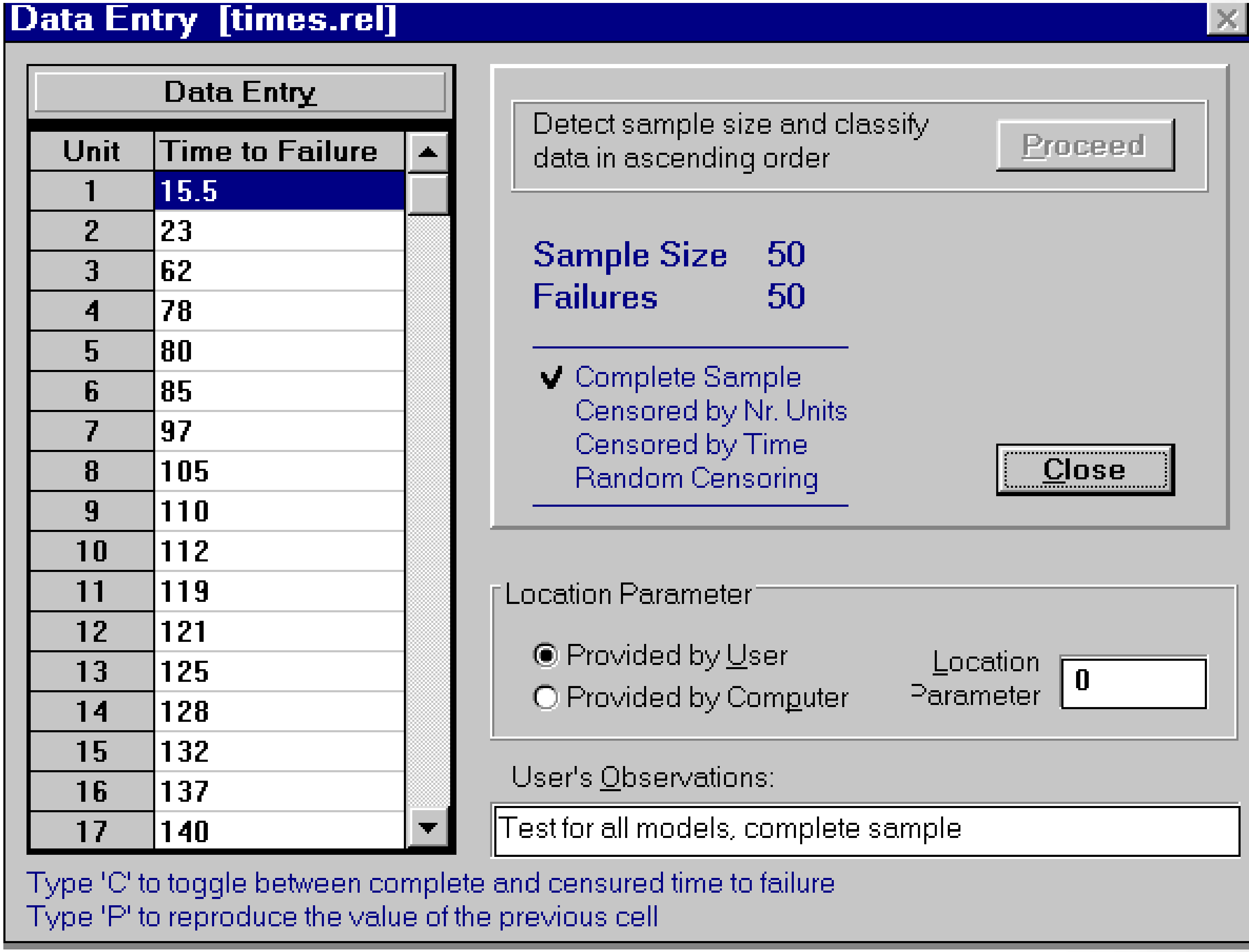
Exponential Distribution: Another Example

Given failure data:

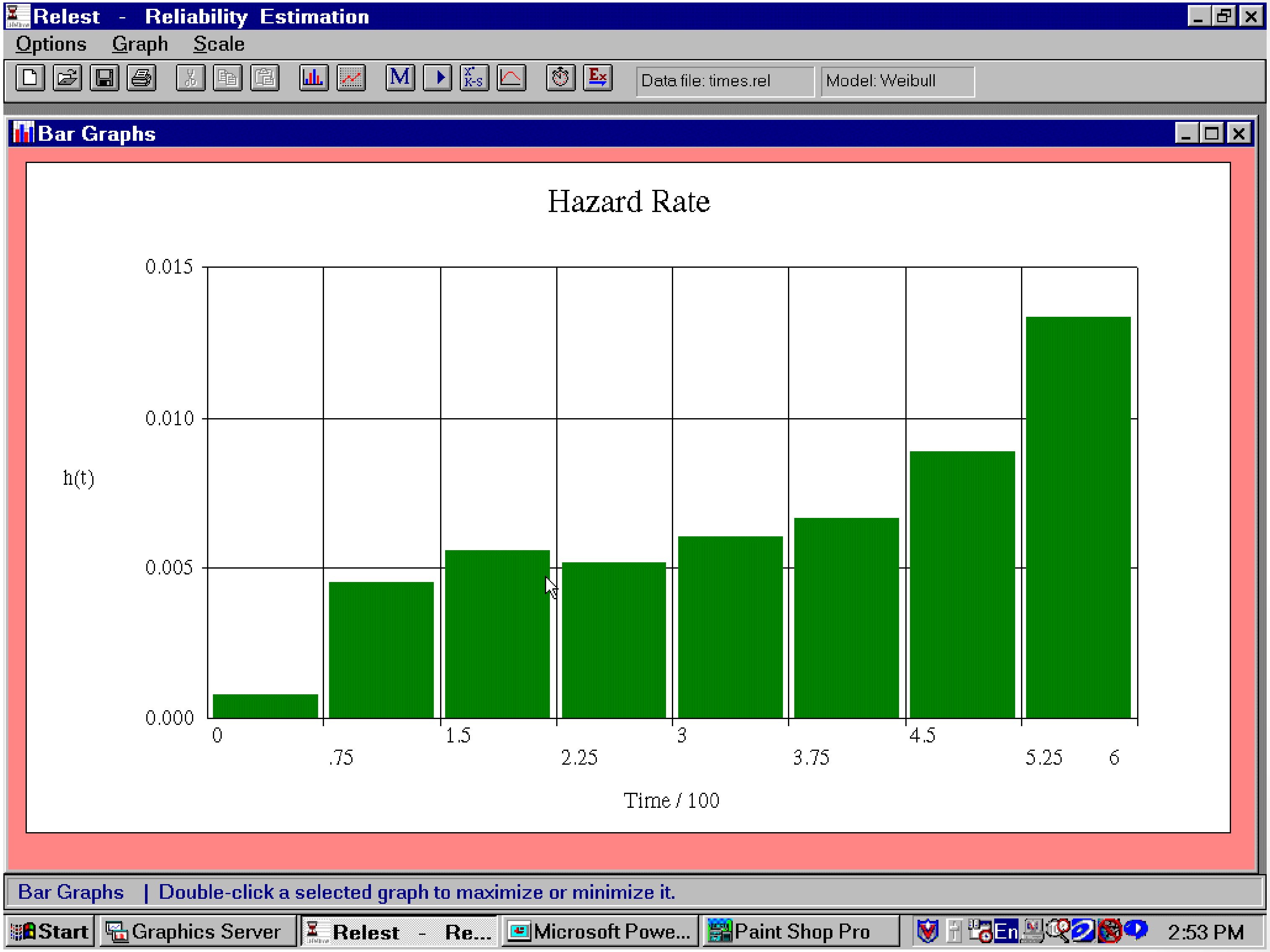
Plot the hazard rate, if constant then use the exponential distribution with $f(t)$, $R(t)$ and $h(t)$ as defined before.

We use a software to demonstrate these steps.

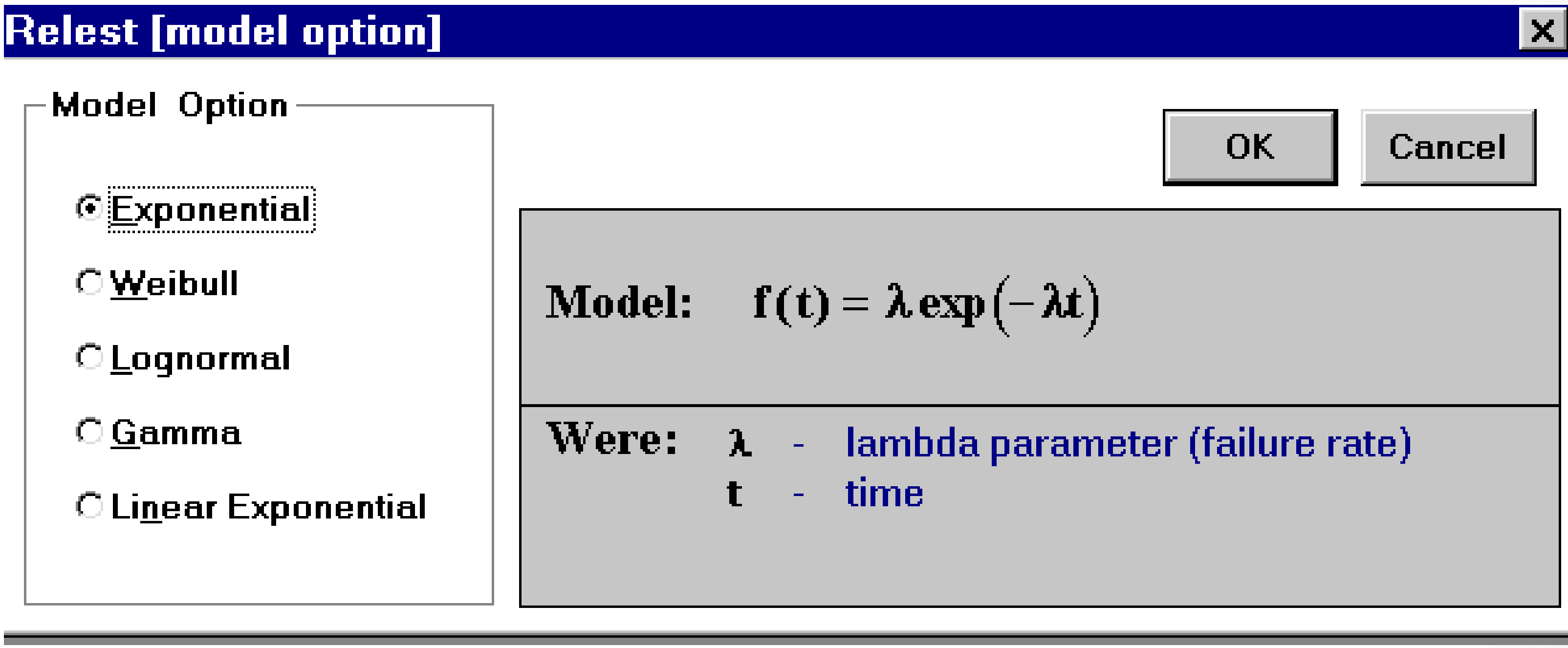
Input Data


Data Entry [times.rel]
Data Entry
Unit
Time to Failure
1 15.5
2 23
3 62
4 78
5 80
6 85
7 97
8 105
9 110
10 112
11 119
12 121
13 125
14 128
15 132
16 137
17 140
Detect sample size and classify data in ascending order
Proceed
Sample Size 50
Failures 50
Complete Sample
Censored by Nr. Units
Censored by Time
Random Censoring
Close
Location Parameter
Provided by User
Provided by Computer
Location Parameter
0
User's Observations:
Test for all models, complete sample
Type 'C' to toggle between complete and censured time to failure
Type 'P' to reproduce the value of the previous cell

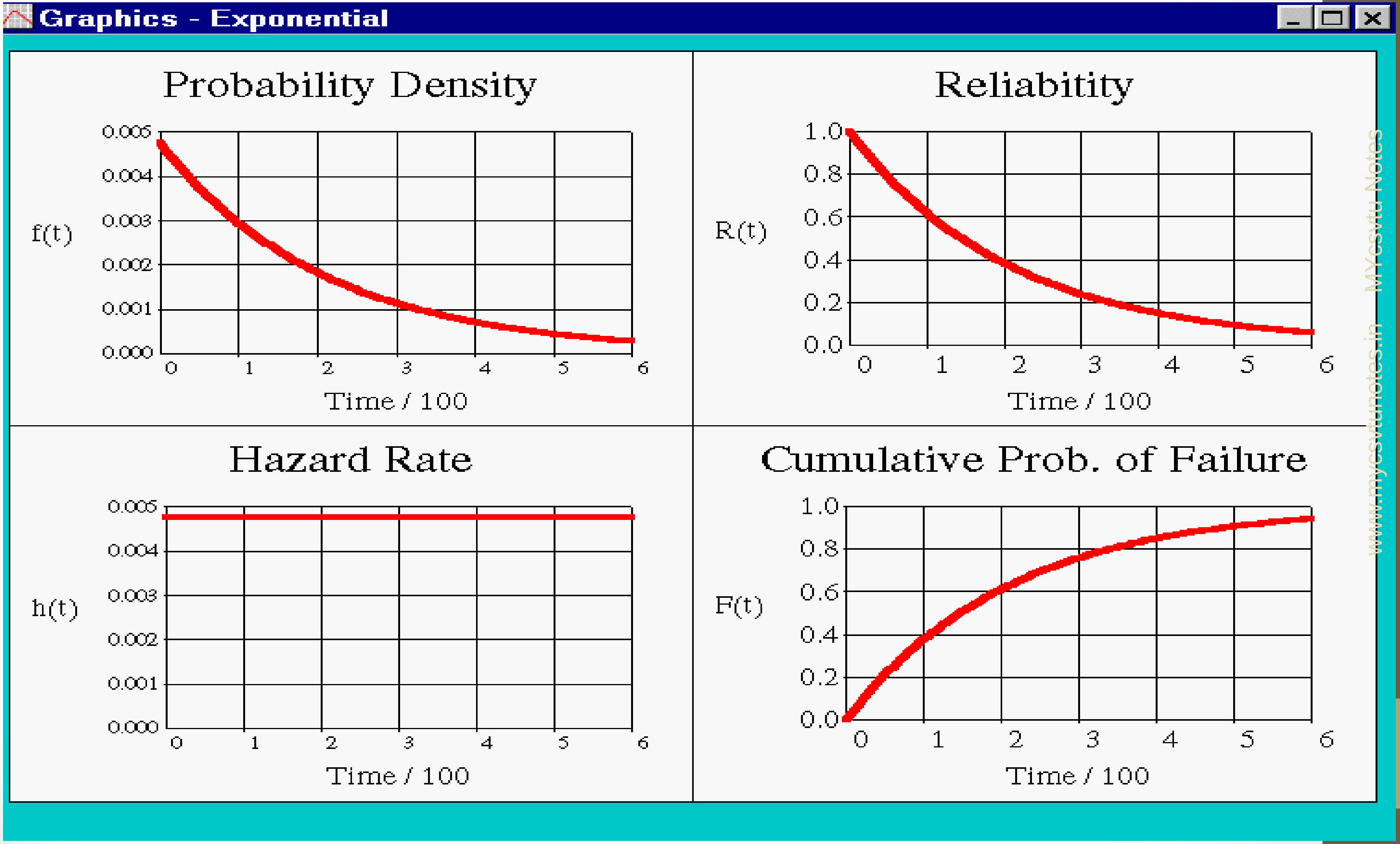
Plot of the Data



Exponential Fit



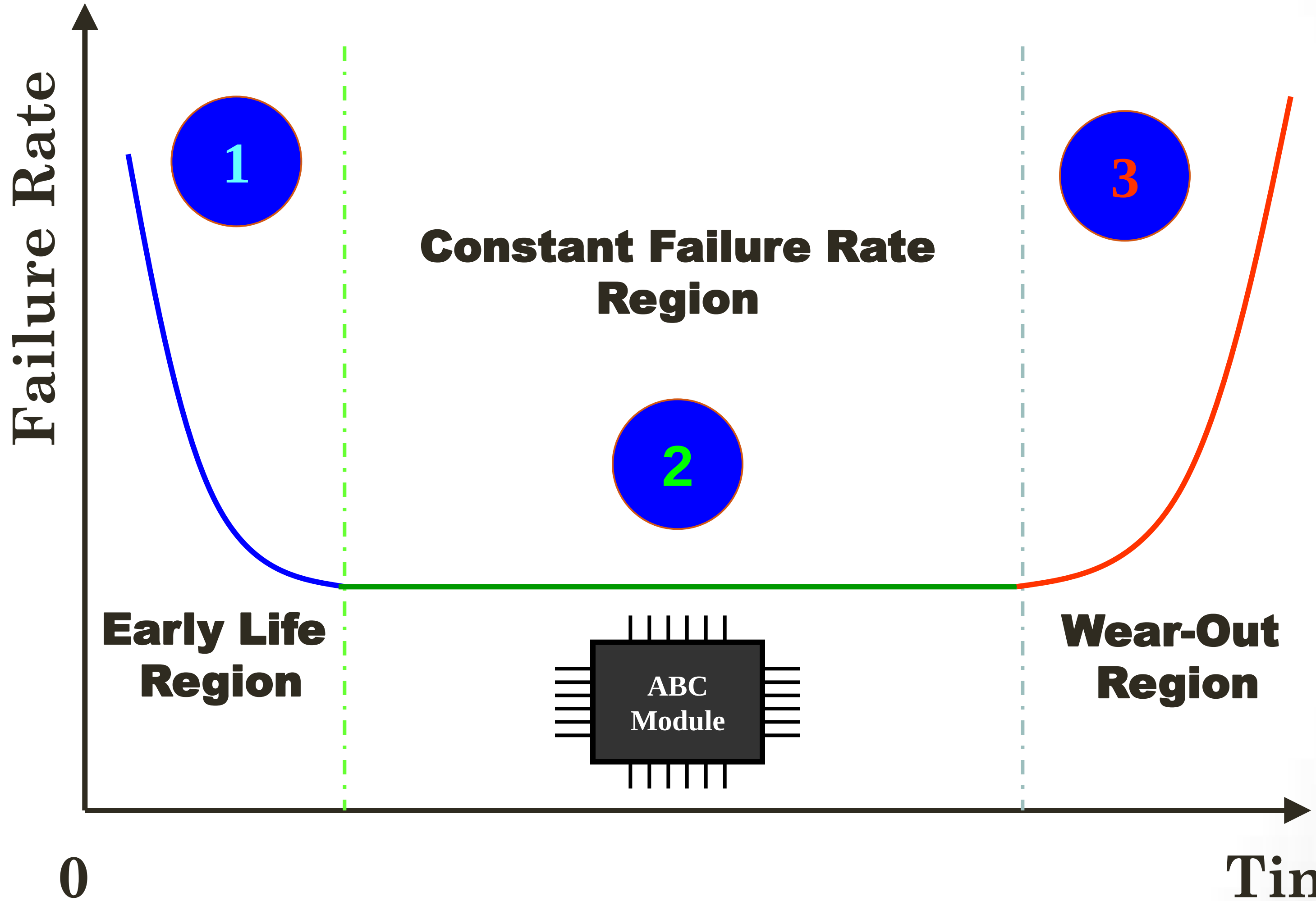
Exponential Analysis



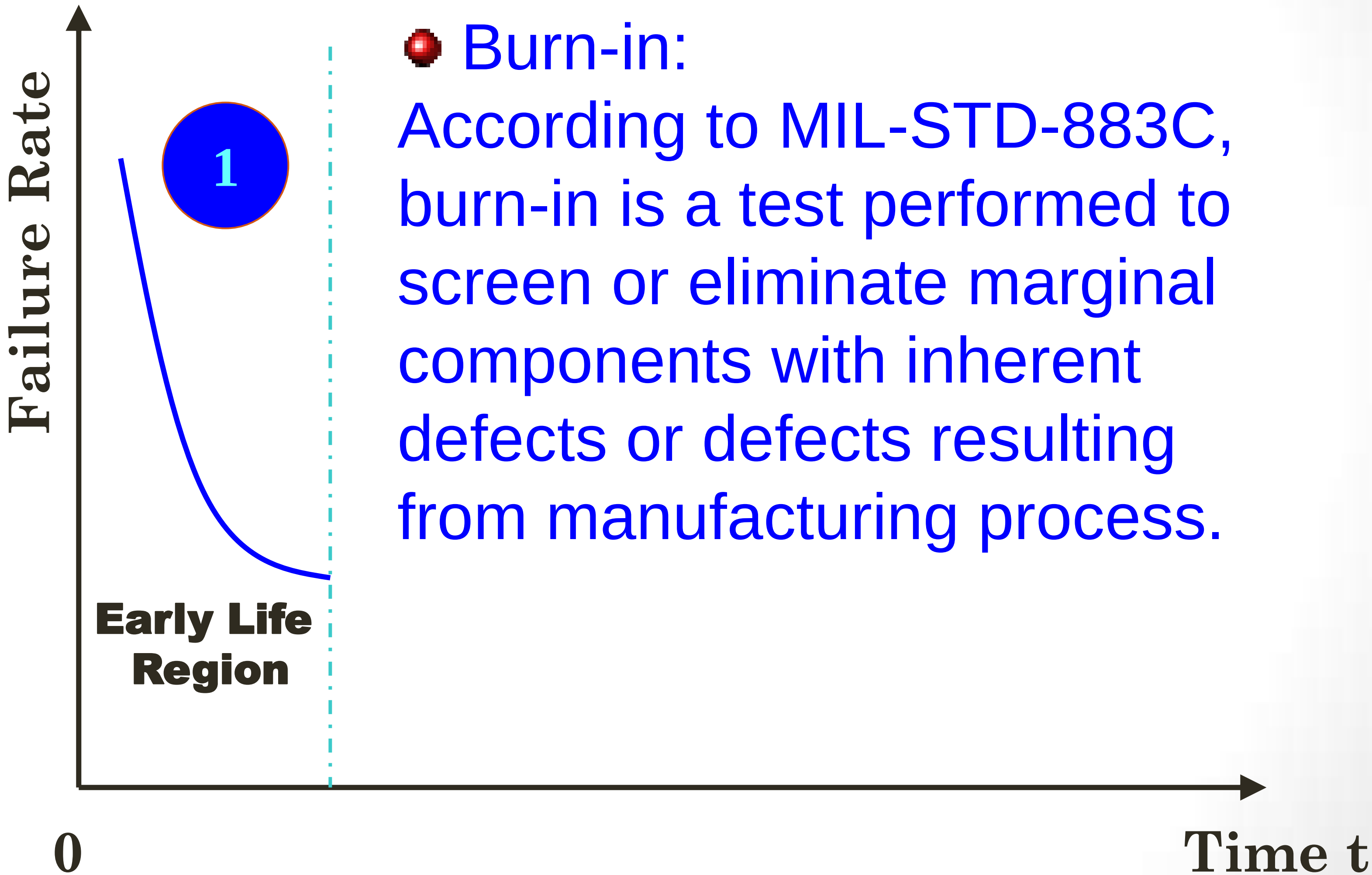
Go Beyond Constant Failure Rate

- Weibull Distribution (Model) and Others

The General Failure Curve



Related Topics (1)



- Burn-in:

According to MIL-STD-883C, burn-in is a test performed to screen or eliminate marginal components with inherent defects or defects resulting from manufacturing process.

Motivation – Simple Example

- Suppose the life times (in hours) of several units are:

$$\underbrace{1\;\;2\;\;3\;\;5\;\;10\;\;15\;\;22\;\;28}$$

$$MTTF = \frac{1+2+3+5+10+15+22+28}{8} = 10.75 \text{ hours}$$

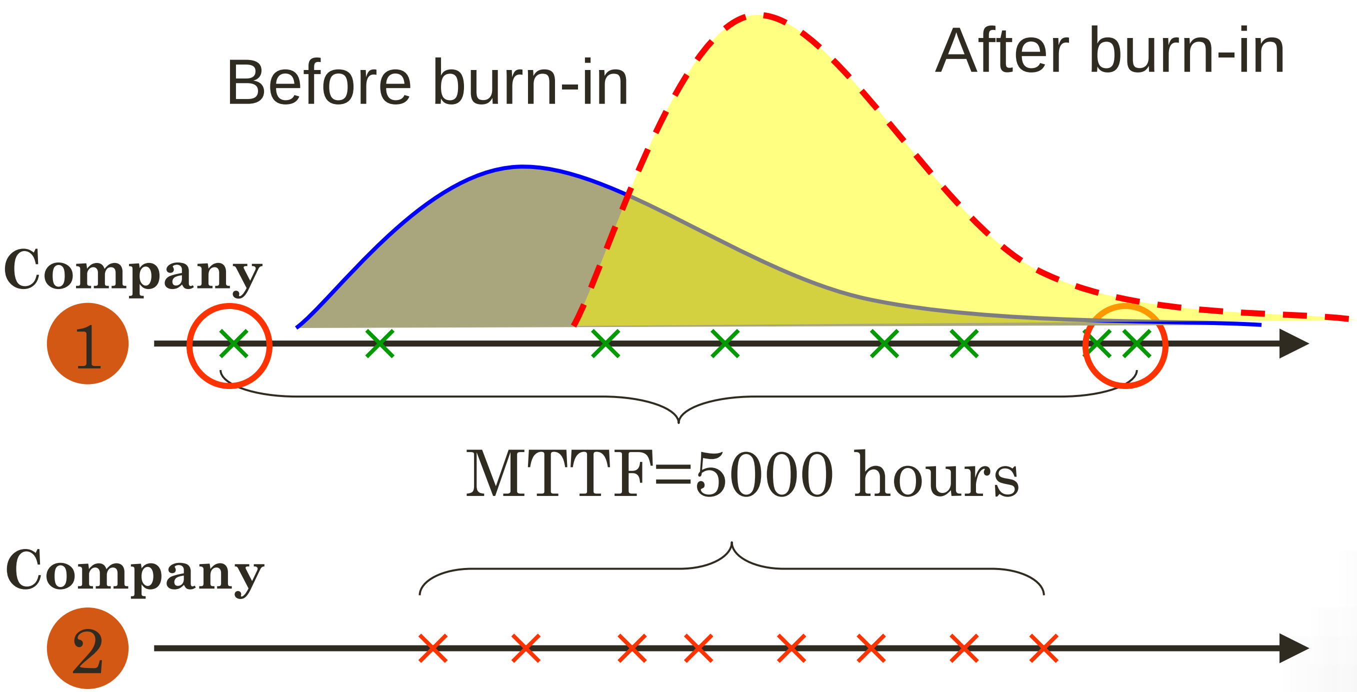
After 2 hours of burn-in

$$1 \quad 2 \;\vdots\; \underbrace{3\text{-}2=1 \quad 5\text{-}2=3 \quad 10\text{-}2=8 \quad 15\text{-}2=13 \quad 22\text{-}2=20 \quad 28\text{-}2=26}$$

$$MRL(\text{after 2 hours}) = \frac{1+3+8+13+20+26}{6} = 11.83 \text{ hours} > MTTF$$

Motivation - Use of Burn-in

- Improve reliability using "cull eliminator"

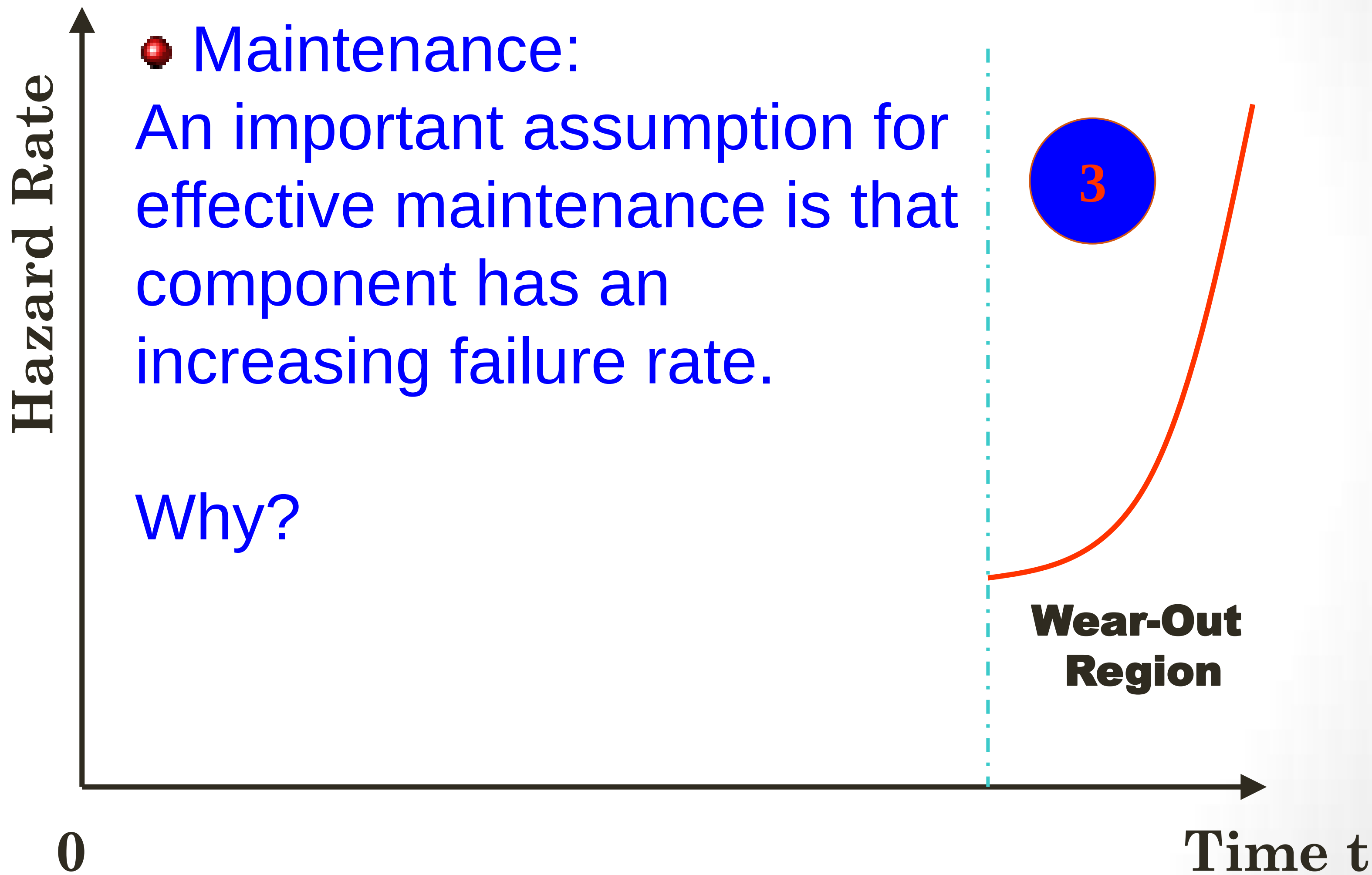


Related Topics (2)

- Maintenance:

An important assumption for effective maintenance is that component has an increasing failure rate.

Why?



Weibull Model

- **Definition**

$$f(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} \exp\left[-\left(\frac{t}{\eta}\right)^{\beta}\right] \qquad \beta > 0,\ \ \eta > 0,\ \ t \geq 0$$

$$\lambda(t) = f(t)/R(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1}$$

$$R(t) = \exp\left[-\left(\frac{t}{\eta}\right)^{\beta}\right] = 1 - F(t)$$

Weibull Model Cont.

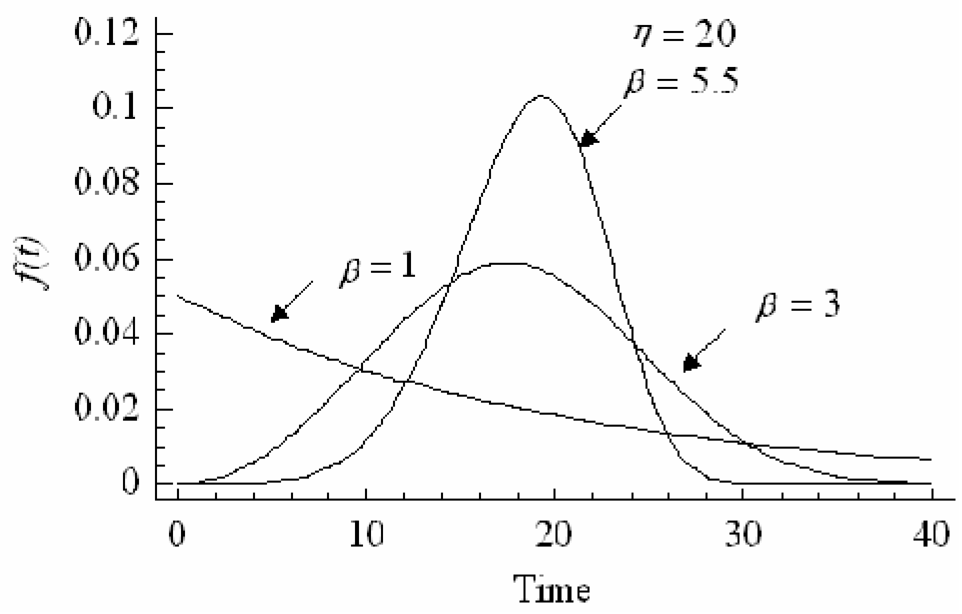
- Statistical properties

$$MTTF = \eta \int_0^{\infty} t^{1/\beta} e^{-t} dt = \eta \Gamma(1+\frac{1}{\beta})$$

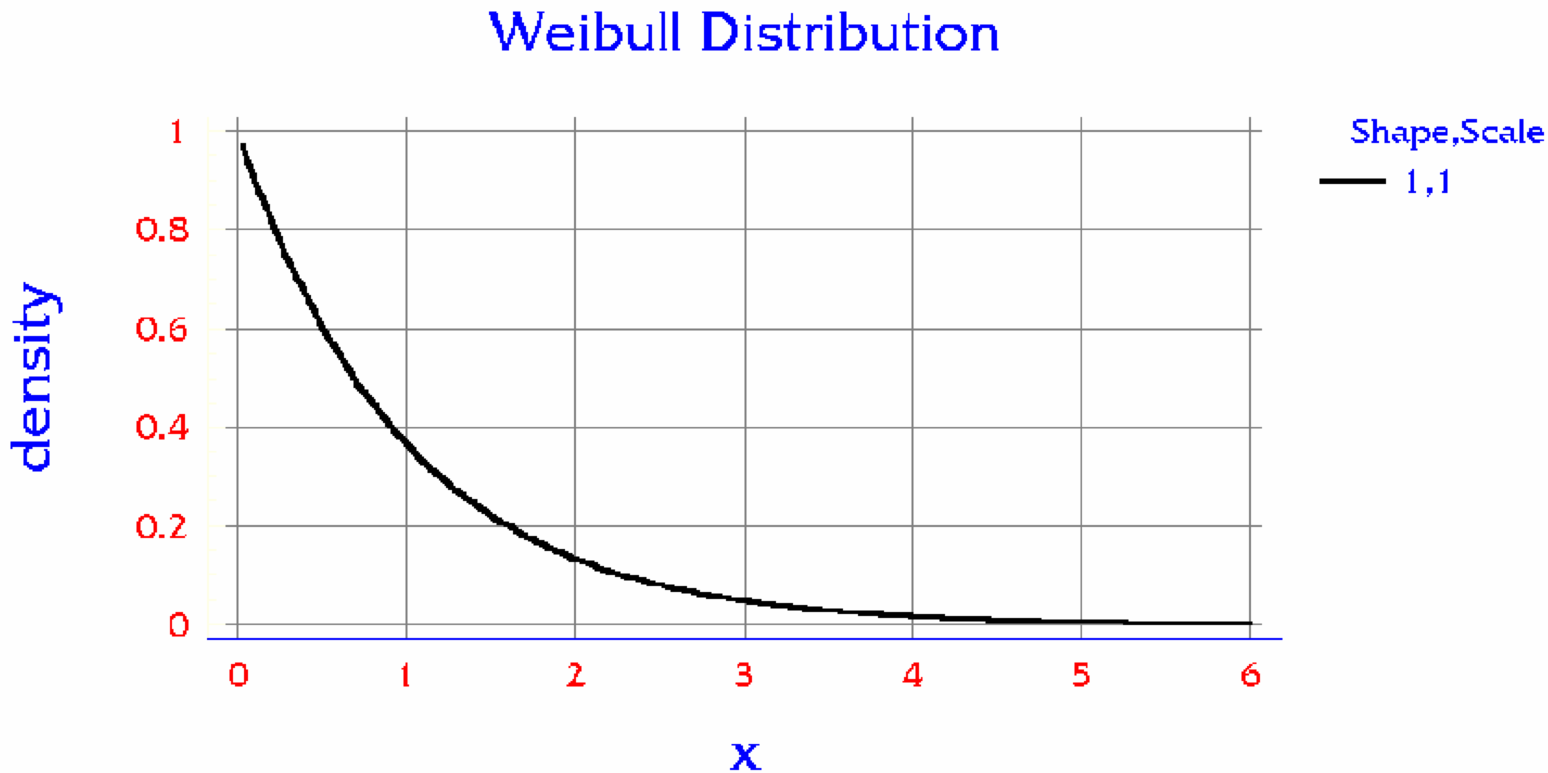
$$Var = \eta^2 \left[\Gamma(1+\frac{2}{\beta}) - \left(\Gamma(1+\frac{1}{\beta}) \right)^2 \right]$$

$$\text{Median life} = \eta((\ln 2)^{1/\beta})$$

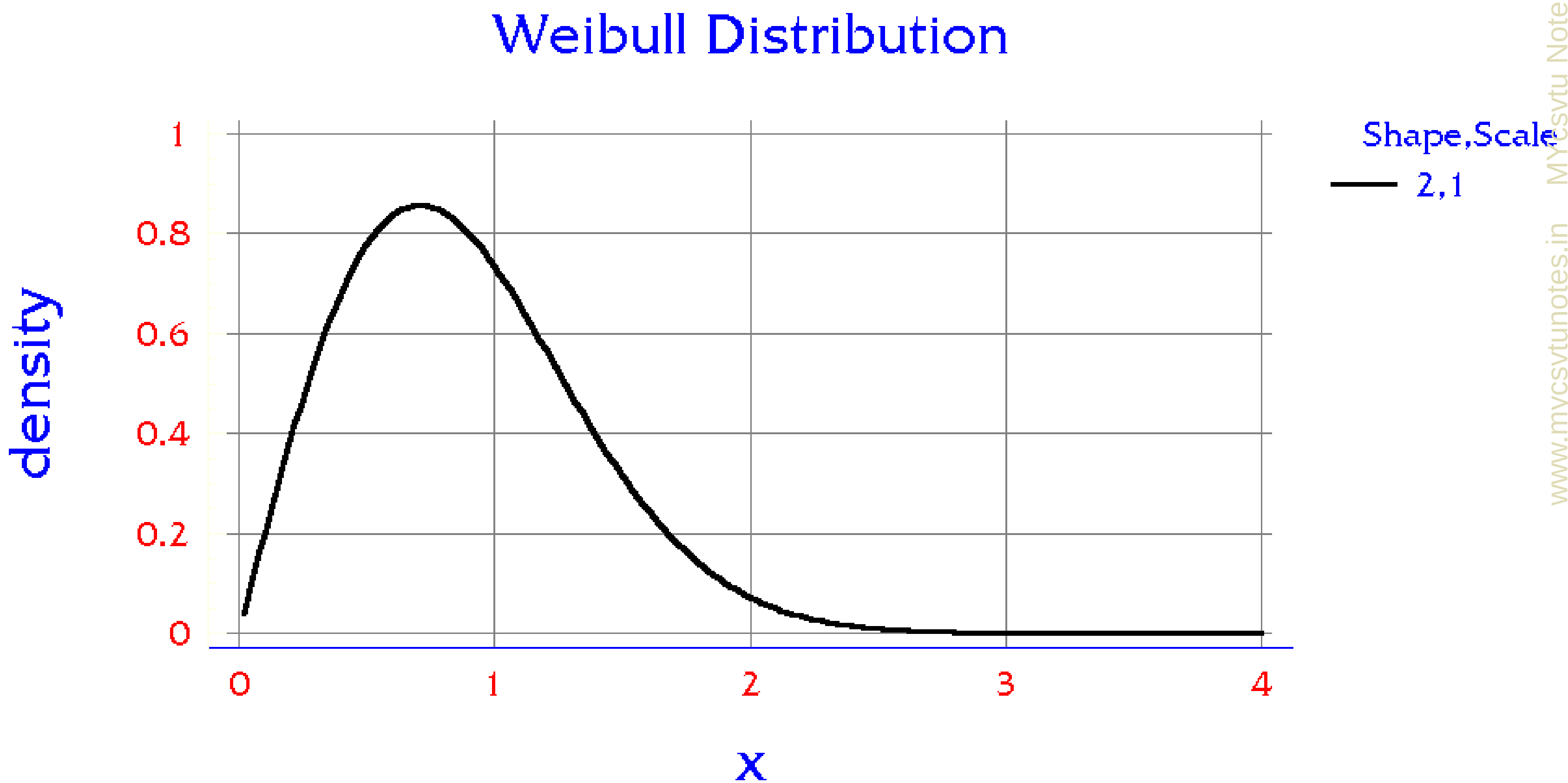
Weibull Model



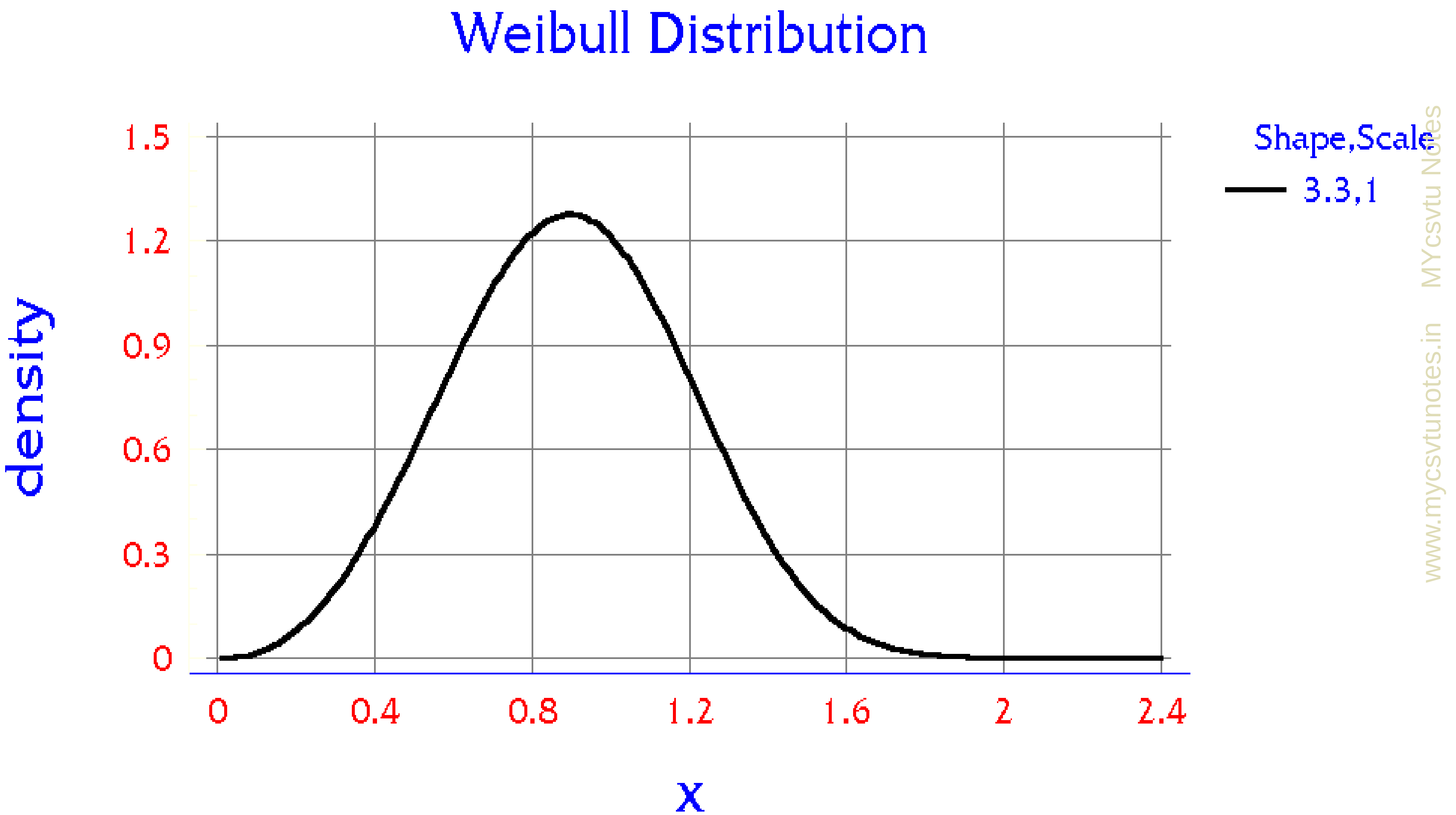
Weibull Analysis: Shape Parameter



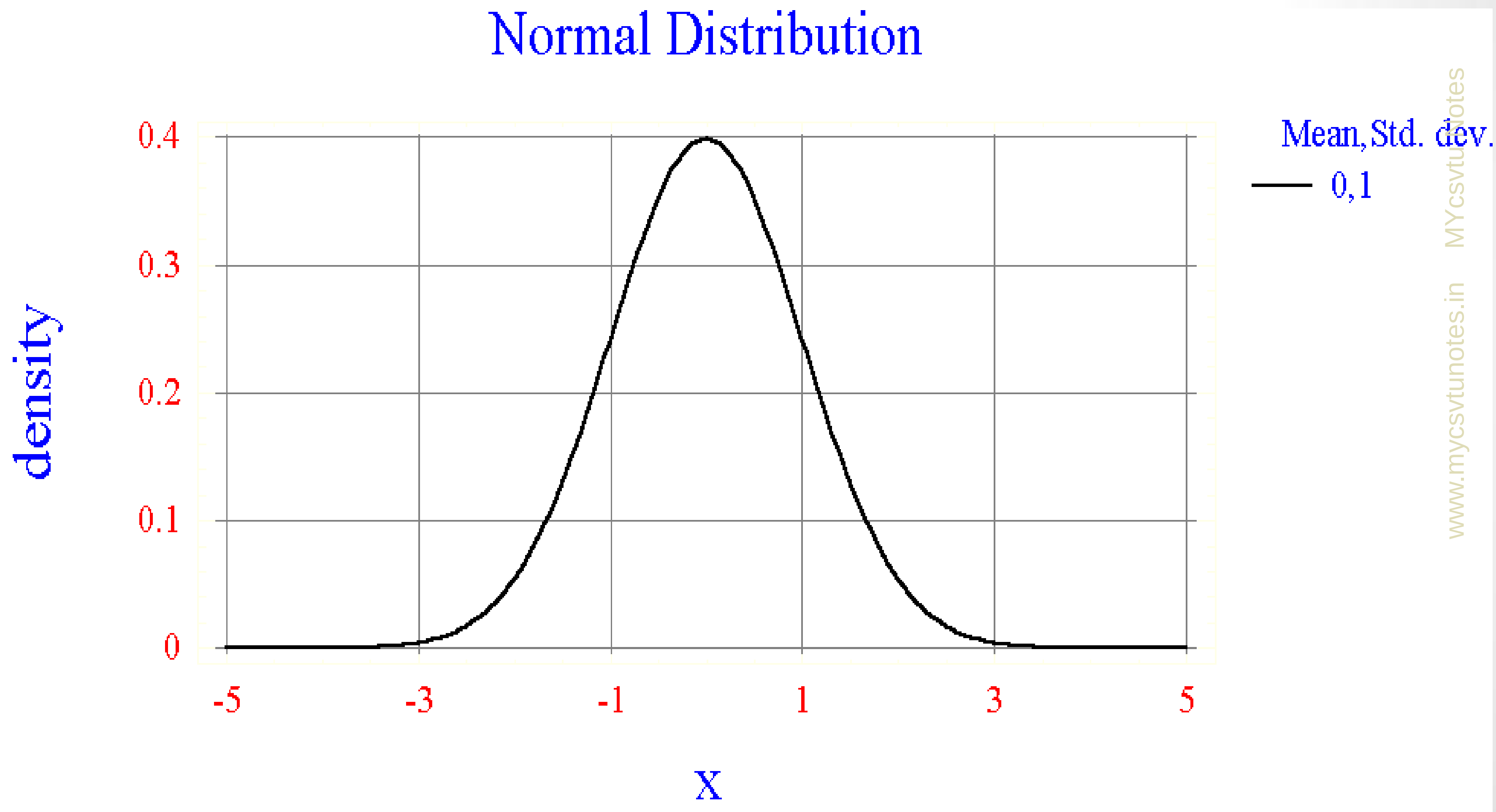
Weibull Analysis: Shape Parameter



Weibull Analysis: Shape Parameter



Normal Distribution



Weibull Model

$$h(t)=\frac{\gamma}{\theta}(\frac{t}{\theta})^{\gamma-1}.$$

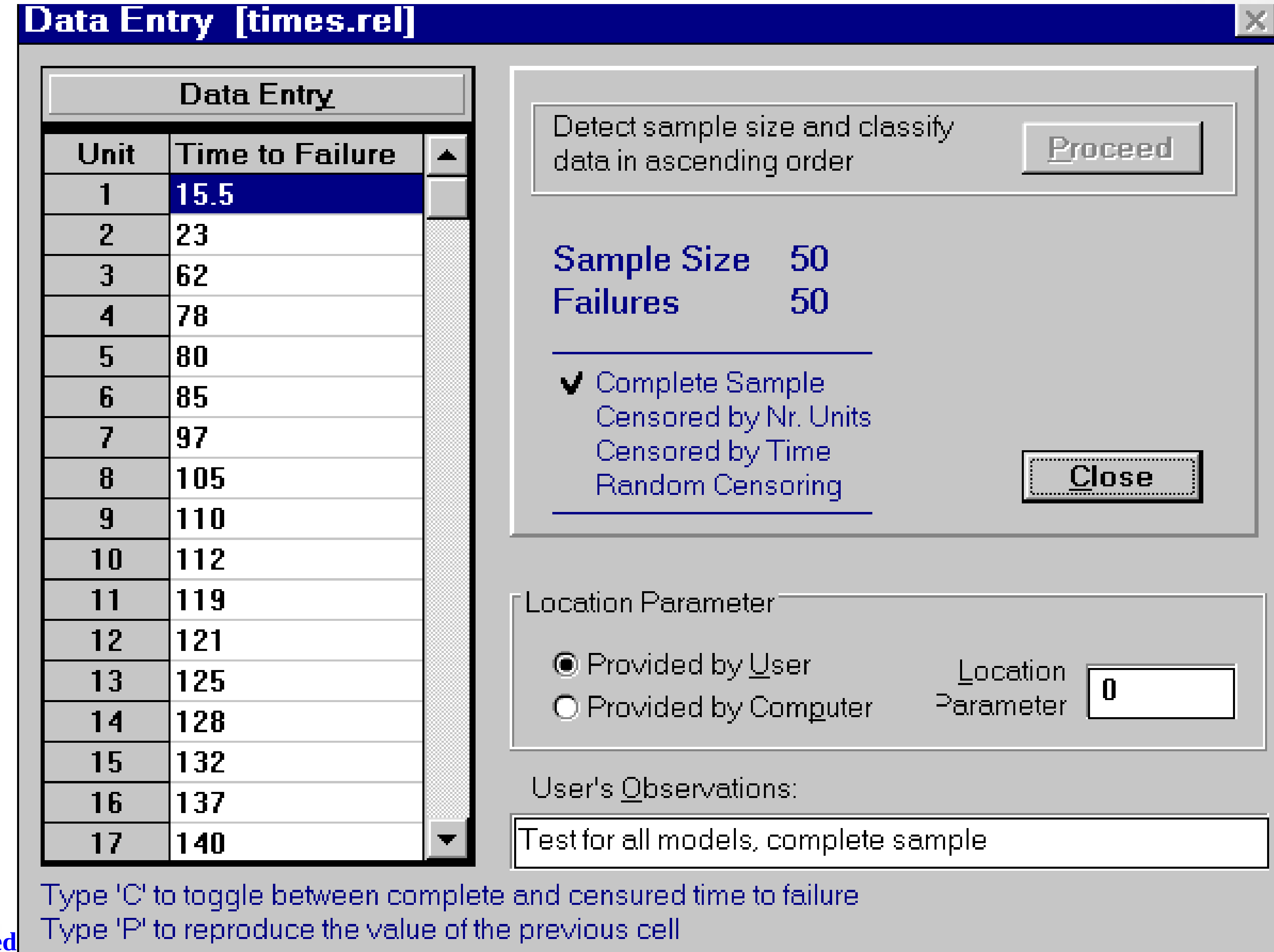
$$f(t)=\frac{\gamma}{\theta}(\frac{t}{\theta})^{\gamma-1}e^{-(\frac{t}{\theta})^{\gamma}}$$

$$F(t)=\int_0^t\frac{\gamma}{\theta}(\frac{\zeta}{\theta})^{\gamma-1}e^{-(\frac{\zeta}{\theta})^{\gamma}}d\zeta$$

$$F(t)=1-e^{-(\frac{t}{\theta})^{\gamma}}$$

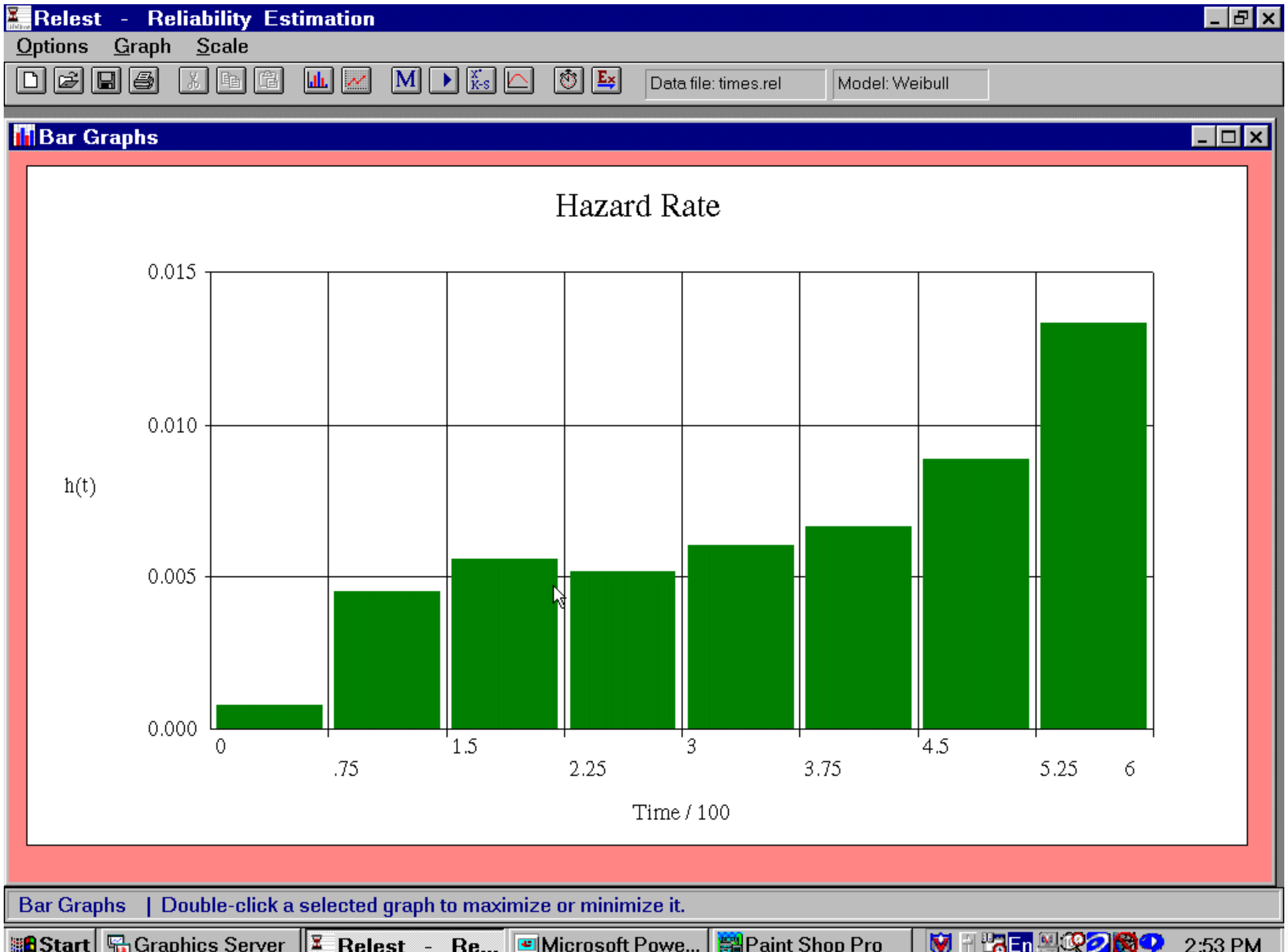
$$R(t)=e^{-(\frac{t}{\theta})^{\gamma}}$$

Input Data

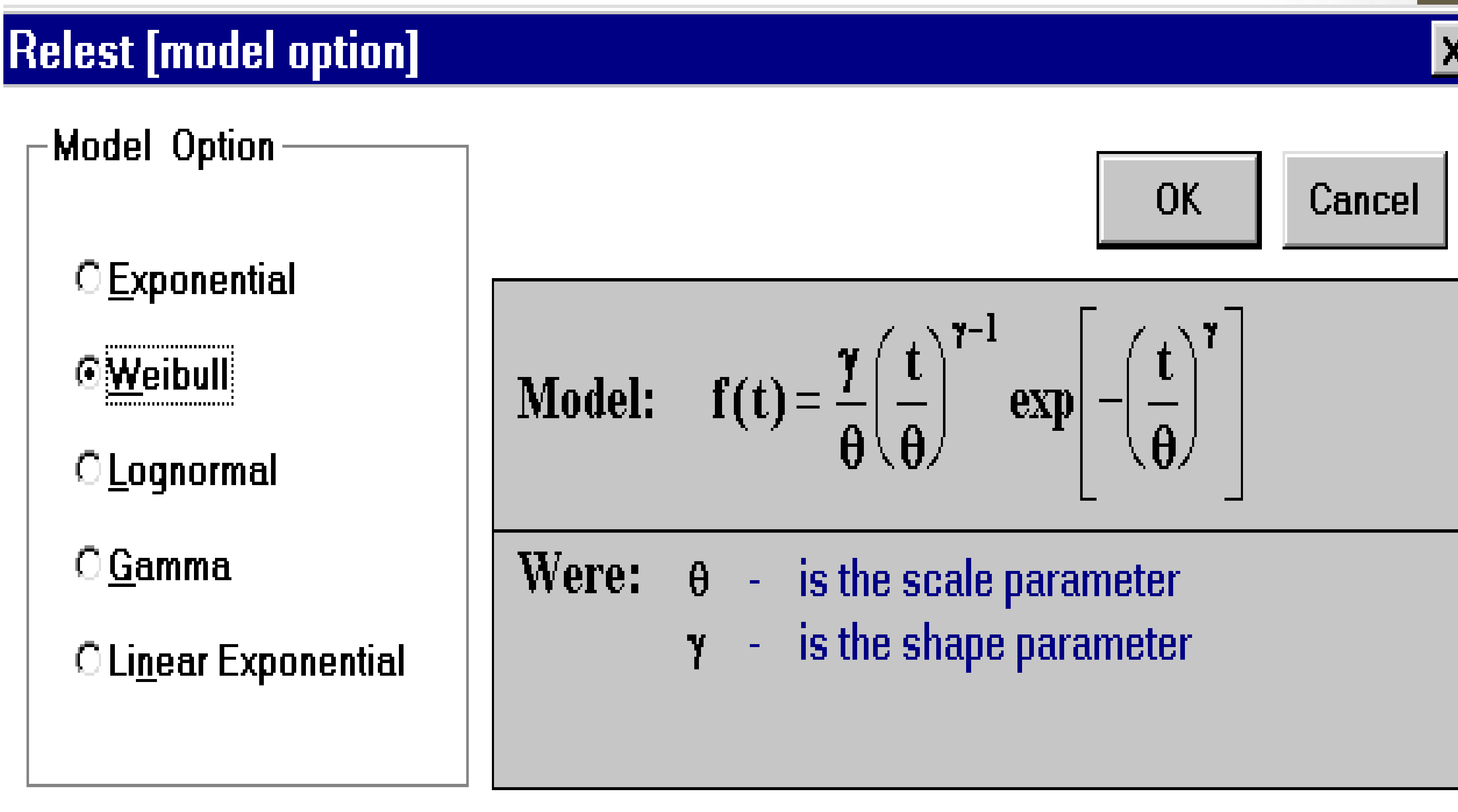


Unit	Time to Failure
1	15.5
2	23
3	62
4	78
5	80
6	85
7	97
8	105
9	110
10	112
11	119
12	121
13	125
14	128
15	132
16	137
17	140

Plots of the Data



Weibull Fit


Relest [model option]
X
Model Option
Exponential
Weibull
Lognormal
Gamma
Linear Exponential
OK
Cancel
Model: $f(t) = \frac{\gamma}{\theta}\left(\frac{t}{\theta}\right)^{\gamma-1} \exp\left[-\left(\frac{t}{\theta}\right)^{\gamma}\right]$
Were: θ - is the scale parameter
γ - is the shape parameter

Test for Weibull Fit

Goodness of Fit Tests

Chi-square Test ($\chi 2$)

Lower Limit	Upper Limit	Observed Frequency	Expected Frequency	Chisquare Value
0	74.978	3	6.2	1.64
74.978	149.956	16	11.9	1.39
149.956	224.934	12	12.	0.00
224.934	299.913	8	9.1	0.13
299.913	374.891	5	5.7	0.08
374.891	449.869	3	3.	0.00
449.869	524.847	2	1.4	0.31
524.847	above	1	0.8	0.05

Chi-square = 3.59 with 5 d.f.
Significance level = 0.6101

Close

Kolmogorov-Smirnov Test (K-S):

Estimated Kolmogorov DN = 0.07293
Approximate significance level = 0.2972

The hypothesis of Weibull distribution can not be rejected.

Parameters for Weibull

Distribution Fitting

Unit	Time to Failure	Without Location	Time Interval
1	15.5	15.5	15.5
2	23.	23.	7.5
3	62.	62.	39.
4	78.	78.	16.
5	80.	80.	2.
6	85.	85.	5.
7	97.	97.	12.
8	105.	105.	8.
9	110.	110.	5.
10	112.	112.	2.
11	119.	119.	7.
12	121.	121.	2.
13	125.	125.	4.
14	128.	128.	3.
15	132.	132.	4.
16	137.	137.	5.
17	140.	140.	3.

Weibull Without Censoring

Close

Confidence Interval: 90%

Change

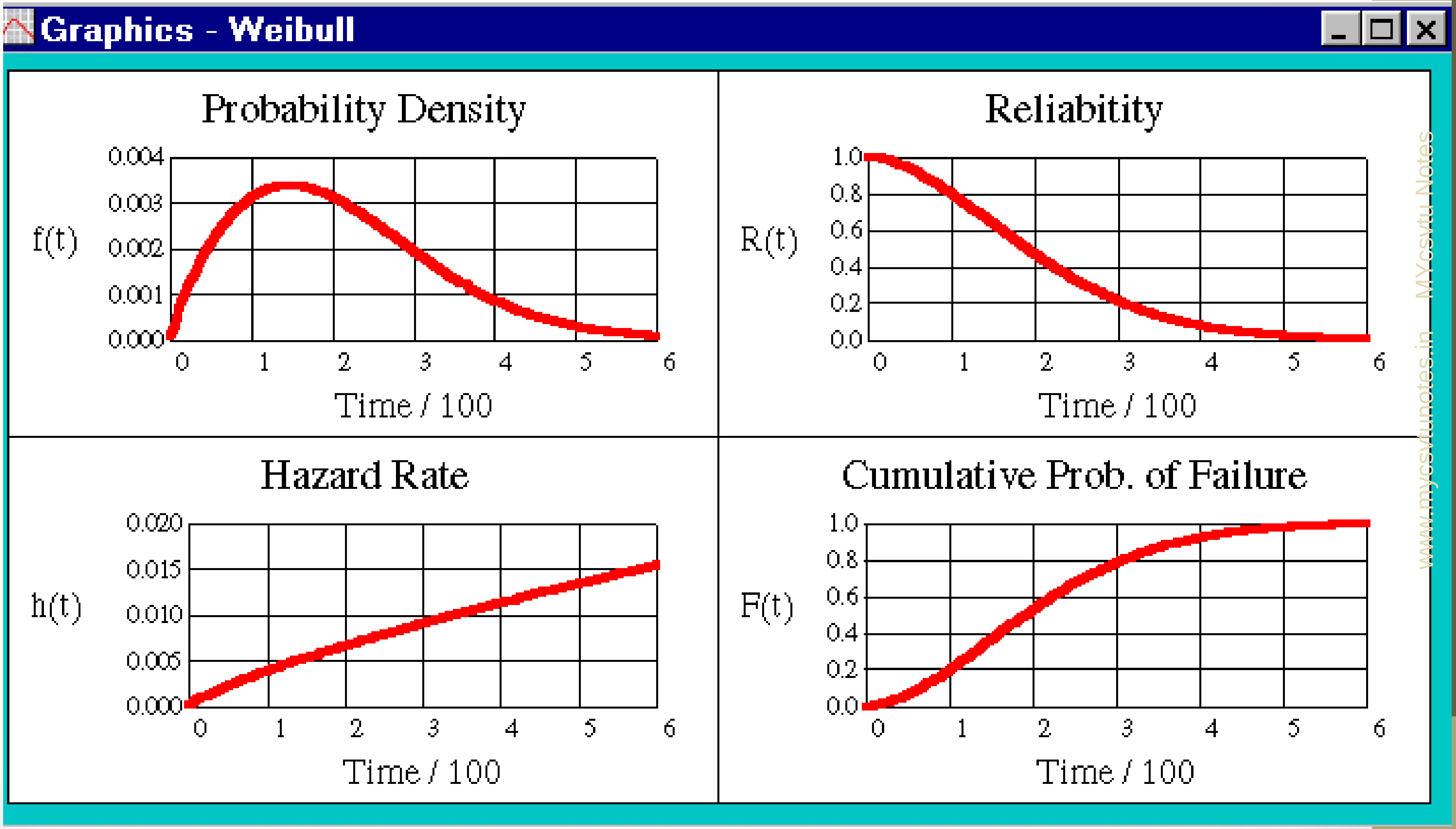
Location Parameter = 0

MLE:
γ = 1.8185
θ = 237.141
Var(γ) = 0.04021
Var(θ) = 377.181
Cov(θ, γ) = 1.21891

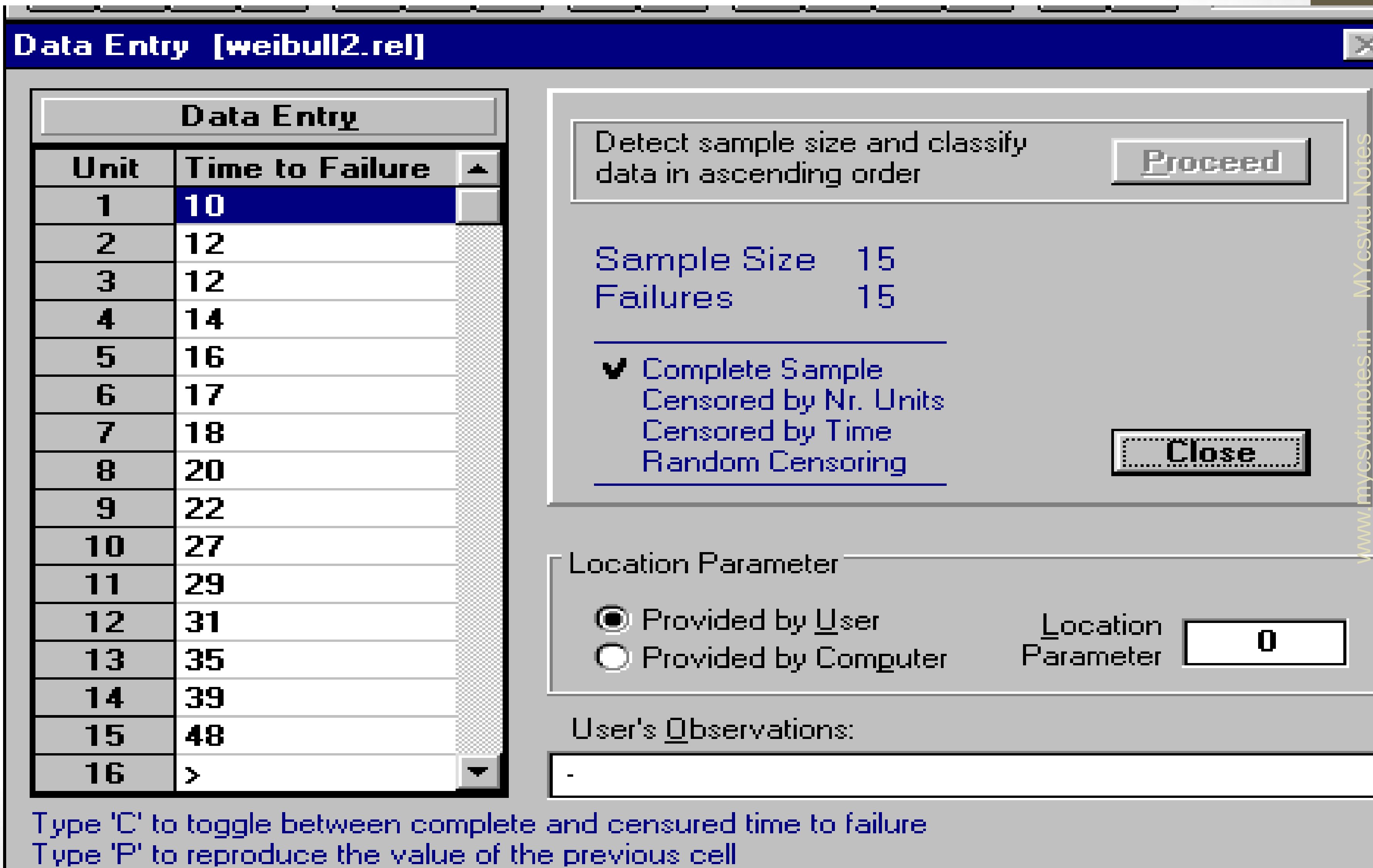
Unbiased estimate:
γ = 1.76894
Var(γ) = 0.04092
θ = 235.581

Confidence Interval: (Low) (Up)
for γ = 1.41274; 2.05453
for θ = 204.26; 272.005

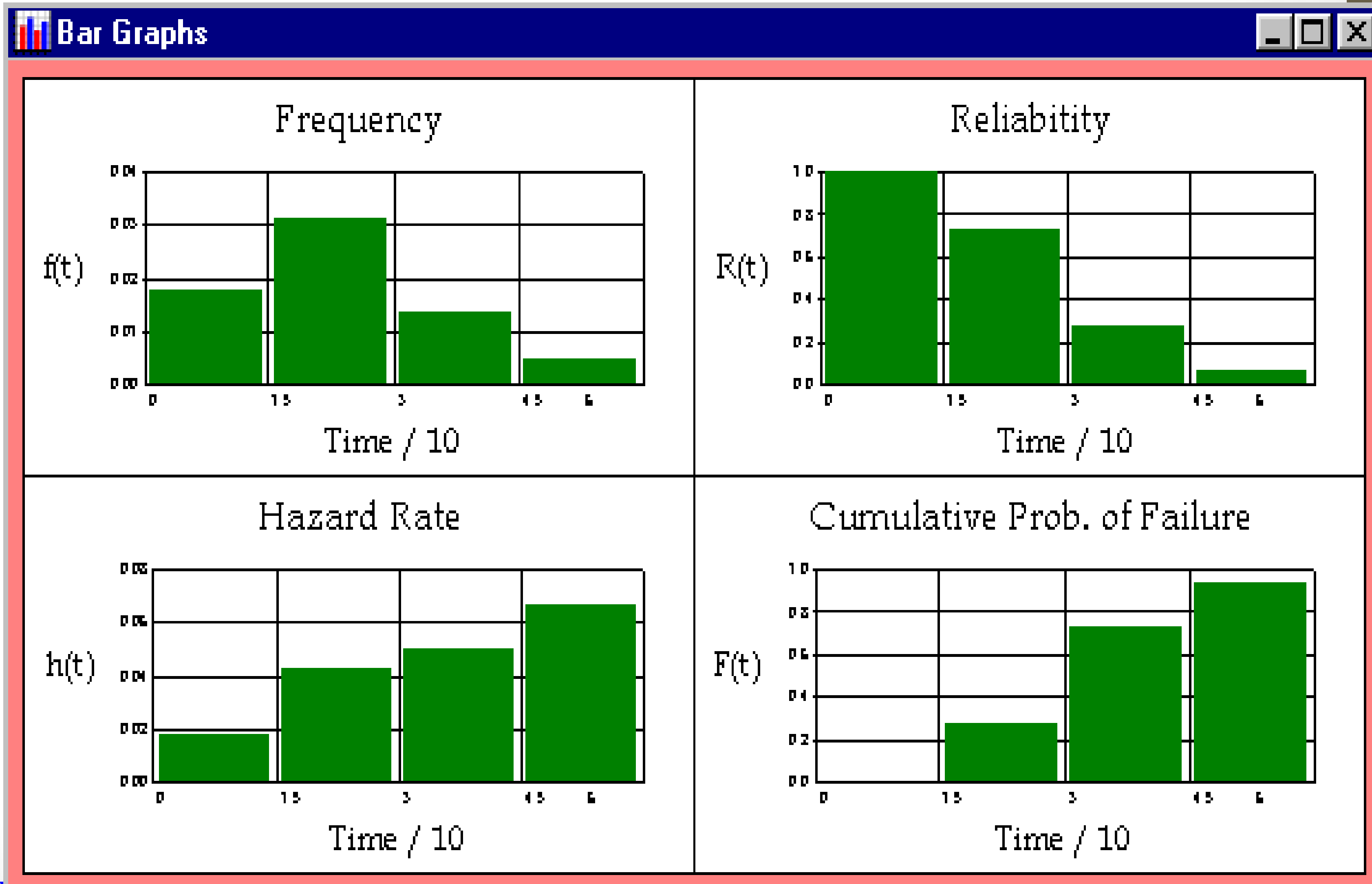
Weibull Analysis



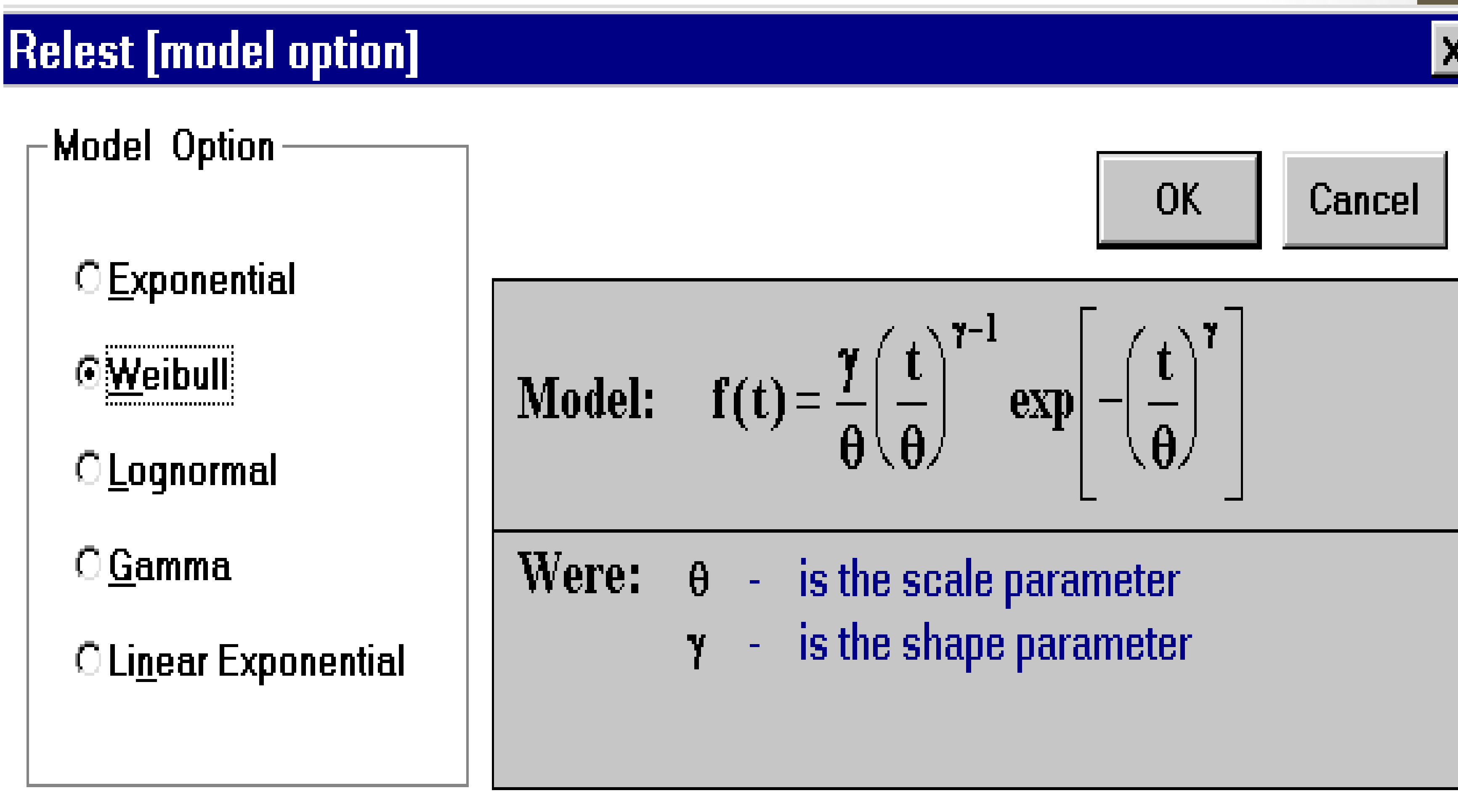
Example 2: Input Data



Example 2: Plots of the Data



Example 2: Weibull Fit



Example 2:Test for Weibull Fit

Goodness of Fit Tests

Chi-square Test (χ2):

Lower Limit	Upper Limit	Observed Frequency	Expected Frequency	Chisquare Value
8.1	18.55	7	5.8	0.23
18.55	29.	4	4.9	0.17
29.	39.45	3	2.9	0.00
39.45	above	1	1.3	0.09

Chi-square = 0.50 with 1 d.f.
Significance level = 0.4804

Close

Kolmogorov-Smirnov Test (K-S):

Estimated Kolmogorov DN = 0.12521
Approximate significance level = 0.3038

The hypothesis of Weibull distribution can not be rejected.

Example 2: Parameters for Weibull

Distribution Fitting

Unit	Time to Failure	Without Location	Time Interval
1	10.	10.	10.
2	12.	12.	2.
3	12.	12.	0
4	14.	14.	2.
5	16.	16.	2.
6	17.	17.	1.
7	18.	18.	1.
8	20.	20.	2.
9	22.	22.	2.
10	27.	27.	5.
11	29.	29.	2.
12	31.	31.	2.
13	35.	35.	4.
14	39.	39.	4.
15	48.	48.	9.

Weibull Without Censoring

Close

Confidence Interval: 90%

Change

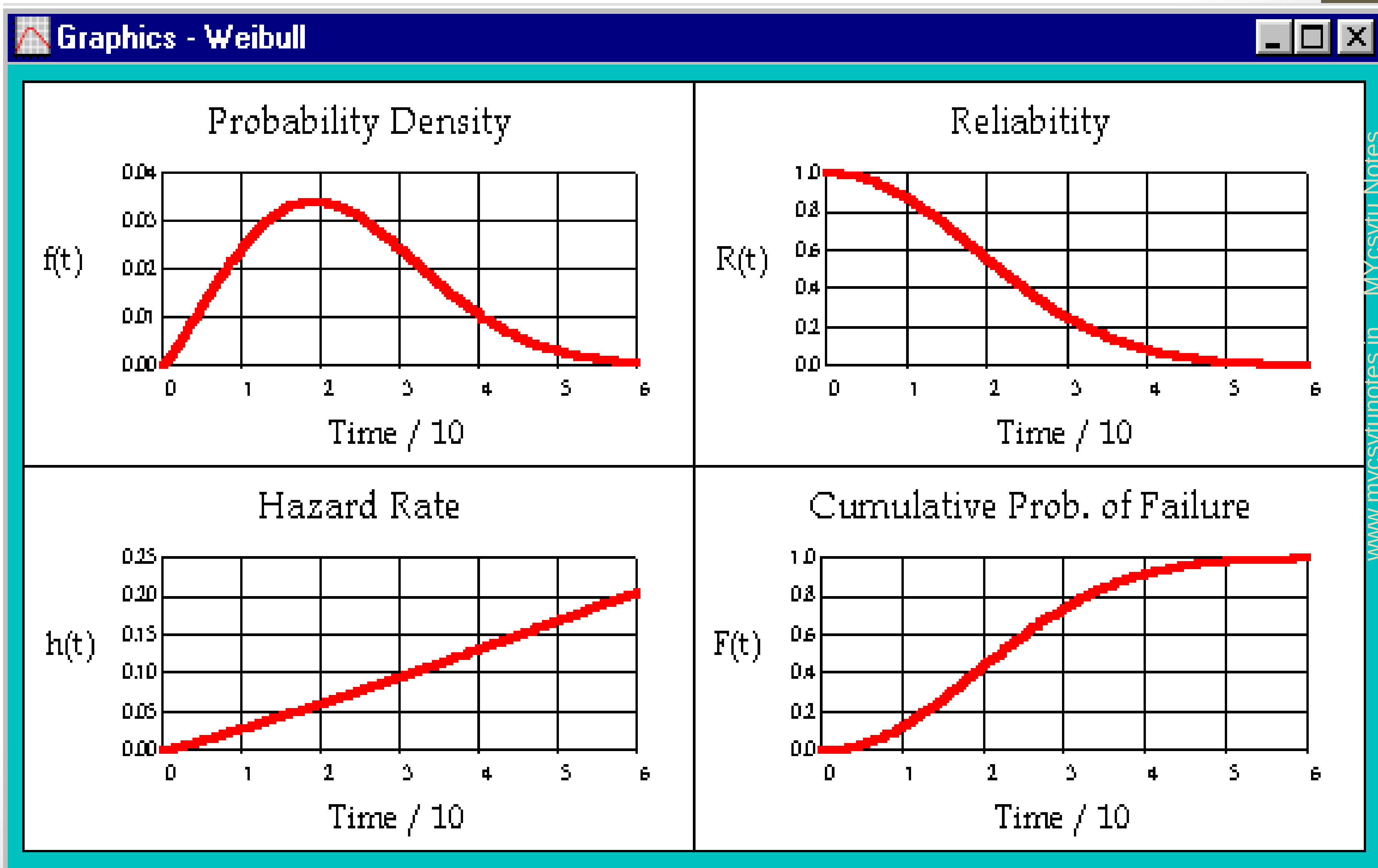
Location Parameter = 0

MLE:
γ = 2.3255
θ = 26.464
Var(γ) = 0.21917
Var(θ) = 9.57451
Cov(θ, γ) = 0.45342

Unbiased estimate:
γ = 2.10821
Var(γ) = 0.2253
θ = 25.965

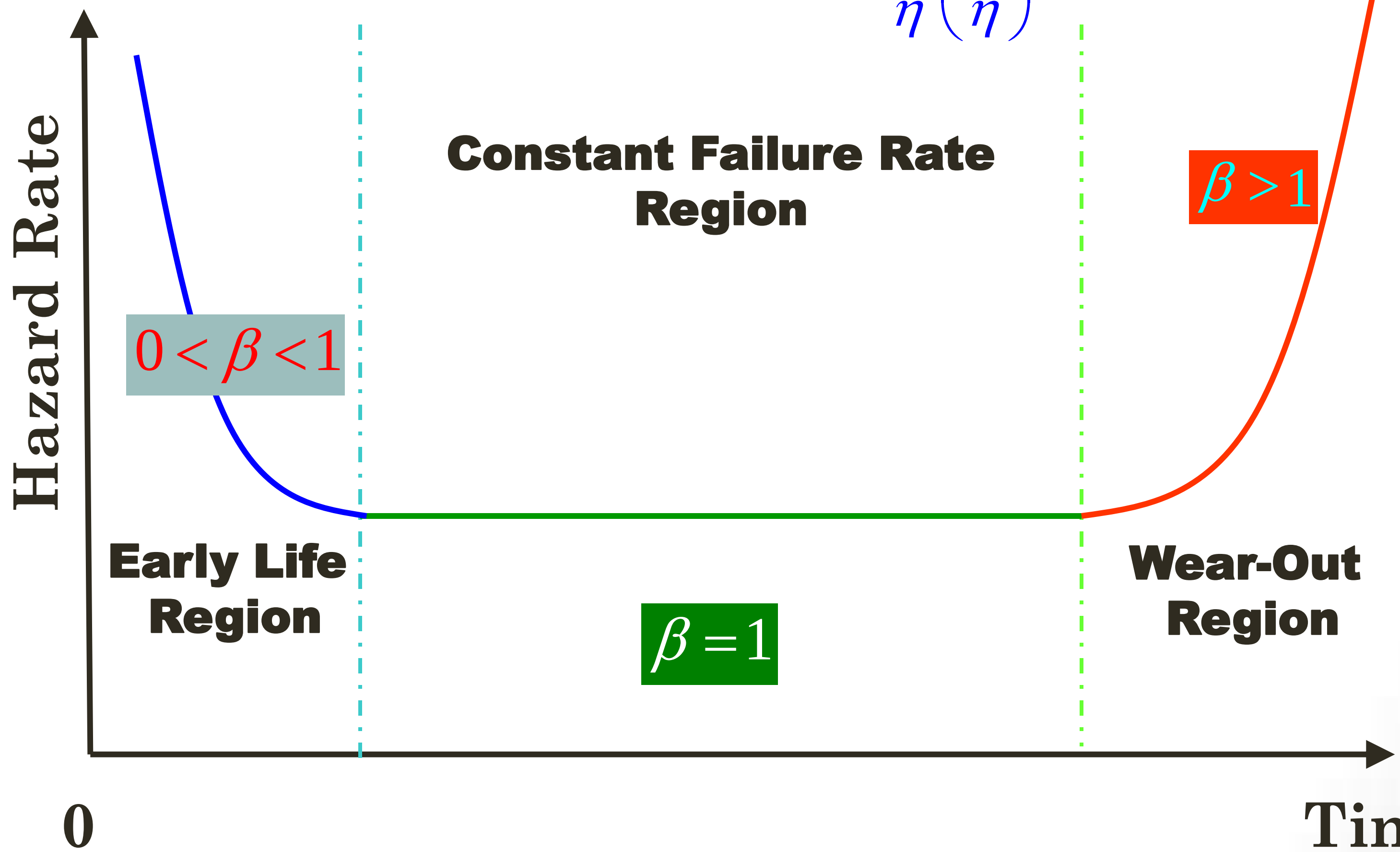
Confidence Interval: (Low) (Up)
for γ = 1.37217; 2.75212
for θ = 20.502; 33.086

Weibull Analysis



Versatility of Weibull Model

Hazard rate: $$\lambda(t) = f(t)/R(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1}$$



Graphical Model Validation

- Weibull Plot

$$F(t) = 1 - R(t) = 1 - \exp\left[-\left(\frac{t}{\eta}\right)^{\beta}\right]$$

$$\Rightarrow \quad \ln\ln\frac{1}{1-F(t)} = \beta\ln t - \beta\ln\eta$$ is linear function of ln(time).

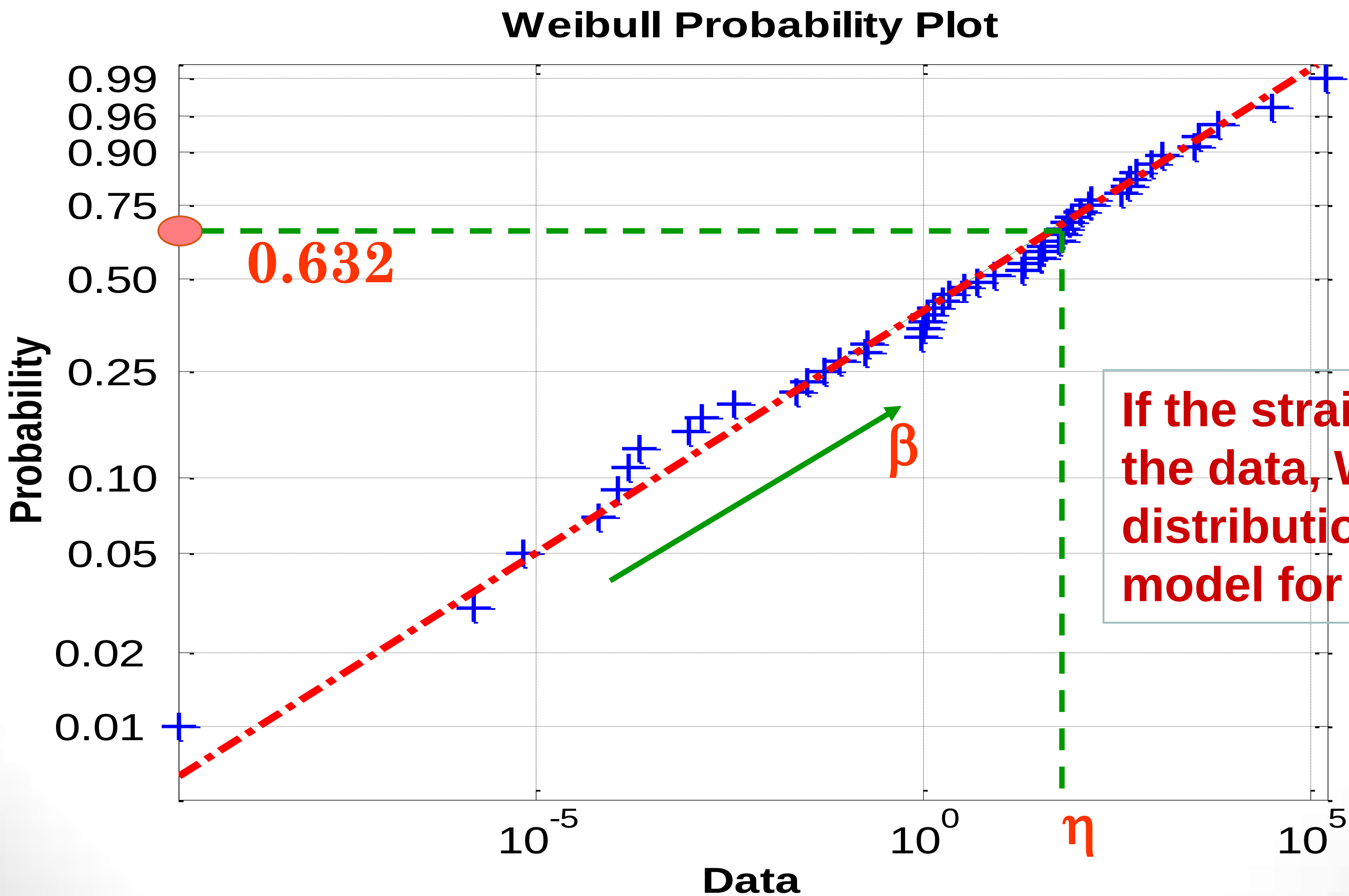
- Estimate $\hat{F}(t_i)$ at t_i using Bernard's Formula

For n observed failure time data $(t_1, t_2, \ldots, t_i, \ldots t_n)$

$$\hat{F}(t_i) = \frac{i - 0.3}{n + 0.4}$$

Example - Weibull Plot

- T~Weibull(1, 4000) Generate 50 data



If the straight line fits the data, Weibull distribution is a good model for the data